

# Contents

<i>Preface</i>	vii
1. Introduction	1
2. Graphs, Networks, Laplacian Matrices and Algebraic Connectivity	5
2.1 Graphs and digraphs . . . . .	5
2.2 Matrices and graphs . . . . .	8
2.3 Algebraic connectivity . . . . .	18
2.3.1 Undirected graphs . . . . .	18
2.3.2 Directed graphs . . . . .	19
2.3.3 Basic properties of $a_1$ and $b_1$ . . . . .	21
2.3.4 Basic properties of $a_2$ . . . . .	24
2.4 Locally connected graphs . . . . .	25
2.4.1 Basic properties of $a_3$ and $a_4$ . . . . .	27
2.5 Examples . . . . .	27
2.6 Hypergraphs . . . . .	28
2.7 Further reading . . . . .	29
3. Graph Models	31
3.1 Examples of complex networks . . . . .	31
3.1.1 Neural networks . . . . .	31
3.1.2 Transportation networks . . . . .	32
3.1.3 World Wide Web . . . . .	33
3.1.4 Erdős number and scientific collaboration network . . . . .	33
3.1.5 Film actor network . . . . .	33

3.1.6	Citation network . . . . .	34
3.2	Classical random graph models . . . . .	35
3.2.1	Algebraic connectivity of random graphs . . . . .	35
3.3	Small-world networks . . . . .	37
3.3.1	Watts-Strogatz model . . . . .	38
3.3.2	Newman-Watts model . . . . .	38
3.3.3	Algebraic connectivity of small-world networks . . . . .	38
3.4	Scale-free networks . . . . .	39
3.5	Random geometric graphs . . . . .	40
3.6	Graphs with a prescribed degree sequence . . . . .	41
3.7	Algebraic connectivity and degree sequence . . . . .	45
3.7.1	Regular graphs . . . . .	45
3.7.1.1	Construction 1: graph with low $\lambda_2$ and $r$ . . . . .	45
3.7.1.2	Construction 2: graph with high $\lambda_2$ and $r$ . . . . .	47
3.7.2	Graphs with prescribed degree sequence . . . . .	47
3.7.2.1	Construction 1: graph with low $\lambda_2$ and $r$ . . . . .	48
3.7.2.2	Construction 2: graph with high $\lambda_2$ and $r$ . . . . .	49
3.8	Further reading . . . . .	49
4.	Synchronization in Networks of Nonlinear Continuous-time Dynamical Systems . . . . .	51
4.1	Static coupling topology . . . . .	53
4.1.1	Properties of $\mu(G)$ . . . . .	54
4.1.2	Computing $\mu(G)$ . . . . .	59
4.1.3	Zero row sums matrices . . . . .	60
4.1.4	Matrices in $\mathcal{W}$ . . . . .	61
4.1.5	Synchronization and algebraic connectivity . . . . .	62
4.2	Coupling topology with a spanning directed tree . . . . .	62
4.3	Time-varying coupling topology . . . . .	66
4.4	Coupling between delayed state variables . . . . .	67
4.4.1	Choosing the factorization $B_1 B_2 = U G_\tau \otimes V D_\tau$ . . . . .	69
4.4.2	Choosing the matrix $U \in \mathcal{W}_s$ . . . . .	70
4.4.3	Choosing the matrix $K$ . . . . .	70
4.4.4	The case $D = 0$ . . . . .	71
4.5	Synchronization criteria based on algebraic connectivity . . . . .	72
4.6	Further reading . . . . .	74
5.	Synchronization in Networks of Coupled Discrete-time Systems . . . . .	75

5.1 Synchronization of coupled scalar maps via contractivity of operators . . . . .	81
6. Synchronization in Network of Systems with Linear Dynamics	83
6.1 Autonomous coupling . . . . .	83
6.2 Nonautonomous coupling: continuous-time case . . . . .	84
6.2.1 Slowly varying coupling . . . . .	85
6.3 Nonautonomous coupling: discrete-time case . . . . .	86
6.3.1 A discrete-time consensus problem . . . . .	86
6.4 Ergodicity of inhomogeneous Markov chains . . . . .	87
6.5 Contractive matrices . . . . .	96
6.5.1 Pseudocontractivity and scrambling stochastic matrices . . . . .	98
6.5.2 Set-nonexpansive and set-contractive operators . . . . .	99
6.5.3 Set-contractivity under the max-norm . . . . .	103
6.5.4 Set-contractivity under the Euclidean norm . . . . .	104
6.5.5 Set-contractivity under a weighted Euclidean norm . . . . .	104
6.5.6 Set-contractivity and coefficient of ergodicity . . . . .	108
6.6 Further reading . . . . .	109
7. Agreement and Consensus Problems in Groups of Interacting Agents	111
7.1 Continuous-time models . . . . .	111
7.1.1 Rate of exponential convergence . . . . .	113
7.1.2 Dynamic coupling topology . . . . .	113
7.2 Discrete-time models . . . . .	114
7.2.1 Follow the leader dynamics and leadership in coordinated agents . . . . .	118
7.3 A nonlinear model for consensus . . . . .	119
7.4 Agreement in random networks . . . . .	120
7.4.1 Synchronization in random networks without the scrambling condition . . . . .	122
7.5 Further reading . . . . .	123
Appendix A Algebraic Connectivity and Combinatorial Properties of a Graph	125
A.1 Properties of algebraic connectivity . . . . .	125
A.2 Vertex and edge connectivity . . . . .	135

A.3 Graph partitions . . . . .	135
A.3.1 Maximum directed cut . . . . .	135
A.3.2 Edge-forwarding index . . . . .	136
A.3.3 Bisection width . . . . .	136
A.3.4 Isoperimetric number . . . . .	137
A.3.5 Minimum ratio cut . . . . .	138
A.3.6 Independence number . . . . .	138
A.4 Semibalanced graphs . . . . .	139
<i>Bibliography</i>	143
<i>Index</i>	153