context. In Section 6 we give relevant references to the literature.

1.2 Historical background

In different disguises, problems of financial engineering have been discussed since classical times in a variety of sources stretching from Plato's Dialogues to the Old and New Testaments. Many illustrious noble houses in Medieval Italy made their fortunes by astute dealings in money lending and foreign exchange. An intimate relation between money lending (fixed income in modern language) and money changing (foreign exchange) was used by Italian bankers collectively known as the Lombards, in order to circumvent the church prohibition on charging interest on loans. They lent money in one currency and received it back in another one with the FXR artificially lowered to accommodate the interest. The blossoming of the Netherlands in the seventeenth century is, at least partly, due the introduction of new financial instruments such as forward contracts, calls and puts, etc. The rise of the British Empire resulted in further advances in finance, particularly, on the fixed income side. The economic growth in the United States was facilitated by the unprecedented growth of the financial system, especially, by the introduction of limited liability companies.

The modern development of financial engineering starts with the work of the French mathematician Louis Bachelier who, in the year 1900, published the now famous memoir entitled "*Theorie de la spéculation*". Bachelier's achievements are remarkable in several respects. To mention just one, he developed (before Einstein and others) the first theory of Brownian motion which he used in order to quantify the evolution of stock prices. In modern terms, Bachelier assumed that the stock price follows an arithmetic Brownian motion and, consequently, is distributed normally at any given time. He derived the pricing formulas for call and put options on such stocks. However, since Bachelier's theory predicted that stock prices can become negative and because of the sheer complexity of its mathematical apparatus, the theory was neglected by the mainstream economists for more than fifty years.

Fundamental contributions to modern financial engineering were made in the 1950s by several authors. Arrow (1953) and Debreu (1959) extended the existing economic models by incorporating uncertainty and showed how to solve the corresponding asset allocation problem. Modigliani and Miller (1958) proved that the financial structure of the firm, i.e., the firm's choice between equity and debt financing, does not affect its value. The method of financial arbitrage they used turned out to be even more useful than the theorem itself and became the method of choice for generations of financial engineers. Finally, Markowitz (1959) developed the mean-variance portfolio selection theory.

In the 1960s financial engineering continued to grow rapidly. In particular, Sprenkle (1961), Boness (1964), and Samuelson (1965) proposed a more adequate description of the stock price evolution by assuming that this price follows the geometrical Brownian motion and, consequently, is distributed lognormally which guarantees its positivity. Although they obtained a closed form formula for pricing options on lognormally distributed stocks, this formula was difficult to use in practice because it contained too many free parameters, namely, the volatility of the stock, and the growth rates of the stock and option. In the meantime, Sharpe (1964), Lintner (1965), and Mossin (1966) extended the Markowitz theory and created the so-called capital asset pricing model (CAPM).

The major breakthrough was achieved the early 1970s when Black and Scholes (1973) and Merton (1973) discovered a consistent pricing formula for stock options depending on the volatility of the underlying stock and the riskless interest rate at which the money can be borrowed overnight (rather than the growth rates of the underlying stock and option). The Black-Scholes-Merton pricing methodology is based on the idea of dynamic hedging of derivatives which allows the seller of an option to become indifferent to the changes in the underlying stock price. In order to hedge himself, the seller of an option has to maintain positions in both stock and bond and to adjust them dynamically when the stock price changes. For their discovery Scholes and Merton were awarded the Nobel Prize for Economics in 1997 (Black died in 1995). The Black-Scholes formula instantly became extremely popular among practitioners and academics alike, and within a few years helped to create a multi-trillion dollar market in financial derivatives, currently estimated at $65 trillion. (The impact of Black-Scholes discovery on financial markets is a great example of the influence of mathematics on society at large.)

Many important contributions to financial engineering were made after publication of the seminal papers by Black and Scholes and Merton. For instance, Merton himself and later Rubinstein and Reiner (1991), used the well-known method of images in order to price the so-called barrier options which disappear when the price of the underlying hits a predetermined barrier. Black (1976) derived the formula for the valuation of options on futures. Ross (1976) developed the arbitrage-pricing theory (APT) as an alternative to CAPM. Margrabe (1978) valued the right to exchange one risky asset for another (via an elegant application of the principle of homogeneity). Harrison and Kreps (1979) and Harrison and Pliska (1981) developed an approach to pricing and hedging of derivatives which complements the one developed by Black and Scholes and Merton. They showed that the price of an option (provided that it is hedged appropriately), can be written as the expectation (with respect to the so-called risk-neutral probability measure) of its discounted payoff at maturity. In probabilistic terms, they showed that the discounted value of an option is a martingale so that options can be priced via probabilistic methods (which complement partial differential equations methods originally used by Black-Scholes). Mathematical aspects of this approach were elucidated by Delbaen and Schachermayer (1994). Garman and Kohlhagen (1983) extended the Black-Scholes valuation formula in order to incorporate options on forex.

Although the original approach to problems of financial engineering was predominantly analytical, numerical methods necessary to solve more complicated problems were developed as well. The earliest were the so-called binomial, explicit finite difference, and Monte Carlo methods introduced by Cox, Ross and Rubinstein (1979), Schwartz (1977) and Brennan and Schwartz (1978), and Boyle (1977), respectively. Although these methods were very intuitive, they were not sufficiently refined and could not price certain derivatives accurately. In a due course they were complemented by the implicit finite difference method which became the method of choice for solving more advanced problems.

In spite of its many successes, the Black-Scholes formula is too idealized and does not capture certain features of the market. The most important of those features are the non-lognormal distribution of the underlying stock prices, transaction costs, liquidity, and discontinuous nature of trading. When this formula is used in practice, different volatilities are used to price options with different strikes which gives rise to the so-called volatility skew (in equity markets) and smile (in forex markets). Appropriate modifications of the Black-Scholes paradigm which would account for skews and smiles observed in actual markets is a major challenge which is only partly met at present.

It proved to be more difficult to model fixed income derivatives, such as bond options, caps, floors, etc., than forex and equity derivatives. This disparity is due to the fact that the dynamics of the underlying short term interest rate (or the bond price) is much more complicated than the forex and stock dynamics and cannot be approximated by the geometrical Brownian motion. (Indeed, empirical observations suggest that interest rates are mean-reverting, while, by definition, bond prices exhibit the so-called pull-to-par property and approach bond's notional value at maturity). In addition, one has to deal with bonds of all maturities (the so-called yield curve) at once. Vasicek (1977) was the first to propose an analytically tractable model taking into account the mean reversion property of the instantaneous interest rate and the pull-to-par property. He assumed that the interest rate follows the Ornstein-Uhlenbeck process and obtained analytical expressions for prices of bonds and bond options. However, Vasicek model is not without some limitations; namely, it cannot be fitted sufficiently accurately to the market data, and it allows interest rates to become negative, i.e., it suffers from the same difficulty as the original Bachelier model for equities. While the first drawback is easy to rectify, as was done by Hull and White (1990), the second one is an inherited feature of the model and cannot be helped. In order to guarantee the positivity of interest rates one needs to use other stochastic processes with mean reversion, such as the Feller square-root process used by Cox, Ingersoll and Ross (1985), or the log-Ornstein-Uhlenbeck process used Black and Karasinski (1991). An alternative idea of dealing with fixed income derivatives is based on studying the yield curve in its entirety, as described by Heath, Jarrow and Morton (1992) and Brace, Gatarek and Musiela (1997) in continuous and discrete settings, respectively. In spite of the progress made by the above mentioned authors and many others, an adequate versatile model for pricing and hedging of fixed income derivatives is still missing.

1.3 Forex as an asset class

As was mentioned earlier, the daily turnover of forex markets is approximately $1.5 trillion. The majority of transactions with foreign exchange are executed from London and New York (about 1/3 and 1/5 of the total, respectively), as well as Tokyo, Singapore, Frankfurt, Zurich, etc. Market participants include governments, banks, international corporations, mutual and hedge funds, and individual investors. With increasing globalization of the world financial system, the role of forex as an important asset class in its own right on a par with more traditional equity and fixed income instruments becomes more and more apparent. Indeed, it is necessary to have an exposure to forex in order to be able to invest in the global markets and create a well-balanced portfolio.

When investors based in a particular country put their money in domestic bonds and equities, they are not (directly) affected by the forex rate fluctuations and all the uncertainties they have to face are domestic in nature. However, if they decide to invest their money in foreign bonds and equities, they first have to convert the domestic currency (say US dollars) into the foreign currency (say euros) at the deterministic spot rate in order to purchase foreign securities, and then, at some time in the future, they have to convert the foreign currency generated by these securities into the domestic currency (at the unknown rate prevailing at that time). Thus, investors have to deal with uncertainties both at home and overseas, and, in addition, with changes in forex rate. Nonetheless, foreign investments can generate returns which are