Preface

This text is based on the lecture notes of a graduate-level course "Mathematical Theory of Financial Derivatives", which I gave first at Department of Mathematics, University of Iowa, U.S.A., and later at Department of Applied Mathematics, Tongji University, Shanghai, China. In this course, I intended to present a systematic and in-depth introduction to the Black-Scholes-Merton's option pricing theory from the perspective of partial differential equation theory. It is the author's hope that this text may contribute to filling a gap in the existing literature.

Option is a financial derivative. Therefore its price depends on the underlying asset's price movement. In the case of the continuous time model, the movement of the underlying asset's price can be described by a stochastic differential equation. Consequently, according to the idea of Black and Scholes, the option price can be modeled as a terminal-boundary problem for a partial differential equation (PDE). Therefore it is reasonable to adopt the existing theory and methods of PDE as a fundamental approach to the study of the option pricing theory. This includes establishing the PDE models for various types of options, deriving the pricing formulas as solutions of the corresponding PDE problems, making qualitative and in-depth analysis of the structure of the option price, and designing efficient algorithms for solving option pricing problems from the viewpoint of numerical solutions of PDE problems.

As a textbook for graduate students in applied mathematics, the depth and scope of this book must be appropriate. In order to limit the prerequisites, we tried our best to make this text self-contained when topics of modern mathematics are involved. In fact, we only assume a basic knowledge of calculus, linear algebra, elementary probability theory, and mathematical physics equations. When topics of stochastic analysis, numerical methods of PDE and free boundary problems are encountered in the text, only a brief presentation of the basic concepts and results is given. That is, the conclusion is stated, the basic idea of the proof is explained, but the details are not presented, and references are provided to guide the reader for further study. Furthermore, we restrict our discussion to those financial topics whose option pricing can be formulated as a PDE problem via the Δ-hedging technique, to illustrate the basic idea of the PDE approach.

The book is organized as follows: Fundamental concepts of financial derivatives are introduced in Chapter 1, and basics of stochastic analysis are covered in Chapter 4. Chapters 2, 3, and Chapter 5 form the core of this book. In these three chapters, in addition to presenting the mathematical models, algorithms and formulas of option pricing, we expound the basic ideas behind the Black-Scholes-Merton option pricing theory from several perspectives and levels: starting from the arbitrage-free assumption, via the Δ-hedging technique, put the investors in a risk-neutral world where all risky assets have the same expected return—the risk-free interest rate, then option as a contingent claim is given a fair market price independent of the risk preference of each individual investor. In the case of the continuous time model, the pricing formula for European vanilla option is the well-known Black-Scholes formula. In Chapter 6 and §7.7, we study American option pricing problems. Since American option offers early exercise, the holder needs to select the optimal exercise strategy to get optimal returns. Mathematically, this is modeled as a free boundary problem, where the free boundary is the optimal exercise boundary of an American option. Since it is a nonlinear problem, explicit closed form solution does not exist in general, hence qualitative analysis and quantitative numerical solution play an important role. Naturally, American option pricing as free boundary problem becomes the central topic and apex of the book, where the power of the theory and methods of PDE are fully demonstrated. In Chapters 7-9, we study the models and solution methods for multi-asset options and various types of path-dependent options. New multi-dimensional PDE pricing models are introduced in those chapters, which include not only the multi-dimensional Black-Scholes equation, but also various types of terminal-boundary problems for hyperparabolic equations. In addition to studying various methods of numerical solution, we are particularly interested in the possibility of reducing a multi-dimensional problem to a one-dimensional problem. Finally, in Chapter 10, we study the inverse problem of option pricing, that is, how to recover the volatility of the underlying asset from the information of its option market. It is called the implied volatility problem. We first derive the Dupire's method from the PDE viewpoint, and then proceed to work in the optimal control framework, thus obtain a system of partial differential equations and propose a well-posed algorithm for recovering the implied volatility.

I would like to thank my colleagues and students at Financial Mathematics Group in Tongji University, who have read earlier versions of the manuscript and made helpful suggestions. My special thanks go to Mrs. Xiaoping Zhang, my editor of the original Chinese edition of this text at the Higher Education Press(Beijing), for her expertise and enthusiastic work, and to Dr. Canguo Li for his elegant and painstaking translation work. The publication of the English edition of this text would not be possible without their efforts.

Lishang Jiang
Tongji University, 2004