Chapter 1

The Fisher Model with Certainty

The model introduced here was originally described by Fisher (1930) under conditions of perfect foresight. It is summarized here in preparation for the subsequent chapters on uncertainty.[1] This model provides three results that form the foundations for the theory of corporate finance. The model shows how one may develop the notion of the time value of money or the interest rate in a financial market model, how one may separate the savings decisions of individuals from the investment decisions of firms, and finally why the net present value rule is appropriate for decision making. All of these results flow from the Fisher model and so make it of central importance in the development of corporate finance. The certainty version of the model does not explain the existence of different rates of return on securities or other matters of concern but an uncertainty version of the model might be expected to provide insights and is developed in subsequent chapters; those chapters are all founded on what is developed here. Some remarks are included at the end of this chapter to outline what may be expected from a more general Fisher model.

Financial markets perform the role of allowing individuals and corporations to transfer money between dates. The individual may save by transferring dollars from the present to the future. The corporation may invest and finance the investment by transferring dollars from the future to the present. The following model provides the theoretical foundation for the net

[1]Those familiar with the Fisher model should proceed to the next chapter.

present value rule and by extension the corporate objective function. It also demonstrates why we look to the financial markets to find the cost of capital.

In its simplest setting, this model of individual behavior incorporates only one time period and does not include uncertainty. The model is developed in three steps. First, allowing the consumer to participate in the financial market, the individual's savings decision is characterized.[2] Second, the investment frontier is introduced and consumer is given the role of firm proprietor; although the net present value rule has not been derived yet, we assume that the proprietor makes an investment decision on behalf of the firm using that rule. This investment decision is in capital goods rather than financial assets. Third, the individual's savings decision is restored and the individual is given two roles. One role is firm proprietor and the other is consumer. The individual makes both decisions to maximize expected utility subject to the appropriate constraints. It should be noted that no objective function for the firm such as net present value is assumed here. Allowing the individual to make an investment decision as firm proprietor and to make a savings decision in the financial market as a consumer, the individual's savings and investment decisions are characterized. In the first and third steps, the individual is assumed to behave in accordance with her own self-interest. It is important to note that we are not assuming that the individual makes any decisions to maximize net present value. If any model is to demonstrate the importance of the net present value rule, then that model must show that the individual finds it optimal to use the rule. This result is demonstrated in case three.

Savings Decisions

Suppose the consumer stands at date zero and makes choices that will allocate income and consumption across two dates $t = 0$ and 1, that we refer to as *now* and *then* respectively. The consumer is endowed with some income *now* and *then*. Let $m = (m_0, m_1)$ denote the income pair, similarly let $c = (c_0, c_1)$ denote the consumption pair; each pair represents dollars *now* and *then*. Suppose the consumer can borrow and lend at the known interest rate r. Then the consumer selecting a consumption pair also makes a savings decision $s_0 = m_0 - c_0$; the savings choice yields $(m_0 - c_0)(1 + r)$ dollars *then*. The consumer must make these consumption choices consistent with a budget constraint

$$c_0 + \frac{c_1}{1+r} = m_0 + \frac{m_1}{1+r} \tag{1.1}$$

[2]Since there is no uncertainty, all financial assets must yield the same rate of return. Hence it is logical to suppose that there is only one financial asset and market. This will change when uncertainty is introduced.

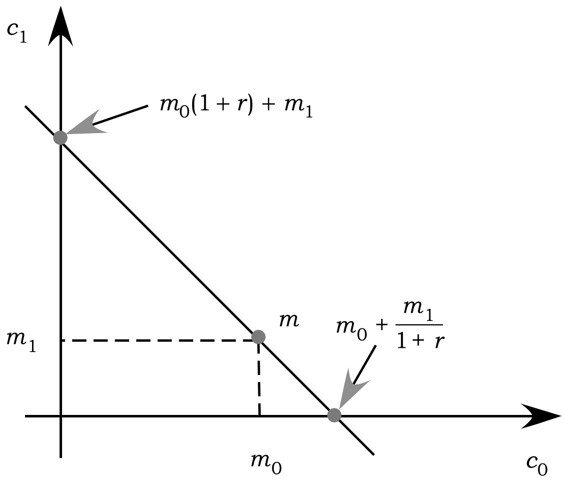

Figure 1.1 The budget constraint.

The budget constraint represents all the consumption pairs that equate the present value of the consumption plan with that of the income stream. The point $m = (m_0, m_1)$ in Figure 1.1 represents the time pattern of the income. Note that $(1 + r)$ is the absolute value of the slope of the budget constraint and corresponds to the increase in consumption *then* from each dollar saved *now*. A greater income either *now* or *then* yields a higher budget line through the new income pair. A greater interest rate yields a steeper budget line, since giving up a unit of consumption *now* would permit even more consumption *then*.

The consumer's preferences are represented by an intertemporal utility function $u(c_0, c_1)$. The utility maps consumption pairs into real numbers, i.e., the larger the number the better the consumption pair. The consumer is assumed to prefer more to less and so utility increases with more consumption *now* and *then*. The utility function also contains information concerning the consumer's preference for more consumption *now* versus *then* and this preference is consumer specific.

The preferences indicated by the utility function may be represented with intertemporal indifference curves; consumption pairs on an indifference curve, of course, indicate the same utility while the higher indifference curves represent greater utility. The absolute value of the slope of these curves at any consumption pair yields the individual's intertemporal marginal rate of substitution, i.e., *mrs*, and measures the value of consumption *now* in terms of consumption *then*. A steeper indifference curve corresponds to greater

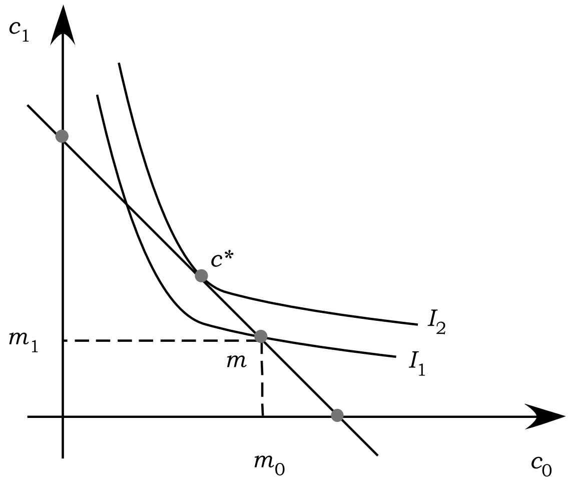

Figure 1.2 ■ The optimal consumption.

desire for consumption *now*. The indifference curves in Figure 1.2 also illustrate the notion of a decreasing marginal rate of substitution, i.e., as the individual increases consumption *now*, the value of consumption *now* in terms of consumption *then* decreases.

The individual's choice problem may be characterized as a constrained maximization problem. The consumer selects the consumption pair to

$$\begin{aligned} &\text{maximize } u(c_0, c_1) \\ &\text{subject to } c_0 + \frac{c_1}{1+r} = m_0 + \frac{m_1}{1+r} \end{aligned} \tag{1.2}$$

The Lagrange function for this problem is

$$L(c_0, c_1, \lambda) = u(c_0, c_1) - \lambda\left(c_0 + \frac{c_1}{1+r} - m_0 - \frac{m_1}{1+r}\right) \tag{1.3}$$

The first order conditions for a maximum are

$$\frac{\partial L}{\partial c_0} = \frac{\partial u}{\partial c_0} - \lambda = 0 \tag{1.4}$$

$$\frac{\partial L}{\partial c_1} = \frac{\partial u}{\partial c_1} - \lambda\frac{1}{1+r} = 0 \tag{1.5}$$

$$\frac{\partial L}{\partial \lambda} = m_0 + \frac{m_1}{1+r} - c_0 - \frac{c_1}{1+r} = 0 \tag{1.6}$$

The solution to this problem is demonstrated in Figure 1.2. The marginal rate of substitution is the ratio of the marginal utility of consumption *now* to the

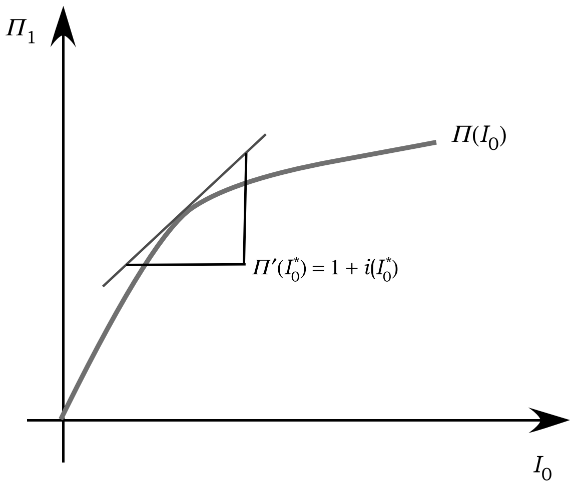

Figure 1.3 The investment frontier.

marginal utility of consumption *then*. From (1.4) and (1.5) it follows that

$$mrs = \frac{\frac{\partial u}{\partial c_0}}{\frac{\partial u}{\partial c_1}} = \frac{\lambda}{\lambda \frac{1}{1+r}} = 1 + r \qquad (1.7)$$

Note that at the consumption bundle c^*, the consumer's marginal rate of substitution equals one plus the rate of interest, i.e., $mrs^* = 1 + r$. The optimality condition has a simple interpretation; it says that at the margin c^* the individual values consumption *now* in terms of consumption *then* at its opportunity cost.

The Investment Frontier

Next suppose the time pattern of income may be altered by investing in capital goods. Let I_0 denote the dollar investment in capital goods and Π_1 denote the total dollar return on the investment; let $\Pi_1 = \Pi(I_0)$ and suppose $\Pi(I_0)$, i.e., the investment frontier, is a function which increases at a decreasing rate in the dollar investment. The function Π is shown in Figure 1.3. The slope of the investment frontier at a point is $\Pi'(I_0) \equiv 1 + i(I_0)$, where i is an interest rate called the marginal efficiency of investment. Since the payoff Π_1 increases at a decreasing rate, the marginal efficiency of investment also decreases as I_0 increases.

The net present value and internal rate of return

Although the firm proprietor is not necessarily concerned with the net present value, or equivalently, the net future value, of the investment project, it is appropriate at this point to identify the investment level which maximizes the net present value. Let *npv* and *nfv* denote net present and future value, respectively. Then

$$npv(I_0) = -I_0 + \frac{\Pi(I_0)}{1+r} \tag{1.8}$$

and

$$\begin{aligned} nfv(I_0) &\equiv (1+r)npv(I_0) \\ &= -(1+r)I_0 + \Pi(I_0) \end{aligned} \tag{1.9}$$

Maximizing *npv* and *nfv*, of course, yields the same investment level. The derivative of net future value with respect to the investment level is

$$\begin{aligned} \frac{dnfv}{dI_0} &= -(1+r) + \Pi'(I_0) \\ &= -(1+r) + (1+i(I_0)) \\ &= 0 \end{aligned} \tag{1.10}$$

At the investment level which maximizes *nfv*, this derivative is zero and so the interest rate in the financial market equals the marginal efficiency of investment, i.e., $r = i(I_0^*)$. This condition simply says that the last dollar invested must yield the same rate of return as is available in the financial market. The investment I_0^* is shown in Figure 1.4. Note that the vertical distance between $\Pi(I_0)$ and $(1+r)\, I_0$ is the net future value and I_0^* maximizes this distance.

It is also possible to provide a graphical interpretation of the internal rate of return, i.e., *IRR*, on the project. The internal rate of return is implicitly defined as that rate of return which yields a zero net present value, or equivalently, a zero net future value. Hence, the $IRR(I_0)$ is implicitly defined by the condition

$$nfv(I_0) = -\,(1 + IRR(I_0))\, I_0 + \Pi(I_0) = 0 \tag{1.11}$$

or equivalently, by the condition

$$1 + IRR(I_0) = \frac{\Pi(I_0)}{I_0} \tag{1.12}$$

This shows that one plus the internal rate of return can be interpreted graphically as the slope of a cord from the origin to a point on the investment

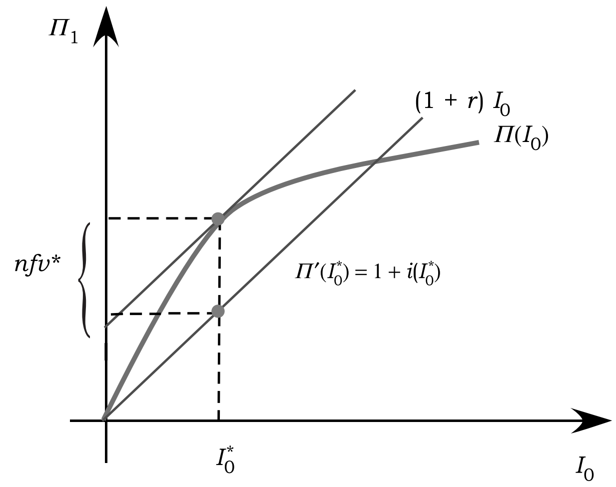

Figure 1.4 ■ The investment that maximizes *npv* and *nfv*.

frontier. Note that if, as assumed, $\Pi(I_0)$ increases at a decreasing rate then the internal rate of return decreases in I_0.

The Optimal Investment and Savings Decisions

Finally, suppose the individual not only selects a consumption plan but also an investment plan in capital goods. The consumer then effectively becomes a single proprietor. The ability to invest in capital goods alters the individual's income pair from (m_0, m_1) to $(m_0 - I_0, m_1 + \Pi(I_0))$. The constrained maximization problem becomes a choice not only of an optimal consumption plan c^* but also an optimal investment. Hence, the constrained maximization problem is

$$\begin{gathered} \text{maximize } u(c_0, c_1) \\ \text{subject to } c_0 + \frac{c_1}{1+r} = m_0 - I_0 + \frac{m_1 + \Pi(I_0)}{1+r} \end{gathered} \tag{1.13}$$

The budget constraint is simply the condition that the present value of consumption plan equals the present value of income stream. The position of the budget constraint is determined by the investment decision because that decision alters the income pair. The individual has two roles. One of the roles is as the proprietor of a firm. In that capacity the individual makes the investment decision. The other role is that of a consumer. In this capacity, the individual selects the pair (c_0, c_1), or equivalently, a savings level. The

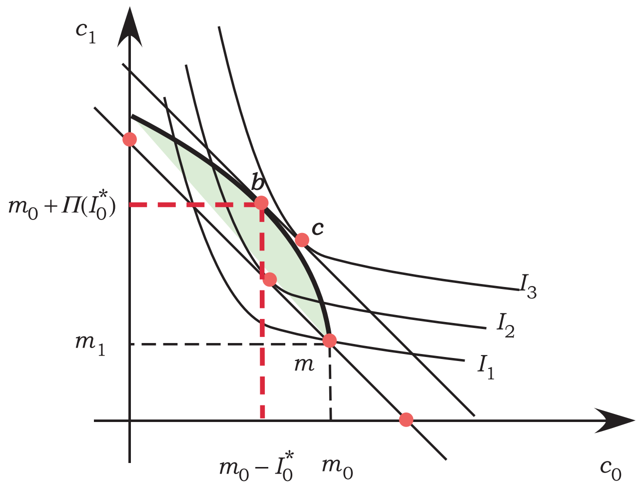

Figure 1.5 Fisher separation.

feasible investment and consumption decisions are represented by the investment frontier and the associated budget line, respectively, in Figure 1.5.

The Lagrange function for the constrained maximization problem in (1.13) is

$$L(c_0, c_1, I_0, \lambda) = u(c_0, c_1) - \lambda\left(c_0 + \frac{c_1}{1+r} - (m_0 - I_0) - \frac{m_1 + \Pi(I_0)}{1+r}\right) \tag{1.14}$$

The first order conditions are (1.4), (1.5), and

$$\frac{\partial L}{\partial I_0} = -\lambda\left(1 - \frac{\Pi'(I_0)}{1+r}\right) = 0 \tag{1.15}$$

$$\frac{\partial L}{\partial \lambda} = (m_0 - I_0) + \frac{m_1 + \Pi(I_0)}{1+r} - c_0 - \frac{c_1}{1+r} = 0 \tag{1.16}$$

Note that first order condition (1.15) yields $i(I_0^*) = r$ as the condition for an optimal investment; this is the familiar marginal efficiency of investment equal the interest rate common in other economic models. The individual, in the role of proprietor, selects the investment level indicated by the pair $(m_0 - I_0^*, m_1 + \Pi(I_0^*))$. Then, the individual, in the role of consumer, selects the consumption bundle indicated by the point c^*. Note that, at b

the condition $i(I_0^*) = r$ holds, while at c the condition $mrs = 1 + r$ holds. Also, observe that the condition $i(I_0^*) = r$ does not depend on the individual's preferences and that it is the condition for a maximum net present value.

An alternative intuitive explanation is as follows: The individual has preferences consistent with the observation that more is preferred to less. The individual selects the investment plan $(I_0^*, \Pi(I_0^*))$ which maximizes the present value of total income, because by doing so the individual obtains the highest possible budget line in the financial market. To see this, note that the budget line for any investment decision intersects the horizontal axis at

$$\begin{aligned} m_0 - I_0 + \frac{m_1 + \Pi(I_0)}{1+r} &= m_0 + \frac{m_1}{1+r} - I_0 + \frac{\Pi(I_0)}{1+r} \\ &= m_0 + \frac{m_1}{1+r} + npv(I_0) \end{aligned} \tag{1.17}$$

and the budget line maximizes this value and so it yields the greatest capability for consumption *now* and *then*. This is the Fisher separation result, i.e., all individuals, irrespective of their preference for consumption *now* versus *then*, select the same investment plan. Maximizing the present value of income is equivalent to maximizing the net present value of the investment. Recall that the analysis was not begun with the objective of maximizing the net present value. The objective was to maximize the individual's utility subject to a budget constraint and this yielded the result that any individual makes the investment decision to maximize net present value. Hence, the roles of proprietor and consumer can be separated.

Remarks

The Fisher model is remarkably robust for such a simple construct. It provides for a determination of an interest rate, for a Fisher separation theorem and for a derivation of the net present value rule. It is possible to derive the supply of and demand for savings based on the model developed here and so it is possible to determine an equilibrium rate of interest; that has not been pursued here because the analysis focuses on the development of corporate finance theory. For this development, the separation theorem plays an important role because it shows that an individual making a savings decision on personal account and a capital investment decision on firm or proprietorship account will separate the two decisions, in the sense that the capital investment decision will be made without reference to intertemporal preferences for consumption *now* versus *then*. Equivalently, the separation theorem shows that the investment decision is driven by the more is preferred to less assumption but not intertemporal preferences because the financial market allows the individual to reallocate consumption across time by borrowing or lending. Finally, the model also provides a decision rule for the single proprietor and that rule is to make decisions that maximize net present value.

The certainty version of the Fisher model has its limitations. The certainty model cannot explain different rates of returns for securities. As one would expect, the introduction of uncertainty provides the basis for explaining different rates of return and much more. Portfolio Theory (Markowitz 1952), the Capital Asset Pricing Model (Sharpe 1964; Mossin 1966; Garman 1979), the Arbitrage Pricing Model (Garman 1979), etc., have all been developed to provide various explanations in corporate finance but none have the capability of integrating the results in one framework. The Fisher model does as the subsequent chapters show. Arrow's work on the allocation of risk (1963) provides the foundation for a generalized version of the Fisher model. The savings decision becomes a portfolio decision as is shown in the next chapter. The subsequent chapters provide a demonstration of some of the key results and insights in corporate finance reframed and motivated in the context of the Fisher model.

Suggested Problems

1. Suppose the individual has intertemporal preferences specified by $u(c_0, c_1) = \min\{c_0, c_1\}$. Sketch the indifferences curves and show the optimal consumption bundle.
2. Suppose the individual has the intertemporal preferences specified in the last problem and an income pair such that $m_0 > m_1$. Does the individual lend or borrow in the financial market? How does the lending or borrowing decision change given an increase in the interest rate?
3. Let the investment frontier be specified as $\Pi(I_0) = \min\{\kappa I_0, M\}$ where κ and M are positive constants. Sketch this frontier. Also provide a sketch of the marginal efficiency of investment and the internal rate of return.
4. Show that $nfv(I_0) > 0$ implies $IRR(I_0) > r$ and that $nfv(I_0) < 0$ implies $IRR(I_0) < r$.
5. Suppose $\Pi' > 0$ and $\Pi'' < 0$. Show that $i < IRR$ for all positive investment levels.
6. Provide a sketch of an investment frontier which would yield a negative net present value for any positive level of investment.
7. Why is it reasonable to say that the financial market rate of return r is the cost of capital?
8. Sketch the case in which any positive investment in a project yields a negative *npv* and show that the consumer\proprietor chooses not to invest.

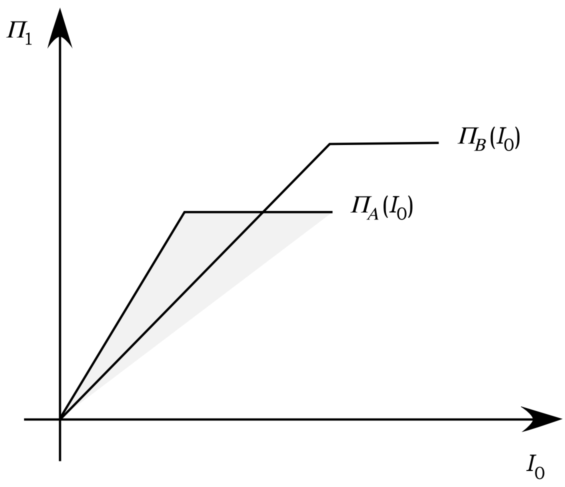

Figure 1.6 ■ Mutually exclusive projects.

9. Show how the proprietor's investment choice is affected by a reduction in the interest rate r.
10. Suppose that the proprietor selects an investment level either greater than or less than the level b shown in Figure 1.5. Show that, in either case, the individual is worse off and that this result does not depend on whether the individual is a borrower or lender.
11. Suppose the proprietor can invest in one of two mutually exclusive projects. Let Π_A and Π_B denote the investment frontiers for the two projects, so that $\Pi_A(I_0)$ is the revenue generated *then* if project A is selected and $\Pi_B(I_0)$ if project B is selected. The investment frontiers are shown in Figure 1.6. Show and explain the following:
    a. The Fisher separation result holds in terms of which project is selected as well as the scale at which the project is operated.
    b. Specify the conditions under which project B will be selected over project A.
    c. Which project has the larger internal rate of return? If you select project A or B on the basis of which has the larger *IRR*, is your choice consistent with your analysis in (b)?