Section I
Sets and Functions

Several basic notions must be built up to lay a proper foundation for further study. An idea that lies at the foundation of modern mathematics is that of a set. While everyone knows what a set is, the following intuitive definition is given to provide a common starting point. You should be aware, however, that though this idea seems simple and clear enough, there are deep difficulties hidden within this concept.

I.1. Definition A **set** is a collection of objects where each object in the set exhibits at least one feature common to all other elements in the set.

Example The set of all things which have four legs includes cows, pigs, horses, tables, but not man. On the other hand, the set of all four legged animals includes cows, pigs, horses, dogs, cats, but not table or man.

Notation The following notation for sets will be used.

Sometimes, if the set is small enough, we will write out all the elements of the set. $\{1, 2, 3, 4, 5\}$ is the set of integers from one to five inclusive.

If the set is infinite, or just large, we may define the set based on the property that the elements have in common. For example, $\{x|x$ is an integer from one to five inclusive$\}$ is the same set we had above. The notation is read this way.

"$\{$" is read "the set of"
"x" is read as "x"
"$|$" is read "such that"

and the qualifier follows the vertical bar. The qualifier tells the trait that all elements of the set have in common. So an element qualifies for our set if it satisfies the qualifier.

There are some symbols that are common and useful.

Symbol	Meaning
$\in$	"is an element of"
$\rightarrow$	"implies"
$\forall$	"for each" or "for every"
$\exists$	"There exists"
R	"the set of all real numbers"
:	"such that"
iff	"if and only if"
$\subseteq$	"is a subset of"
$\emptyset$	"the empty set"

Suppose that A is a set. We would write $a \in A$ if the element a is in the set A. If we write $\forall\, a \in A$, we mean "for every a that is an element of A". If we write $\exists\, x \in X$ so that $x > y$, we mean "there exists an element of X, namely x, so that $x > y$ holds." If we write $A \subseteq R$, we mean that the set A is a subset of the set of all real numbers. We formally define this latter concept now.

I.2. Definition We say that a set A is a **subset** of a set B, and write $A \subseteq B$, if every element of A is also in B.

I.3. Definition We say that two sets A and B are **equal** if $A \subseteq B$ and $B \subseteq A$.

Example The set of integers is a subset of the set of real numbers. The set of rational number (numbers expressible as the ratio of two integers) is a subset of the real numbers. The set of complex numbers (numbers of the form $a + bi$, where $i = \sqrt{(-1)}$) is not a subset of the set of real numbers.

I.4. Definition An **ordered pair,** (x, y), is a set with two elements, $\{[x], (x, y)\}$ where the $[x]$ denotes that the x comes before the y in the pair of numbers.

I.5. Definition The **Cartesian Product** of a set A with a set B, denoted $A \times B$, is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$.

We can also write $A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$.

Exercise 1 Suppose $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{3, 7, 8\}$. Write out $A \times B$.

Remark We can extent our definition of an ordered pair to an ordered 3-tuple and then to an ordered n-tuple. We may use this concept to define the n-fold Cartesian Product.

I.6. Definition An **ordered 3-tuple** is an ordered pair $([x, y], z)$ where the pair (x, y) comes before z.

I.7. Definition An **ordered n-tuple** is an ordered pair, $([x_1, x_2, \ldots, x_{n-1}], x_n)$ where $x_1, x_2, \ldots, x_{n-1}$ comes before x_n.

I.8. Definition Suppose we have sets $A_1, A_2, A_3, \ldots, A_n$. Then the **$n$-fold Cartesian Product** of $A_1, A_2, A_3, \ldots, A_n$ is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ with $a_i \in A_i$, $i = 1, \ldots, n$. We denote this set $A_1 \times A_2 \times A_3 \times \cdots \times A_n$.

Example In the case that $A_i = R$, the set of all real numbers, R^n is the Cartesian product of R with itself n times. The elements of R^n are elements of the form $(x_1, x_2, x_3, \ldots, x_n)$ where $x_i \in R$ for each i. R^2 is the real plane. R^3 is three-dimensional space.

I.9. Definition A **relation**, Z, is a set of ordered pairs.

Remark Note that a relation, Z, is a subset of $A \times B$. Suppose that Z is any relation. We write $x\,Z\,y$ to mean that x is Z related to y or $(x, y) \in Z$. If X is a set, and if Z is a relation in X, then $Z \subseteq X \times X$. Note that if Z is a relation in R, then Z is a set of ordered pairs in the plane, $R \times R$.

Example

a. Consider the following set of ordered pairs. $\{(1, 3), (3, -4), (-2, -6)\}$. This is a relation in R.
b. The set of (x, y) so that $y = 5x - 22$ is a relation in R.
c. The set of (x, y) so that $y^2 = 6x - 32$ is also a relation in R.

Remark There are some relations that have an added restriction that make them useful. The added restriction is the condition of single valuedness. In this case we have a function.

I.10. Definition A **function, f,** is a set of ordered pairs with the following property. If $(x, y) \in f$, and $(x, z) \in f$, then $y = z$ must hold.

Remark The defining property of a function is that for each first element in the ordered pair, there is one and only one second element. One way to see if a relation is a function is to draw a graph. If every vertical line intersects the graph of the function just once, then we have a function.

Example
a. $f = \{(0, 1), (2, 1), (1, 6)\}$ is a function.
b. $f = \{(1, 2), (2, 3), (1, -1)\}$ is not a function.
c. $f = \{(x, y)|y = 5x + 22, x \in R\}$ is a function.
d. $f = \{(x, y)|y^2 = 5x - 1, x \in R\}$ is not a function.

I.11. Definition Let f be a function. The set of all possible first elements of f is called the **domain** of f.

I.12. Definition Let f be a function. Let S be the domain of f. The **image** of f is the set of values $f(x)$ for all $x \in S$.

I.13. Definition[a] Let f be a function. The set of all possible second elements of f is called the **range** of f.

Exercise 2 Which of the following are functions? For each function, what is the domain and the range?

a. $\{(3, 1), (4, 1), (1, 6), (4, 2)\}$.
b. $\{(x, y) \mid x = y, x \in R\}$.
c. $\{(x, y) \mid y^2 = x, x \in R\}$.
d. $\{(x, y) \mid y = (x + 4)/(x - 1), x \in R\}$.

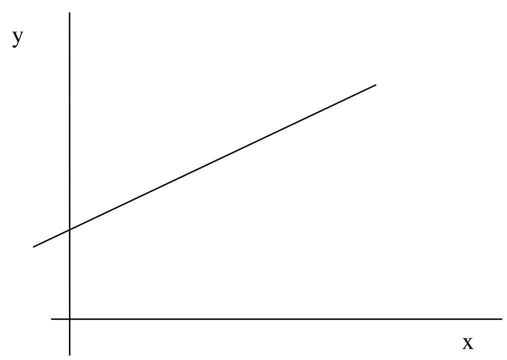
Remark Our initial focus will be on functions whose domain is R and whose range is R. But note that the definition of a function

[a] The experienced student will observe that we have defined f so that the image of f is a subset of its range. An alternative definition of the range of f is the image of f. In that case f is onto its range. A function $f(x)$ is **onto** a set Y, contained in the range of $f(x)$, if $f(x) \in Y$ for all x in the domain of $f(x)$, and if for each $y \in Y$ there is x in the domain so that $y = f(x)$.

allows more general settings. For example, as we will do later, the domain could be R^n and the range R. In this case the ordered pair is $\{(x_1, x_2, \ldots, x_n), y\}$. The first element of the pair is $(x_1, x_2, \ldots, x_n)$, an element of R^n.

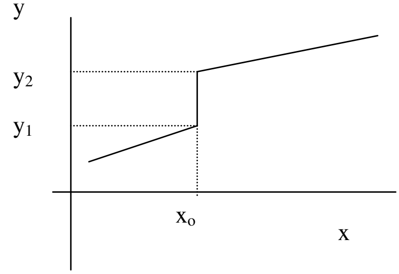
Example Suppose that we have a domain of R and a range of R. Then the following is a function.

$$y = f(x) = ax + b.$$



For each x value, there is one and only one y value. You can easily check this by computation. How can you do that? Compute the corresponding y value for a given x. What did you get for the y values? Is it possible to get two different y values for a given x? Hence we have a function.

Example Suppose we have a domain of R and a range of R. Then the following is not a function.



In this case, you can see that we do not have a function, but we do have a relation. At the value x_0 any value of y between (and including) y_1 and y_2 are part of the relation. So at x_0 there is more than one y, and we do not have a function.

I.14. Definition A function f is said to be a **real-valued function of a real variable** if the domain of f and the range of f are both R.

Remark The statement that a function is real-valued means that the values of the function, the range, are in the real number system. To say that a relation is a function of a real variable means that the domain of the function is the set of real numbers.

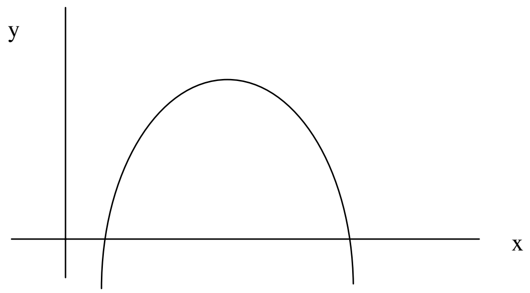
I.15. Definition A **sequence** is a function defined on the natural numbers (that is, the integers starting with 1).

Remark We will not have much need for sequences in these notes. However, much of the work we do on limits (later) could be done in terms of sequences.

I.16. Definition A real-valued function of a real variable is said to be **one to one** if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Remark You can see that the straight line in the example above is one to one. An easy way to see if a function is one to one is to graph it. Then every horizontal line must intersect the graph of the function just once.

Example Suppose that we have the relation shown below. Is it a function? Is it one to one? It is a function because each vertical line only cuts the relation once. But many horizontal lines cross the function more than once.



Exercise 3 Which of the following are one to one? (Hint: draw a graph.)

a. $y = x$
b. $y = x^2$
c. $y = x^3$
d. $y = 16$

Exercise 4 Suppose that we define a function with the domain of the positive integers. For example, $f(n) = 10^n$. Is this a function (again remember that the domain is the set of integers, not the real numbers). Is this function one to one?

Exercise 5 Let $f(n)$ be defined on the positive integers. Suppose $f(n) = 1/n$. Is this a function? Is it one to one?