

- cumulative distribution:

$$F_X(x) = \Phi \left[ b \sinh^{-1} \left( \frac{x - m_1}{d} \right) \right]$$

- inverse cumulative distribution:

$$F_X^{-1}(\alpha) = m_1 + d \sinh \left[ \frac{\Phi^{-1}(\alpha)}{b} \right] \quad (2)$$

- density function:

$$f_X(x) = \frac{b}{\sqrt{x^2 + d^2}} \phi[b \sinh^{-1}(x/d)]$$

Let  $\mu_2(X) = \mu_2$  and  $\gamma_2(X) = \gamma_2$  be, respectively, the variance and kurtosis coefficients of the random variable  $X$ . If  $X$  is a symmetrical Johnson  $S_U$  distribution, then we proved in Castagliola (1998) that parameters  $b$  and  $d$  are related to  $\mu_2$  and  $\gamma_2$  using the following equations (see the appendix for the proof):

$$b = \sqrt{\frac{2}{\ln(\sqrt{2(\gamma_2 + 2)} - 1)}} \quad (3)$$

$$d = \sqrt{\frac{2\mu_2}{\sqrt{2(\gamma_2 + 2)} - 2}} \quad (4)$$

### 3. Application to Control Charts

Let  $X_1, \dots, X_n$  be a sample of  $n$  independent random variables corresponding to training data taken when the process is considered to be “in control”, from which we have to compute control limits. Let  $\hat{m}_1$ ,  $\hat{\mu}_2$ ,  $\hat{\gamma}_1$ , and  $\hat{\gamma}_2$  be, respectively, the (moment) estimators of the mean, variance, skewness, and kurtosis. We will assume now (see the appendix) that a first statistical test leads to the conclusion  $\gamma_1 = 0$  (the data distribution seems symmetrical) and a second one leads to the conclusion  $\gamma_2 \geq 0$  (the data distribution seems leptokurtic). If these conditions are verified, we suggest to compute control limits as presented below:

- Compute  $\hat{b}$  and  $\hat{d}$  using Eqs. (3) and (4) in which  $\mu_2$  and  $\gamma_2$  have been replaced by their estimators.
- Transform each new observation  $X$  to a quasi-normal  $N(0, 1)$  random variable  $Z$  using the following equation:

$$Z = b \sinh^{-1} \left( \frac{X - m_1}{d} \right) \quad (5)$$

- Use “classical” control limits (mean, median, standard deviation, range, EWMA, CUSUM, etc.) corresponding to a normal  $N(0, 1)$  distribution.