

## Chapter 0

### A KIND OF INTRODUCTION

Often in life we find out that our goals are mutually incompatible: we have to renounce some of them and we feel frustrated. For example, I may want to be a friend of both Mr. White and Mr. Smith. Unfortunately, they hate each other: it is then rather difficult to be a good friend of both of them (a very frustrating situation).

The situation is more complex when many individuals are present. In a classical tragedy the scenario may be the following: there is a fight between two groups and the various characters on the scene have to choose sides. In addition they all have strong personal feelings, positive or negative, towards each other (it is a tragedy!). Some of them are friends and some are enemies. For simplicity we will assume that all feelings are reciprocal; otherwise the system may never reach equilibrium (this more general case, though much more complicated can be studied. See Reprint 34 for one particular example). Let us consider three characters ( $A$ ,  $B$  and  $C$ ); if  $A$  and  $B$ ,  $B$  and  $C$ ,  $A$  and  $C$  do like each other, there is no problem: they will all choose the same side. In a similar way, if  $A$  and  $B$  are friends and  $C$  is an enemy of both, then  $A$  and  $B$  can be on one side and  $C$  will be on the other. Frustration follows, instead, if  $A$ ,  $B$  and  $C$  hate each other because two personal enemies must then fight on the same side.

This analysis can be formalized by assigning to each pair a number  $J_{AB}$  which is  $+1$  if  $A$  and  $B$  are friends and  $-1$  if they hate each other; the relation among three characters is frustrated if (Ref. 1)

$$J_{AB} \cdot J_{BC} \cdot J_{CA} = -1. \quad (0.1)$$

When many triples are frustrated, evidently the situation on the scene is unstable and many rearrangements of the two fields are possible.

At a given moment of the tragedy it is possible to define the "dramatic tension" as

$$\text{Number of frustrated triples/Total number of triples.} \quad (0.2)$$

Detailed studies<sup>2</sup> have shown that in many Shakespeare's plays the dramatic tension has a small value at the beginning of the tragedy, reaches a maximum in the middle and decreases by the end.

Mathematically we could say that we have  $N$  variables  $s_i$ , one for each character;  $s_i$

takes values  $+1$  or  $-1$  depending on which side the  $i$ th character stays; for a given set of the  $J_{ik}$  we are interested in minimizing the “discomfort” function

$$H_J[s] = - \sum_{i>k} J_{ik} s_i s_k, \quad (0.3)$$

where the square parenthesis are used to stress that  $H$  is a function of all the  $s_i$  together.

The function  $H$  is often called (depending on the context) the cost function or the Hamiltonian (the energy).

It is a well-known, and hard, mathematical problem to find a fast algorithm which, for a given instance of the  $J$ 's, computes the set of  $s_i$  that minimize  $H_J$ . From the point of view of complexity theory this problem is  $NP$  complete, which means that very likely there is no algorithm that can find the minimum using a computer time increasing as a power of  $N$  for large  $N$ . In other words it is generally believed that the time needed by the best algorithm increases as  $\exp(cN)$  with  $c$  constant when  $N \rightarrow \infty$ .

In this book we will describe an approach to this problem that makes essential use of the tools of statistical mechanics: we suppose that the  $J$ 's are chosen at random (according to a probability law) and we are interested in finding analytical properties of both the absolute minimum of  $H$  and of other local minima. We will try to find out how much these minima are different and which are the relations among them. We are less concerned about constructing a fast algorithm for finding the absolute minimum, though it is clear that an analytical knowledge of the properties of these minima may be useful in this task.

A typical result that has been obtained is the following: if each  $J_{ik}$  is chosen  $+1$  or  $-1$  with equal probability and independently of the others, in the limit  $N \rightarrow \infty$ , the minimum of  $H_J/N^{3/2}$  takes the same value for almost all realizations of the  $J$ 's and this value is about  $-.7633$ . In other words, the best possible arrangement is that in average the number of friends minus the number of enemies among our allies be  $(.7633/2)N^{1/2}$ . For 10 000 persons, and to the best of everybody's effort, on my same side, I will have to bear with approximately 2462 enemies, thirty eight less than if the choice of side had been decided by tossing a coin! This sobering result is related to the large proportion of frustrated triples ( $1/2$  of the total number), a consequence of the uncorrelated character of the  $J_{ik}$ . A different result, also discussed in this book follows if

$$J_{ik} = \sum_{p=1}^P \xi_i^p \xi_k^p, \quad (0.4)$$

where now the  $\xi_i^p$  are independent random  $\pm 1$  variables<sup>3</sup>. For  $P = 1$  there are no frustrated triples while for  $P \gg N$  it reduces to the previous case. There are interesting different regimes in  $P/N$  (see Chap. XIII).

The properties of those configurations  $\{s_i\}$  which have low energy but are not minima are also interesting. They can be treated by assigning to each configuration  $s_i$  a probability

$$P_J[s] = \exp(-\beta H_J[s])/Z_J, \quad (0.5)$$

where the “partition function”  $Z_J$  is such that the sum of the probabilities is 1

$$Z_J = \sum_{\{s\}} \exp(-\beta H_J[s]). \quad (0.6)$$

In statistical mechanics  $H_J$  is the energy of the system and  $P_J[s]$  is the Gibbs probability distribution ( $\beta = 1/(kT)$ , where  $k$  is Boltzmann constant and  $T$  is the absolute temperature).

We will denote by  $\langle g \rangle$  the expectation value of a function  $g[s]$  according to the probability distribution (0.5)

$$\langle g \rangle = \sum_{\{s\}} P_J[s] g[s] \quad (0.7)$$

(sometimes we will omit the  $J$  subindex to lighten the notation).

These are simple examples in which we want to find the minimum of a function in the presence of requirements that push towards different directions and it is difficult to establish which could be the best compromise. An essential ingredient, in addition to frustration is the so-called disorder. In this context this means that the structural parameters defining the problem (here the  $J_{ik}$ ) are themselves the outcome of a random choice—or at least have large complexity. It is generally believed that the results obtained for one model are relevant for other disordered frustrated systems though to what degree they share the same properties is still unclear. Although this version of the model has its own interest, historically it was first studied in relation with spin glasses.

Spin glasses (also called amorphous magnets) are magnetic substances in which the interaction among the spins is sometimes ferromagnetic (it tends to align the spins;  $J_{ik} > 0$ ), sometimes antiferromagnetic (it antialigns the spins;  $J_{ik} < 0$ ). The sign of the interaction is supposed to be random.

In some spin glasses the spins can take only two values  $\pm 1$  (Ising spins) and the form of the energy is the same as in Eq. 0.3. The probability law for the  $J$ 's distinguish various models. If all the  $J$ 's have the same probability distribution, all spins interact with the others and the infinite range model so defined is called the Sherrington-Kirkpatrick (in short SK) model. Another possibility is to choose  $J_{ik}$  different from zero only for nearby pairs of spins (short range model: a spin interacts only with its neighbours).

In the following chapters we will discuss almost exclusively the SK model in the limit  $N \rightarrow \infty$ . We will show that the mean field theory can be considered to be satisfactorily understood. The replica theory with its strange  $n \rightarrow 0$  limit will be discussed at length stressing the fact that though unusual, once one masters the different steps, it is a formidable method to derive the mean field approximation for disordered frustrated systems. The TAP approach (Chap. II) which was originally proposed as an alternative to replicas, when correctly complemented leads to the same result obtained by breaking replica symmetry (Chap. V). The main conclusion of this part concerns the complicated structure of the configuration space landscape of the

*SK* free energy which at low temperatures has an infinite number of valleys which are organized ultrametrically (Chap. IV). A chapter is dedicated to the dynamics of spin glasses (Chap. VI).

In the second part dedicated to Optimization, after two introductory chapters dedicated to the presentation of the subject, we demonstrate the power of the methods discussed in the first part by showing how they can be applied to the matching problem (Sec. IX.2), the travelling salesman problem (Sec. IX.3), the assignment problem (Sec. IX.4) and graph partitioning (Sec. IX.5).

In the third part we discuss applications to the problem of biological organization. The general strategy is presented in Chap. X while in Chap. XI we discuss why an infinite valley landscape can be useful to model some aspects of prebiotic evolution. In Chap. XII we introduce the ideas behind brain modelling while in Chap. XIII we show that one particular model, the Hopfield model, can again be analyzed as a spin glass.

## References

1. Toulouse (1977).
2. Marcus, contribution to the *Seminaire de Semiotique Theatrale*, Urbino (1974), unpublished and Doreian (1971).
3. These examples are relevant to the so-called *voting problem* discussed in sociology (R. Axelrod, private communication).