CHAPTER 1

# SOME ARGUMENTS CONCERNING NUCLEAR FORCES

The properties of nuclei in their ground states resemble, in important respects, those of ordinary solids and liquids. The binding energy is approximately proportional to the number of nucleons, A, and the mean density is is nearly independent of A. The curve of binding energy per particle versus A, shown in Fig.1.1, is similar to that of a charged liquid drop. If the liquid has a particle density $\rho$, an energy density $-\epsilon\rho$, a surface tension $\sigma$ and a charge Ze, the energy per particle would be

$$E/A=-\epsilon+\sigma S/A+(3/5)Z^2e^2/RA, \qquad (1.01)$$

where S is the surface area and R is the radius. The first term is the volume energy, the second the surface energy and the third the Coulomb energy.

The decrease in $-E/A$ for small A is interpreted as due to the repulsive surface term which, since

$$A=(4\pi/3)\rho R^3, \quad S=4\pi R^2, \qquad (1.02)$$

is proportional to $1/A^{1/3}$. The decrease for large A is due

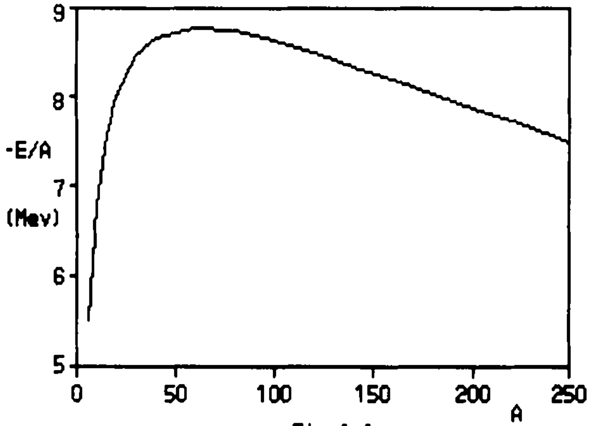


Fig.1.1

to the repulsive Coulomb term which, since Z/A decreases with increasing A, goes a little less rapidly than $A^{2/3}$. Using observed nuclear radii we can easily check that the decrease in $-E/A$ of 1.2 Mev between A=60 and A=240 is in the right ball-park. The first of the relations (1.02) can be written

$$R=r_0A^{1/3}.$$

We take $r_0=1.4$ f, which has the convenient property that it is just half of the classical electron radius, $r_0=(1/2)e^2/mc^2$. The Coulomb energy becomes $(3/5)(Z^2/A^{4/3})\cdot 2mc^2=(3/5)(Z^2/A^{4/3})$ Mev. This gives 1.9 Mev for ${}^{26}Fe^{56}$, 3.4 Mev for ${}^{92}U^{238}$, a difference of 1.5 Mev.

One might infer from this that the forces between nuclei are similar to those between atoms (Fig.1.2), an inference that is indeed true. However in simpler minded days (1932) Heisenberg[1] argued that a potential such as that of Fig.1.2 would only arise in the interaction of complex structures, and nucleons, being "elementary particles", should show more "elementary" behavior. A

[1]W.Heisenberg, Z. Physik 77,1 (1932).

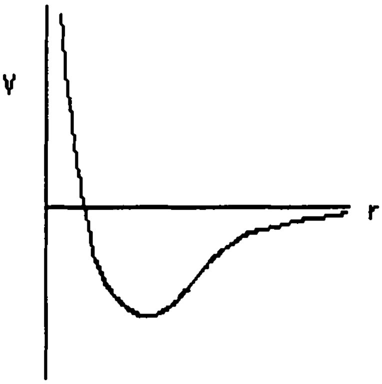


Fig. 1.2

Coulomb-like interaction is an elementary behavior but it would not lead to the required properties. In the liquid (nuclear matter) the energy density is locally determined and does not depend on the size of the system. The potential energy of a particle in the liquid is $\int V(r)\rho d\underline{r}$. If $V(r) \approx 1/r^n$ for large r, the contribution from distant parts of the liquid is $4\pi\rho\int(r^2/r^n)dr$. Thus n must be greater than 3 for the distant parts not to contribute. Since a $1/r^n$ potential with n>3 is too singular at the origin, we conclude that the force has a range.

A simple generalization of the Coulomb interaction, $V=e\Phi$,

$$\Phi=e/4\pi r, \tag{1.03}$$

to one which has a finite range is the Yukawa interaction, $V=-g\Phi$,

$$\Phi=ge^{-\mu r}/4\pi r. \tag{1.04}$$

A minus sign has been inserted to make the force attractive. For an extended source the $\Phi$ of (1.03) satisfies Laplace's equation,

$$\Delta\Phi=-e\rho. \tag{1.05}$$

The $\Phi$ of (1.04) can be regarded as elementary in that it also satisfies a simple differential equation,

$$\Delta\Phi-\mu^2\Phi=-g\rho. \tag{1.06}$$

As Yukawa[a] pointed out, the generalization of (1.06) to a time dependent relativistic equation replaces the $\Delta$ by the

[a]H. Yukawa, Proc. Phys. Math. Soc., Japan 17, 48 (1935).

$\Box$, and the free-field equation,

$$\Box\phi-\mu^2\phi=0,$$

has plane wave solutions, $\phi=\exp[i(\mathbf{p}\cdot\mathbf{r}-Et)]$, with $E^2=\mu^2+p^2$, the right relation for a particle of mass $\mu$. (Here and hereafter we take $h/2\pi=c=1$.) In the lingo of field theory, $\phi$ is a neutral scalar meson field. The range of the nuclear force is thus related to the Compton wavelength of the particle field carrying the interaction.

However, as Heisenberg pointed out, an interaction such as (1.04) does not lead to the proper nuclear properties. In the interior of a nucleus, where $\rho$ is constant, the potential energy of a particle is

$$V=\int V(r)\rho d\mathbf{r}=\rho V_0, \text{ with } V_0=\int V(r)d\mathbf{r}, \qquad (1.07)$$

and thus increases linearly with $\rho$. In order to have equilibrium at a finite $\rho$, the kinetic energy must produce enough pressure to balance the attractive forces. Let us estimate the kinetic energy by treating the nuclear matter as a degenerate Fermi gas.

In the Fermi gas each state occupies a volume $h^3$ in phase space, or in our units, with $h/2\pi=1$, a volume $(2\pi)^3$. If there are A particles in a volume $V_{01}$, and the momentum space is filled up to the Fermi momentum $p_f$, we thus have $A=4(4\pi/3)p_f^3V_{01}/(2\pi)^3$. The factor 4 arises because four particles can be put in each state, protons and neutrons with spins up or down (for simplicity, take the number of protons and neutrons to be the same). We then have

$$\rho=A/V_{01}=(2/3\pi^2)p_f^3. \qquad (1.08)$$

The mean kinetic energy of a nucleon is

$$T=\int_0^{p_f}(p^2/2M)d\mathbf{p}/\int_0^{p_f}d\mathbf{p}=(1/2M)\int_0^{p_f}p^4dp/\int_0^{p_f}p^2dp=(3/5)T_f, \qquad (1.09)$$

with $T_f=p_f^2/2M$. The mean energy per nucleon (neglecting surface and Coulomb energy) is

$$E/A=T+\tfrac{1}{2}V=(3/5)(1/2M)(3\pi^2/2)^{2/3}\rho^{2/3}+\tfrac{1}{2}V_0\rho. \qquad (1.10)$$

The mean potential energy per nucleon is (1/2)V, rather than V, since V would include the interaction of particle A with B plus the interaction of B with A.[3] Equation (1.10) is plotted in Fig.3a; it leads to no stable configuration.

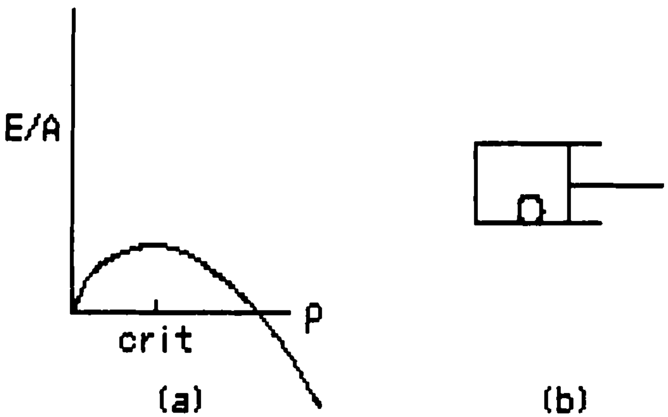


Fig.1.3

[3] The energy density is
$\epsilon=\rho(E/A)=(3/5)(1/2M)(3\pi^2/2)^{2/3}\rho^{5/3}+\tfrac{1}{2}V_0\rho^2$. The energy of a nucleon at the top of the Fermi sea (the binding energy of the last nucleon) is, since $E=\epsilon V_{ol}$, $A=\rho V_{ol}$, $dE/dA=d\epsilon/d\rho=T_f+V$. Consistency requires that at equilibrium this equal (1.10). Setting the two expressions equal leads to the virial relation, $T_f=-(5/4)V$, which shows the nucleus is unbound. The reader can verify that at the maximum in Fig.1.3(a) the virial relation is satisfied.

The physical situation is illustrated by Fig.3b. Imagine the Fermi gas in a cylinder being compressed by a piston. At low density the gas exerts a positive pressure on the piston. At a critical density we reach a point of unstable equilibrium, the attractive forces between nucleons just balancing the Fermi pressure, and the pressure on the piston is zero. If the piston is pushed a little further the system collapses to a droplet on the cylinder floor. In our approximation the density would be infinite; actually the nucleus would collapse to a radius near the range of forces so (1.07) would fail, or $p_f$ would become large enough so that relativistic corrections became important. Lee and Wick[4] have pointed out that if relativistic formulas are used for the kinetic and potential energies there is a solution of the problem, but one which gives a binding energy much larger than that of ordinary nuclei.

This difficulty is avoided by Heisenberg's supposition that the nuclear forces are exchange forces between protons and neutrons. By this is meant that if a proton is an a state (including spin) $u_1(\underline{r})$ and the neutron in a state $u_2(\underline{r})$ the interaction energy, rather than being the ordinary term

$$\int u_1(\underline{r}_P)^* u_2(\underline{r}_N)^* V(\underline{r}_P-\underline{r}_N) u_1(\underline{r}_P) u_2(\underline{r}_N) d\underline{r}_P d\underline{r}_N, \qquad (1.11)$$

is instead an exchange term,

$$\int u_1(\underline{r}_N)^* u_2(\underline{r}_P)^* V(\underline{r}_P-\underline{r}_N) u_1(\underline{r}_P) u_2(\underline{r}_N) d\underline{r}_P d\underline{r}_N, \qquad (1.12)$$

that is, the neutron and proton coordinates and spins are interchanged in the final state. Exchange terms were familiar in atomic and molecular theory, and Heisenberg's justification for the exchange force was based on an analogy with the $H_2^+$ ion. He suggested a model in which the

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[4] T.D. Lee and G.C. Wick, Phys. Rev. D **9**, 2291 (1974).

neutron was regarded as proton plus electron, the exchange force arising from exchange of the electron between the two protons. A more plausible interpretation was supplied by Yukawa. The interaction (1.04) can be viewed as due to the exchange of an uncharged meson, as illustrated in Fig.1.4(a).

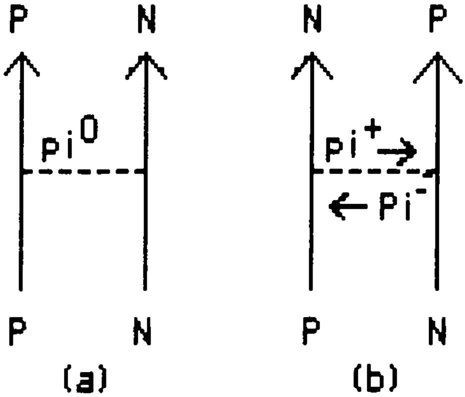


Fig.1.4

If the mesons were charged we would have Fig.1.4(b), which exchanges the proton and neutron in the final state. We then have, in place of (1.04),

$$V_{NP}=\pm(g^2/4\pi)(e^{-\mu r}/r)P^H, \qquad (1.13)$$

where $P^H$ is defined to be an operator which exchanges the coordinates and spins of neutron and proton. Since Yukawa envisaged only charged mesons there would be no N-N or P-P forces of order $g^2$ (single meson exchange), in accord with the Heisenberg model. The ± sign in (1.13) depends on the transformation properties supposed for the meson field, $\phi$. Yukawa originally supposed $\phi$ was the fourth component of a four-vector field, in analogy to the electrostatic potential, in which case the + sign would apply. For a scalar $\phi$ the minus sign is correct. The latter is required for the N-P interaction, as can most easily be seen by considering the deuteron: the potential must be attractive for the bound state, which is $S^3$ and thus has $P^H=1$.

If we write $V_{\underline{q}}$ for the Fourier component of $V(r)$, i.e.

$$V(\underline{r})=\int V_{\underline{q}} e^{i\underline{q}\cdot\underline{r}} d\underline{q}/(2\pi)^3, \qquad V_{\underline{q}}=\int V(\underline{r}) e^{-i\underline{q}\cdot\underline{r}} d\underline{r},$$

one sees that if $u_1$ and $u_2$ are plane waves of momentum $\underline{p}$ and $\underline{p}'$, (1.11) gives $V_0$, while (1.12) gives $V_{\underline{q}}$, with $\underline{q}=\underline{p}-\underline{p}'$. Note that the $V_0$ defined here is the same as that of (1.07).

As an example of the difference an exchange force makes, let us consider the Yukawa potentials, (1.04) or (1.13). Here

$$V_{\underline{q}}=-g^2/(\mu^2+q^2), \tag{1.14}$$

and $V_0=-g^2/\mu^2$ (note that, for $\rho$=const., the solution of (1.06) is $\Phi=(g/\mu^2)\rho$ which gives $V=-(g^2/\mu^2)\rho$, in agreement with (1.07)). In the high density limit, $p_f>>\mu$, the exchange term (1.14) is smaller than $V_0$ by two powers of q. Thus in (1.10) the potential energy term will be proportional to $p_f$ rather than to $\rho\approx p_f^3$. With a repulsive kinetic energy proportional to $p_f^2$ and an attractive potential energy proportional to $p_f$, (1.10) will have a minimum, at negative E/A, for a finite $p_f$.

Heisenberg's exchange force, while ingenious, is not the true explanation of the "saturation of nuclear forces". It is now known from high energy scattering experiments that the nuclear forces are repulsive at small distances, less than 0.5 f, like Fig.1.2. The effect of the "repulsive core" can be estimated by using the results of the theory of a hard-sphere Fermi gas in the form given by Bohr and Mottleson*. The kinetic energy (1.09) is replaced by

* A. Bohr and B.R. Mottleson, Nuclear Structure (Benjamin, New York, 1969), Vol.1, p. 256.

$$T=\frac{3}{5}\frac{p_f^2}{(2M)}\frac{1}{[1-(5/3\pi)r_cp_f]^2}, \tag{1.15}$$

where $r_c$ is the core radius. The form of (1.15) is a little surprising, since one might expect to estimate the correction by reducing $V_{ol}$ in (1.08) by the excluded volume, $(4\pi/3)r_c^3A$. This argument would lead to a correction factor $1/[1-(4\pi/3)r_c^3\rho]^{2/3}$, i.e. a correction proportional to $(r_cp_f)^3$ rather than one linear in $r_cp_f$.

With the nuclear radius written $R=r_0A^{1/3}$ we have

$$\rho=3/(4\pi r_0^3),\quad p_f=(9/8\pi)^{1/3}/r_0, \tag{1.16}$$

the second relation following from (1.08). The excluded volume argument gives T a correction factor $1/[1-(r_c/r_0)^3]^{2/3}$. The value of $r_0$ determined from high energy electron scattering experiments is $r_0=1.1$ f. For $r_c=0.5$ f the correction would be only 7%. This lead to a long-held belief that a core radius of this size was too small to stabilize the nucleus at so low a density, and thus an alternative such as Heisenberg's was necessary. On the other hand, (1.15) gives a factor $1/[1-(5/3\pi)(9\pi/8)^{1/3}(r_c/r_0)]^2=1/[1-0.8r_c/r_0]^2=2.5$, a comparatively enormous effect.

Some feeling for the reason $r_cp_f$ appears linearly in (1.15) may perhaps be found in the following argument. In terms of the variable $r_0$, rather than $p_f$, (1.15) is of the form

$$T=B/(r_0-0.8r_c)^2,$$

with B a constant. Consider a particle inside a sphere of of radius $r_0$, with a wave function $\emptyset(r)=u(r)/r$ satisfying some simple boundary condition at the surface, e.g. $u'(r_0)=0$. The solution is $u=\sin(pr)$, $pr_0=\frac{1}{2}\pi$, and the energy is

$$T=p^2/2M=(1/2M)(\pi/2)^2/r_0^2.$$

If a repulsive core is added at the center (or an excluded shell at the edge) we would have $p(r_0-r_c)=\frac{1}{2}\pi$, and an energy

$$T=(1/2M)(\pi/2)^2/(r_0-r_c)^2.$$

To continue the argument, write the binding energy in terms of $r_0$; it then takes the form (see (1.10)

$$E/A=T+\tfrac{1}{2}V=B/(r_0-a)^2 -C/r_0^3,$$

where we have written $a=0.8r_c$. The condition for a minimum is

$$\frac{d}{dr_0}\left(\frac{E}{A}\right)= -\frac{2T}{r_0-a} - \frac{3V}{2r_0} =0,$$

or

$$1-(a/r_0)=(2/3)T/(-\tfrac{1}{2}V).$$

If we suppose that in nuclear matter the binding energy is a small difference between large kinetic and potential energies, $T/(-\frac{1}{2}V)\approx 1$, and

$$1-a/r_0=2/3,\ a/r_0=1/3,\ r =3a=3\times 0.8r_c.$$

For $r_c=0.5$ f this gives $r_0=1.2$ f, close to the observed $r_0=1.1$ f. Thus a repulsive core of the observed range can account for the actual nuclear density.

For $r_0=1.1$ f, the Fermi energy is $T_f=42$ Mev and T, given by (1.15), is $T=(3/5)\times 2.5\times 42=63$ Mev. The empirical volume binding energy, the constant $\epsilon$ in (1.01), is 15 Mev. Thus $(1/2)V=-78$ Mev. These numbers are not to be taken too literally, but they give an idea of the magnitudes involved.

It should also be observed that these arguments depend only on the the volume integral of the potential, e.g. for a Yukawa potential on $g^2/\mu^2$, and do not separately determine the strength of the interaction or its range.

A more complete but equally simple-minded model, discussing all three terms in (1.01), will be found in Appendix A, "A Simple Nuclear Model".