PART ONE

DISSIPATIVE SYSTEMS

Call Pythagoras and bid
The sage to mark divine the laws which rule
Each planet's course; and when he reads and sees
Such harmony amidst the countless worlds,
Trembling with joy his heart will overflow
Before the sacred concert of high reason.

Hans Christian Oersted, The Soul in Nature.
Taken from Bern Dibner, Oersted (The Burndy
Library, Norwalk, Connecticut, 1961)

1. Introduction

The Greeks at the time of Ptolemy believed all motion could be decomposed into "perfect" circular motions. This idea was successfully applied in the description of planetary motions in terms of epicycles. With the usual acuity of hindsight we might phrase the Platonic ideal this way: all motion is quasiperiodic, such that the Fourier transform of any coordinate consists of sharp spikes. (Figure 1.1) Poincare', near the turn of the century, was perhaps the first person to understand that there are (bounded) systems whose spectra do not have this form, that there are so-called "nonintegrable" systems. Such systems have a broadband, continuous component in their spectra, as indicated in Figure 1.2. Einstein (1917) also recognized this possibility.

Spectra of the type shown in Figure 1.2 are associated with turbulent motion. Imagine two little corks placed near to each other in a flowing fluid, and suppose the flow is laminar (smooth and orderly). As time evolves the positions of the corks are well correlated and their separation grows linearly with time, at a rate proportional to the difference in their velocities. But if the flow is turbulent, the corks separate rapidly, typically exponentially with time. Their locations depend very sensitively on where they started out. We will call a system chaotic if it has this property of very sensitive (exponential) dependence on initial conditions.

Chaotic motion is non-quasiperiodic, having a spectrum like that sketched in Figure 1.2. Over the past few years many different kinds of physical systems have been found to evolve chaotically. Among the recent developments, two are especially exciting. One is the recognition that chaos may appear in systems described by relatively simple rules of evolution, such as three first-order differential equations. The other is the discovery that there are a few prevalent or "universal" ways by which a system can make the transition from regular, quasiperiodic behavior to chaos. These are called routes to chaos, and the same routes have been observed in systems as diverse as fluids, lasers, and semiconductor devices, to name but a few.

One of the seminal papers in the field was published by E.N. Lorenz in 1963. [1.1] Lorenz made a rather severe truncation of some hydrodynamical partial differential equations, ending up with a set of three ordinary, nonlinear differential equations. This Lorenz model exhibits, for certain parameter ranges, what is now called chaos, the property of very sensitive dependence on initial conditions. Lorenz offered a half-serious metaphor, which we can paraphrase as follows. Suppose you are trying to predict the weather using complicated hydrodynamical equations and a really super computer. You know initial conditions for the atmospheric pressure, temperature, etc., and feed all this information into your computer code. But you neglected to account for a butterfly fluttering about somewhere over Taipei! If your system is chaotic, this "uncertainty" makes detailed, long-term weather prediction impossible, because of the extreme sensitivity to initial conditions. What the butterfly does might affect the weather next month in New York.

Lorenz's work, which seemed to remain largely unnoticed for a long time, showed that we needn't study terribly complicated systems of equations to learn something about chaos: systems of just a few simple equations can have very sensitive dependence on initial conditions. This point has taken a long time to sink in.

At about the same time as Lorenz , Buley and Cummings [1.2] made a numerical study of essentially the same system of equations as Lorenz while modeling a single-mode laser. They remarked that "A case has been run...in which the output...appears as a series of almost random spikes." Evidently they were finding the chaotic behavior that Lorenz was wrestling with in a completely different context.

In modeling physical systems we usually work with differential equations (e.g., $F = ma$). Typically we specify initial conditions at some time $t = 0$, and try to predict the state of the system for times $t > 0$. From a computational standpoint things would be much easier if time evolved in discrete steps rather than continuously. Then a typical simulation would involve a discrete mapping like

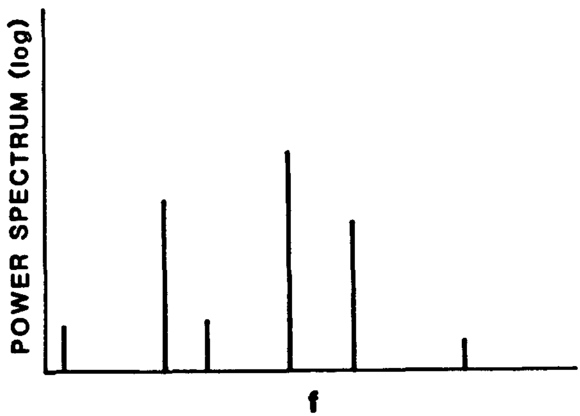

Figure 1.1 Typical frequency spectrum of some coordinate of a quasiperiodic system.

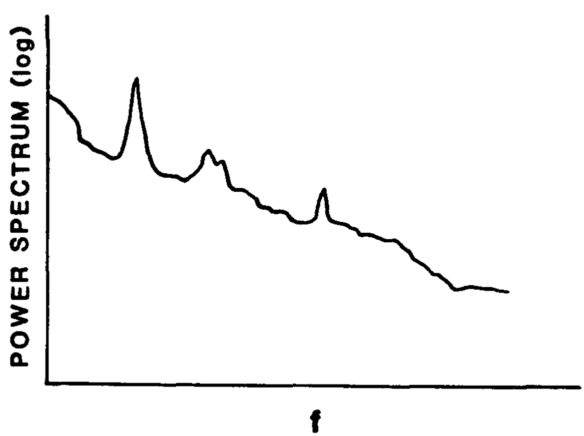

Figure 1.2 Typical frequency spectrum of some coordinate of a chaotic system.

$$x_{n+1} = f(x_n) \tag{1.1}$$

where f(x) is a prescribed function of x. Here the system at "time" n would be characterized by the number x_n, which is determined by (1.1) once the initial condition x_o is given. The mapping (1.1) is one-dimensional. We can easily conceive of higher-dimensional mappings, such as two-dimensional ones of the form

$$x_{n+1} = f(x_n, y_n) \tag{1.2a}$$

$$y_{n+1} = g(x_n, y_n) \tag{1.2b}$$

We might regard discrete mappings as contrived models for the kind of behavior exhibited by differential equations. Actually, however, it is possible, at least in principle, to construct discrete mappings from systems of differential equations. Such a technique was invented by Poincare' as part of his program to geometrize the theory of differential equations. Consider as an example a set of three differential equations of the form

$$\dot{x} = f(x,y,z), \quad \dot{y} = g(x,y,z),$$
$$\dot{z} = h(x,y,z) \tag{1.3}$$

Suppose we plot points $(x_n, y_n) \equiv (x(t_n), y(t_n))$ for times t_n for which $z(t_n) = 0$ and $\dot{z}(t_n) < 0$, as illustrated in Figure 1.3. Then the evolution $(x_n, y_n) \rightarrow (x_{n+1}, y_{n+1})$ defines a <u>Poincare' map</u>, which is a discrete mapping of the form (1.2). Although the only known general way of constructing a Poincare' map is by numerical integration of the differential equations, it is very useful to know that such a discrete mapping exists, for then we can study some aspects of "continuous flows" like (1.3) by looking at discrete mappings. Mappings are much easier to handle than differential equations (i.e., iteration is easier than integration).

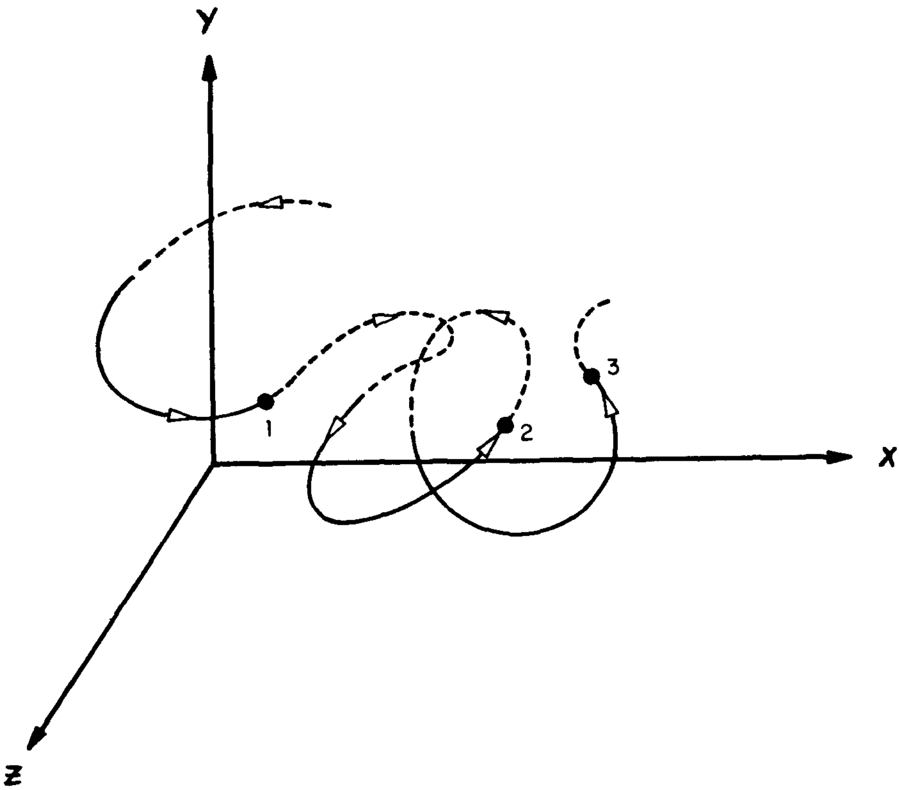

Figure 1.3 Construction of a Poincare' map with the xy plane as the surface of section. In this example point 1 is mapped into point 2, and 2 is mapped into 3.

In some instances the analysis of a physical system leads directly to a discrete mapping. Consider a particle constrained to move in one dimension under the influence of periodic impulses, the force being

$$F = A(x) \sum_{-\infty}^{\infty} \delta(t/T - n) \tag{1.4}$$

Then the Hamilton equations of motion for the position and momentum are

$$\dot{x} = p/m \tag{1.5a}$$

$$\dot{p} = A(x) \sum_{-\infty}^{\infty} \delta(t/T - n) \tag{1.5b}$$

Let us integrate these equations from $t = nT-\epsilon$ to $(n+1)T-\epsilon$, where ϵ is an infinitesimally short time. Using the properties of the delta function, we easily obtain

$$x_{n+1} = x_n + T/m[p_n + TA(x_n)] \tag{1.6a}$$

$$p_{n+1} = p_n + TA(x_n) \tag{1.6b}$$

where x_n and p_n are the values of x and p just before the n<u>th</u> kick. This is a discrete mapping of the form (1.2). Periodically kicked systems of this type, because they are easy to handle numerically, have been useful in studies of "quantum chaos," as we will see later.

A one-dimensional mapping that has played an important role in the recent developments is the <u>logistic map</u>

$$x_{n+1} = 4\lambda x_n(1 - x_n), \qquad 0 \le x_o, \lambda \le 1 \tag{1.7}$$

Before ending our brief introduction, let us illustrate what we mean by very sensitive dependence on initial conditions by considering the case $\lambda = 1$ of the logistic map. In this case the transformation $x_n = \sin^2\pi\theta_n$, plus the identity $\sin 2\pi\theta_n = 2\sin\pi\theta_n\cos\pi\theta_n$, reduces the mapping to the form $\theta_{n+1} = 2\theta_n$, which has the explicit solution $\theta_n = 2^n\theta_o$. Since we can add any integer to θ without changing the value of x, we can write

$$\theta_n = 2^n\theta_o \quad (\mathrm{mod}\ 1) \qquad (1.8)$$

as the solution of the logistic map for $\lambda = 1$.

It is easy to see that the mapping (1.8) has the property of very sensitive dependence on initial conditions: if we change the initial seed θ_o to $\theta_o + \epsilon$, then θ_n changes by $2^n\epsilon = \epsilon e^{n\log 2}$. In other words, there is an exponential separation with "time" n of initially close "trajectories." The rate of exponential separation, namely log2, is called the Lyapunov exponent, and the fact that it is positive in this example means that we have very sensitive dependence on initial conditions, i.e., chaos.

We can understand this extreme sensitivity to initial conditions from another standpoint. Let us write θ_o in base-2 notation. For instance, we can write the number 1/2 + 1/4 + 1/16 + 1/128 +...in base 2 as 0.1101001... In base 2 the algorithm (1.8) amounts to just shifting the "decimal" point. Thus if θ_o = .1101001...then θ_1 = .101001..., θ_2 = .01001..., θ_3 = .1001..., etc. Obviously then θ_n will depend on the nth and higher digits of θ_o, and when n is large the value of θ_n depends extremely sensitively on the precise value of θ_o. We can now begin to better appreciate the Lorenz butterfly metaphor!

This example illustrates another aspect of extreme sensitivity to initial conditions. If we iterate the map (1.8) on a digital computer, then after a relatively small number of iterations (typically ≈ 50 on 16-digit machines) we generate numerical "garbage," simply because in the digit-shifting process we eventually pick up digits representing round-off errors in the computer. (The reader unfamiliar with this sort of thing might find it amusing to program (1.8) on his computer, starting with $\theta_o = 1/7$. The sequence predicted by (1.8) is simply 1/7, 2/7, 4/7, 1/7, 2/7, 4/7, 1/7, ...,i.e., a "3-cycle." But this is not what is found by computer iteration, at least not if the iterations are carried out far enough.) For this reason it has been said that "Chaos will beat any computer!"

This example suggests that detailed, long-time predictions about a chaotic system are impossible in practical terms because (a) we don't know initial conditions with infinite precision, and (b) we can't handle an infinite string of digits in our computations. This practical indeterminability has obvious "philosophical" implications, some of which we will consider later on.

Chaos is often defined qualitatively as a more or less "random" or disorderly behavior that is intrinsic to a system and not due to any externally imposed "noise." That is, the chaotic behavior is described by completely deterministic equations of motion. In the Russian literature this intrinsic chaos is called stochasticity. Our goals are (a) to better understand this deterministic chaos, (b) to see how systems make the transition from orderly to chaotic behavior, (c) to study examples of how chaos arises in the interaction of light and matter, and (d) to consider some important aspects of "quantum chaos."

Aside from things like modular arithmetic or piecewise linearity, chaotic behavior is associated with nonlinear systems. For this reason we will devote the next section to a very brief introduction to nonlinear differential equations, emphasizing how they differ from linear differential equations. Of course the whole area of laser-matter interactions is replete with nonlinearities, and during these lectures we will see that even the simplest models in the field can be chaotic.