# CHAPTER I.

## General Properties of Electromagnetic Waves in Dielectric Media

### §I.1 Plasma as a dielectric medium

In this chapter, we shall discuss some general properties pertinent to linear electromagnetic waves in plasmas. In fact, these properties are sufficiently general and, hence, fundamental that they are applicable to other dielectric media.

Let us assume that the plasma or, equivalently, the dielectric medium to be time-stationary and spatially homogeneous. The perturbed (wave) electric field, $\delta \underset{\sim}{E}(\underset{\sim}{x}, t)$, can then be taken to assume the plane-wave form; i.e.,

$$\delta \underset{\sim}{E} (\underset{\sim}{x},t) = \delta \hat{\underset{\sim}{E}} \exp (i\omega t - i \underset{\sim}{k}\cdot\underset{\sim}{x}) \quad . \qquad (I.1.1)$$

Here, $\underset{\sim}{k}$ is the wave vector and $\omega$ is the frequency. Whether $(\omega,\underset{\sim}{k})$ is real or complex depends on the detailed nature of the problem. For example, $\omega=\omega_o$ is real if we are interested in wave properties driven by an external source oscillating at frequency $\omega_o$. $\underset{\sim}{k}$, however, can become complex as in the case of evanescent waves. On the otherhand, if we are interested in an initial value problem (e.g., stability analysis), $\omega$ is then complex. This point can be realized by interpreting Eq. (I.1.1) as the Laplace-in-time and Fourier-in-space transform. Thus, we have, in this case, Im $\omega > 0$ to ensure causality (i.e., $\delta \underset{\sim}{E} \to 0$ as $t \to -\infty$) and $\underset{\sim}{k}$ is real.

In response to $\delta \underset{\sim}{E}$, charged particles will jiggle and acquire velocities, $\delta \underset{\sim}{V}$, and, hence, there will be perturbed current densities, $\delta \underset{\sim}{J} (\underset{\sim}{x},t)$. Since we are only interested in linear effects and the background plasma is taken to be stationary and homogeneous, it is quite general that

$$\delta\hat{\underset{\sim}{J}}\ (\omega,\underset{\sim}{k}) = \underset{\approx}{\sigma}\ (\omega,\underset{\sim}{k}) \cdot \delta\hat{\underset{\sim}{E}}\ (\omega,\underset{\sim}{k}) \quad . \tag{I.1.2}$$

Here, $\underset{\approx}{\sigma}$ is the conductivity tensor. For the present discussion, we shall simply assume $\underset{\approx}{\sigma}$ and Eq. (I.1.2) are given.

Substituting Eqs. (I.1.1) and (I.1.2) into the following Maxwell's equations

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = -\ (1/c)\partial\underset{\sim}{B}/\partial t\ , \tag{I.1.3}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = (4\pi\underset{\sim}{j} + \partial\underset{\sim}{E}/\partial t)/c\ ; \tag{I.1.4}$$

and noting that $\underset{\sim}{E} = \underset{\sim}{E}_o + \delta\underset{\sim}{E}$ , $\underset{\sim}{\nabla} \times \underset{\sim}{E}_o = 0$ , $\partial\underset{\sim}{E}_o/\partial t = 0$ and etc., we find

$$\underset{\sim}{k} \times \delta\hat{\underset{\sim}{E}} = \omega\ \delta\hat{\underset{\sim}{B}}/c \tag{I.1.5}$$

$$\begin{aligned} \underset{\sim}{k} \times \delta\hat{\underset{\sim}{B}} &= -\ (\omega/c)\ [\underset{\approx}{I} + i4\pi\ \underset{\approx}{\sigma}/\omega]\cdot\delta\hat{\underset{\sim}{E}} \\ &\equiv -\ (\omega/c)\ \underset{\approx}{D} \cdot \delta\hat{\underset{\sim}{E}} \quad . \end{aligned} \tag{I.1.6}$$

Here, $\underset{\approx}{I}$ is the unit tensor and $\underset{\approx}{D} \equiv \underset{\approx}{I} + i4\pi\ \underset{\approx}{\sigma}/\omega$ is the equivalent dielectric tensor. Combining Eqs. (I.1.5) and (I.1.6), we obtain

$$\left[\underset{\sim}{k}\ \underset{\sim}{k} - k^2\ \underset{\approx}{I} + \frac{\omega^2}{c^2}\ \underset{\approx}{D}\right] \cdot \delta\hat{\underset{\sim}{E}} \equiv \underset{\approx}{\varepsilon} \cdot \delta\hat{\underset{\sim}{E}} = 0 \quad . \tag{I.1.7}$$

In deriving Eq. (I.1.7) , we have used the vector identity

$$\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{A}) = \underset{\sim}{\nabla} (\underset{\sim}{\nabla} \cdot \underset{\sim}{A}) - \nabla^2 \underset{\sim}{A} .$$

In order that Eq. (I.1.7) has non-trival solution of $\delta\hat{\underset{\sim}{E}}$, the determinant of $\underset{\approx}{\varepsilon}$ must vanish; i.e.,

$$\varepsilon(\omega,\underset{\sim}{k}) \equiv \| \underset{\sim}{\varepsilon}(\omega,\underset{\sim}{k}) \| = 0 \quad . \tag{I.1.8}$$

Equation (I.1.8) is the so-called dispersion relation. That is, from Eq. (I.1.8) we can obtain a relation between $\omega$ and $\underset{\sim}{k}$, $\omega = \omega(\underset{\sim}{k})$, and, thus, plot the $\omega$ vs. $\underset{\sim}{k}$ dispersion curves.

In general, given a $\underset{\sim}{k}$, the dispersion relation can contain multiple (or, sometimes, infinite) number of $\omega$ roots. Indexing these $\omega$ branches by j, we then have

$$\omega = \omega_j(\underset{\sim}{k}) \text{ for } j = 1,2..... \quad . \tag{I.1.9}$$

In plasma physics literatures, the various $\omega$ branches are sometimes called normal modes. Substituting Eq. (I.1.9) into Eq. (I.1.7), we can then determine the eigenvector $\delta\hat{\underset{\sim}{E}}_j$ corresponding to the eigenvalue $\omega_j$. From $\delta\hat{\underset{\sim}{E}}_j$ the polarization of the perturbed electric (or magnetic) field for the jth branch (normal mode) with respect to $\underset{\sim}{k}$ or the equilibrium magnetic field $\underset{\sim}{B}_o$ is then known.

<u>Homework #1</u>

(1) Illustrate with an example the above mentioned concepts: conductivity tensor, dispersion relation, $\omega$ branches (or modes), eigenvector $\delta\underset{\sim}{E}$ and wave polarization.

(2) Consider the following Inverse Laplace transform $\phi(t) = \int_c (d\omega/2\pi) \times e^{-i\omega t}/D(\omega)$. How will you choose the integration contour c to ensure causality for (i) D ($\omega$) has no zeroes in the upper-half (Im $\omega$>0) plane? (ii) D ($\omega$) has finite number of zeroes in the Im $\omega$ > 0 plane?

(3) (i) what is the condition that $\delta\phi(x,t)$ is Fourier transformable in x; i.e., $\delta\phi(k,t) = \int dx\ \delta\phi(x,t)e^{-ikx}$ exists? (ii) Suppose that $\delta\phi$ at t = 0 is sufficiently smooth and of finite spatial extent; i.e., $\delta\phi(x,t=0) = 0$ for $|x| \geq d$. Is $\delta\phi(x,t)$ Fourier transformable for any <u>finite</u> t and why?

§I.2 <u>Two time-scale analysis</u>

In plasma, due to the presence of weak dissipation or source, waves tend to either grow or decay slowly with time. In many cases of interest, the dissipation or source is sufficiently weak such that the characteristic growth or decay rate is much smaller than the typical oscillation frequency $\omega_o$. The wave process is then said to possess two time scales and we may assume $\delta E$ to be of the following form

$$\delta E\ (t) = \delta\hat{E}\ (\varepsilon t)\ \exp\ (-i\omega_o t). \tag{I.2.1}$$

Here, for simplicity of discussions, we have neglected spatial dependence and taken $\delta E$ to be a scalar and $\varepsilon \ll 1$ is a small parameter. Thus,

$$|\partial\delta\hat{E}/\partial t|/|\omega_o\delta\hat{E}| \sim o(\varepsilon) \ll 1. \tag{I.2.2}$$

Using Eqs. (I.2.1) and (I.2.2), we shall explore some implications of the two time scales. First, we show that given the relation

$$\delta A(\omega) = B(\omega)\delta E(\omega) \tag{I.2.3}$$

with ω being, strictly speaking, the Laplace-transform variable, we have

$$\delta\hat{A}(\epsilon t) = B\,(\omega_o + i\,\frac{\partial}{\partial t})\;\delta\hat{E}(\epsilon t). \tag{I.2.4}$$

Proof of Eq. (I.2.4): From Eq. (I.2.3), we have

$$\delta A(t) = \int \delta A(\omega)\; e^{-\,i\omega t}\, d\omega = \int d\omega\; B\omega \int \frac{dt'}{2\pi}\;\delta E(t')e^{i\omega(t'-t)} \tag{I.2.5}$$

Taylor expanding B(ω) about $\omega=\omega_o$ and noting Eq. (I.2.1), Eq. (I.2.5) becomes

$$\delta A(t) = e^{-i\omega_o t}\int \frac{dt'}{2\pi}\;\delta\hat{E}(\epsilon t')\int d\omega\;[B(\omega_o) + \frac{\partial B}{\partial \omega_o}\,(\omega-\omega_o)\ldots]\;e^{i(\omega-\omega_o)(t'-t)}$$

$$= e^{-i\omega_o t}\int \frac{dt'}{2\pi}\;\delta\hat{E}(\epsilon t')[B(\omega_o) + \frac{\partial B}{\partial \omega_o}\,\frac{i\partial}{\partial t} + \ldots]\int d\omega e^{i(\omega-\omega_o)(t'-t)}$$

$$= e^{-i\omega_o t}[B(\omega_o) + \frac{\partial B}{\partial \omega_o}\;i\,\frac{\partial}{\partial t} + \ldots]\int dt'\;\delta\hat{E}(\epsilon t')\delta(t-t) \quad ;$$

that is,

$$\delta\hat{A}\;(\epsilon t) = [B(\omega_o) + \frac{\partial B}{\partial \omega_o}\;i\,\frac{\partial}{\partial t} + \ldots]\;\delta\hat{E}(\epsilon t). \tag{I.2.4}$$

Equation (I.2.4) can be derived alternatively by noting that

$$L_p^{-1}[-\,i\omega\;\delta E(\omega)] = \frac{\partial}{\partial t}\;\delta E(t) = -\;ie^{-i\omega_o t}(\omega_o + i\,\frac{\partial}{\partial t})\delta\hat{E}\;(\epsilon t). \tag{I.2.6}$$

Thus regarding ω as an operator, Eq. (I.2.3) inversely transforms to

$$\delta A(t) = e^{-i\omega_o t} B(\omega_o + i \frac{\partial}{\partial t}) \delta\hat{E}(\epsilon t)$$

or

$$\delta\hat{A}(\epsilon t) = B (\omega_o + i \frac{\partial}{\partial t}) \delta\hat{E}(\epsilon t). \qquad (I.2.4)$$

Next, we note the existence of two disparate time scales presents an unique opportunity of separating dynamics on the two scales. For example, we may average out the fast oscillations in order to examine only the slow time-scale phenomena. This concept of averaging out the fast oscillations (phase averaging) is essentially the same one used in deriving guiding center motion where the cyclotron motion is the fast oscillation and is averaged out. In the present case, the phase-averaging method is simple. Let

$$\delta A(t) = [\delta\hat{A} (\epsilon t) e^{-i\omega_o t} + c.c.]/2 \quad ,$$

$$\delta B(t) = [\delta\hat{B} (\epsilon t) e^{-i\omega_o t} + c.c.] \quad ; \qquad (I.2.6)$$

we then have

$$\overline{\delta A(t) \ \delta B(t)} \equiv \frac{1}{\Delta_t} \int_o^{\Delta_t} \delta A(t) \ \delta B(t) \ dt$$

$$= [\delta\hat{A} \ \delta\hat{B}^* + \delta\hat{A}^* \ \delta\hat{B}]/4 \quad . \qquad (I.2.7)$$

Here, $\Delta_t$ the integration interval is chosen such that

$$\frac{1}{\varepsilon} \gg \Delta_t \, \omega_o \gg 1; \qquad (I.2.8)$$

i.e., for the $\Delta_t$ time scale, $\delta\hat{A}$ is frozen (adiabatic) and can be taken outside the integral.

We now examine the implication of the above results, Eqs. (I.2.4) and (I.2.7), to the Maxwell's equations, Eqs. (I.1.3) and (I.1.4). First, from $\underset{\sim}{B} \cdot$ Eq. (I.1.3) - $\underset{\sim}{E} \cdot$ Eq. (I.1.4), we have

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{P} = - \frac{1}{4\pi} \left[ (\underset{\sim}{B} \cdot \frac{\partial}{\partial t} \underset{\sim}{B} + \underset{\sim}{E} \cdot (4\pi \underset{\sim}{J} + \frac{\partial}{\partial t} \underset{\sim}{E}) \right] \quad . \qquad (I.2.9)$$

Here

$$\underset{\sim}{P} = c\underset{\sim}{E} \times \underset{\sim}{B}/4\pi \qquad (I.2.10)$$

is the Poynting vector. Equation (I.2.9) is often referred to as Poynting's conservation theorem. Neglecting spatial dependence and noting that $\underset{\sim}{E} = \underset{\sim}{E}_o + \delta\underset{\sim}{E}$, $\underset{\sim}{B} = \underset{\sim}{B}_o + \delta\underset{\sim}{B}$, Eq. (I.2.9) yields, the following expression quadratic in the wave quantities

$$\frac{1}{4\pi} \delta\underset{\sim}{B} \cdot \frac{\partial}{\delta t} \delta\underset{\sim}{B} + \delta\underset{\sim}{E} \cdot (4\pi \, \delta\underset{\sim}{J} + \frac{\partial}{\partial t} \delta\underset{\sim}{E}) = 0. \qquad (I.2.11)$$

We now note that the Laplace transform of $4\pi \, \delta\underset{\sim}{J} + \frac{\partial}{\delta t} \delta\underset{\sim}{E}$ is given by

$$L_p(4\pi\delta\underset{\sim}{J} + \frac{\partial}{\delta t} \delta\underset{\sim}{E}) = 4\pi \, \delta\underset{\sim}{J}(\omega) - i\omega \, \delta\underset{\sim}{E}(\omega) = - i\omega \, \underset{\approx}{D}(\omega) \cdot \delta\underset{\sim}{E}(\omega) \quad . \qquad (I.2.12)$$

Thus, if we let

$$\delta \underset{\sim}{E}(t) = [\ \delta \hat{\underset{\sim}{E}}(\epsilon t) e^{-i\omega_o t} + c.c.]/2 \tag{I.2.13}$$

and similarly for $\delta \underset{\sim}{B}(t)$, we find, using Eq. (I.2.4)

$$4\pi\ \delta \underset{\sim}{J} + \frac{\partial}{\partial t}\ \delta \underset{\sim}{E} = (1/2) e^{-i\omega_o t} [-i(\omega_o + i\ \frac{\partial}{\partial t}) \underset{\approx}{D}(\omega_o + i\ \frac{\partial}{\partial t}) \cdot \delta \hat{\underset{\sim}{E}}]$$

$$+ (1/2) e^{i\omega_o t} [(-\ i)(-\omega_o + i\ \frac{\partial}{\partial t}) \underset{\approx}{D}(-\omega_o + i\ \frac{\partial}{\partial t}) \cdot \delta \hat{\underset{\sim}{E}}^*]$$

$$\simeq (1/2)\{e^{-i\omega_o t} [-\ i\omega_o \underset{\approx}{D}(\omega_o) + \frac{\partial}{\partial \omega_o} [\omega_o \underset{\approx}{D}(\omega_o)]\ \frac{\partial}{\partial t}] \cdot \delta \hat{\underset{\sim}{E}}$$

$$+ e^{i\omega_o t} [i\omega_o\ \underset{\approx}{D}\ (-\omega_o) + \frac{\partial}{\partial \omega_o} [\omega_o\ \underset{\approx}{D}\ (-\omega_o)]\ \frac{\partial}{\delta t}] \cdot \delta \hat{\underset{\sim}{E}}^*\}. \tag{I.2.14}$$

Substituting Eq. (I.2.14) into Eq. (I.2.11) and performing the phase averaging, we have, term by term,

$$\overline{\delta \underset{\sim}{E} \cdot (4\pi \delta \underset{\sim}{J} + \frac{\partial}{\partial t}\ \delta \underset{\sim}{E})} = (1/4)\ \{-\ i\omega_o \delta \hat{\underset{\sim}{E}}^* \cdot \underset{\approx}{D}(\omega_o) \cdot \delta \hat{\underset{\sim}{E}} + i\omega_o\ \delta \hat{\underset{\sim}{E}} \cdot \underset{\approx}{D}(-\omega_o) \cdot \delta \hat{\underset{\sim}{E}}^* +$$

$$\delta \hat{\underset{\sim}{E}}^* \cdot \frac{\partial}{\partial \omega_o}\ [\omega_o \underset{\approx}{D}(\omega_o)]\ \frac{\partial}{\partial t}\ \delta \hat{\underset{\sim}{E}} + \delta \hat{\underset{\sim}{E}} \cdot \frac{\partial}{\partial \omega_o}\ [\omega_o \underset{\approx}{D}(-\omega_o)] \cdot \frac{\partial}{\partial t}\ \delta \hat{\underset{\sim}{E}}^*\}. \tag{I.2.15}$$

Defining

$$\underset{\approx}{D} = \underset{\approx}{D}_h + i\ \underset{\approx}{D}_a,$$

where h and a denote, respectively, Hermitian and anti-Hermitan components; i.e.,

$$\underset{\approx}{D}_h(\omega_o) = (1/2)\{\underset{\approx}{D} + (\underset{\approx}{D}^T)^*\}\ , \tag{I.2.17}$$

and

$$\underset{\approx}{D}_a(\omega_o) = (1/2i)\{\underset{\approx}{D} - (\underset{\approx}{D}^T)^*\}. \tag{I.2.18}$$

Here $(A_{ij})^{T*} = A^*_{ji}$ ; i.e., the complex conjugate of the transpose of $\underset{\approx}{A}$. We note, futhermore, that from realizability condiion

$$\underset{\approx}{D}(-\omega_o) = \underset{\approx}{D}^*(\omega_o). \tag{I.2.19}$$

The proof of Eq. (I.2.19) is straightforward by noting that both $\delta\underset{\sim}{E}$ and $\partial\delta\underset{\sim}{E}/\partial t + 4\pi\delta\underset{\sim}{J}$ are real physical quantities. Equation (I.2.15) then reduces to, assuming $|\underset{\approx}{D}_a| << |\underset{\approx}{D}_h|$,

$$-i\omega_o\delta\hat{E}^*_j D_{ji}\delta\hat{E}_i + i\omega_o\,\delta\hat{E}_i D^*_{ij}\,\delta E^*_j + \delta\hat{E}^*_j \frac{\partial}{\partial\omega_o}[\omega_o D_{ji}]\frac{\partial}{\partial t}\delta\hat{E}_i$$

$$+ \delta\hat{E}_i \frac{\partial}{\partial\omega_o}[\omega_o D^*_{ij}]\frac{\partial}{\partial t}\delta\hat{E}^*_j = 2\omega_o\delta\hat{E}^*_j(D_{ji})_a\,\delta\hat{E}_i + \frac{\partial}{\partial t}[\delta\hat{E}^*_j \frac{\partial}{\partial\omega_o}(\omega_o D_{ji})_h\delta\hat{E}_i]. \tag{I.2.20}$$

All together, phase averaging Eq. (I.2.11) leads to

$$\frac{\partial}{\partial t}\delta\hat{W} + \frac{1}{8\pi}\omega_o\,\delta\hat{\underset{\sim}{E}}^*\cdot\underset{\approx}{D}_a\cdot\delta\hat{\underset{\sim}{E}} = 0, \tag{I.2.21}$$

where

$$\delta\hat{W} = \frac{1}{16\pi}[|\delta\hat{\underset{\sim}{B}}|^2 + \delta\hat{\underset{\sim}{E}}^*\cdot\frac{\partial}{\partial\omega_o}(\omega_o\,\underset{\approx}{D}_h)\cdot\delta\hat{\underset{\sim}{E}}] \tag{I.2.22}$$

is the wave energy. We note that $\delta\hat{W}$ consists of three components: the magnetic field energy $|\delta\hat{\underset{\sim}{B}}|^2$, the electric field energy and coherent particle kinetic (mechanical) energy. The latter two contributions are combined into the second term in Eq. (I.2.22). This is clear by noting Eq. (I.1.6) (i.e., $\underset{\approx}{D} = \underset{\approx}{I} + i4\pi\underset{\approx}{\sigma}/\omega$). Thus,

$$\delta\hat{\underset{\sim}{E}}^* \cdot \frac{\partial}{\partial\omega_o}(\omega_o \underset{\approx}{D}_h)\cdot\delta\hat{\underset{\sim}{E}}^* = |\delta\hat{\underset{\sim}{E}}|^2 + \delta\hat{\underset{\sim}{E}}^* \cdot \frac{\partial}{\partial\omega_o}(i4\pi\underset{\approx}{\sigma})_h\cdot\delta\hat{\underset{\sim}{E}} . \tag{I.2.23}$$

Since $\delta\underset{\sim}{J} = \underset{\approx}{\sigma}\cdot\delta\underset{\sim}{E}$ , which is due to particle coherent dynamics, the meaning of particle kinetic (mechanical) energy is transparent.

Equation (I.2.21) clearly shows that in plasmas wave energy may either decrease or increase depending on the anti-Hermitian part of $\underset{\approx}{D}$. As a simple example, we take $\underset{\sim}{k}=0$ and $\underset{\approx}{D} = D_s \underset{\approx}{I}$ . We then have, noting $\delta\underset{\sim}{B}=0$,

$$\frac{1}{16\pi}\frac{\partial}{\partial t}\left[\frac{\partial}{\partial\omega_o}(\omega_o D_{sh})|\delta\hat{\underset{\sim}{E}}|^2\right] + \frac{1}{8\pi}\omega_o D_{sa}|\delta\hat{E}|^2 = 0. \tag{I.2.24}$$

Equation (I.2.24) has the solution

$$|\delta\hat{\underset{\sim}{E}}|^2 = |\delta\hat{\underset{\sim}{E}}_o|^2 \exp(2\gamma t) \tag{I.2.25}$$

with

$$\gamma/\omega_o = -D_{sa}/[\partial(\omega_o D_{sh})/\partial\omega_o]. \tag{I.2.26}$$

Noting also that in this case $D_{sh} = \mathrm{Re}(D_s) \equiv D_{sr}$ and $D_{sa} = D_{si}$. Thus, for waves with positive wave energy $\partial(\omega_o D_{sr})/\partial\omega_o > 0$, we would have wave growth (decay) if the dissipation, i.e., $\omega_o D_{si}$, is negative (positive). The converse is true for negative-energy waves. Finally, Eqs. (I.2.21) and (I.2.22) indicate that the consistency of the two-time-scale analysis requires the previously stated assumption

$$|\underset{\approx}{D}_a| \;/\; |\underset{\approx}{D}_h| \sim O\,(\varepsilon) \ll 1. \tag{I.2.27}$$

§I.3 Analysis with two time and space scales

In plasmas, spatial dependence plays a crucial role. In particular, it gives rise to the wave dispersion; i.e., $\omega$ depends on the wave vector $\underset{\sim}{k}$. As in the case of temporal variations, wave spatial variations also often exhibits two scales, one rapid and one slow. Such two-space-scale phenomena can occur due to various reasons; e.g., dissiation or source, propagation of wave packet, weak nonuniformity, etc.

Thus, in general, we would allow fast and slow variations in both time and space. $\delta\underset{\sim}{E}(\underset{\sim}{x},t)$ can then be expressed in the following form

$$\delta\underset{\sim}{E}(t,\underset{\sim}{x}) = \delta\underset{\sim}{E}(\varepsilon t,\varepsilon\underset{\sim}{x})\,\exp(i\underset{\sim}{k}_o \cdot \underset{\sim}{x} - i\omega_o t). \tag{I.3.1}$$

We can then carry out analyses similar to those done in Sec. §I.2. For example, given the following relation in $(\omega,\underset{\sim}{k})$ space

$$\delta\underset{\sim}{A}(\omega,\underset{\sim}{k}) = \underset{\approx}{B}\,(\omega,\underset{\sim}{k})\cdot\delta\underset{\sim}{E}(\omega,\underset{\sim}{k}) \quad ; \tag{I.3.2}$$

where $(\omega,\underset{\sim}{k})$ are the Laplace and Fourier transform variables, we can find, in the $(t,\underset{\sim}{x})$ space, that

$$\delta\hat{\underset{\sim}{A}}(\varepsilon t,\varepsilon\underset{\sim}{x}) = [\underset{\approx}{B}(\omega_o + i\,\frac{\partial}{\partial t},\,\underset{\sim}{k}_o - i\,\frac{\partial}{\partial \underset{\sim}{x}})]\cdot\delta\hat{\underset{\sim}{E}}\,(\varepsilon t,\,\varepsilon\underset{\sim}{x}). \tag{I.3.3}$$

Expanding Eq. (I.3.3) to first order, we obtain

$$\delta\hat{A}_i(\epsilon t,\epsilon \underset{\sim}{x}) \cong [B_{ij}(\omega_o, k_{o\ell}) + \frac{\partial B_{ij}(\omega_o, k_{o\ell})}{\partial \omega_o} (i \frac{\partial}{\partial t})$$

$$+ \frac{\partial B_{ij}(\omega_o, k_{o\ell})}{\partial k_{om}} (- i \frac{\partial}{\partial x_m})] \delta\hat{E}_j(\epsilon t, \epsilon \underset{\sim}{x}). \tag{I.3.4}$$

Here, we have used indicial tensor representation to clarify the operations. We can of course carry out a similar phase averaging both in time and space. This is straightforward and we shall not discuss it.

In stead, we shall substitute Eqs. (I.3.1) and (I.3.4) into Maxwell's equations, Eqs. (I.1.5) and (I.1.6), and examine the results order by order. For this purpose, we note that

$$\underset{\approx}{D} = \underset{\approx}{D}_h + i\, \underset{\approx}{D}_a \tag{I.3.5}$$

and assume, more specifically,

$$|\delta\hat{\underset{\sim}{E}}^* \cdot [\partial(\omega_o \underset{\approx}{D}_a/\partial\omega_o] \cdot \delta\hat{\underset{\sim}{E}}| / |\delta\hat{\underset{\sim}{E}}^* \cdot [\partial(\omega_o \underset{\approx}{D}_h)/\partial\omega_o] \cdot \delta\hat{\underset{\sim}{E}}| \sim O(\epsilon) << 1. \tag{I.2.27)'$$

In the zeroth order, O(1), we obtain the expected result

$$\underset{\approx}{\epsilon}_h(\omega_o, \underset{\sim}{k}_o) \cdot \delta\hat{\underset{\sim}{E}} = [\underset{\sim}{k}_o \underset{\sim}{k}_o - k_o^2 \underset{\approx}{I} + (\omega_o/c)^2 \underset{\approx}{D}_h] \cdot \delta\hat{\underset{\sim}{E}} = 0. \tag{I.3.6}$$

Equation (I.3.6) gives the zeroth-order dispersion relation

$$\epsilon_h(\omega_o, \underset{\sim}{k}_o) \equiv \| \underset{\approx}{\epsilon}_h (\omega_o, \underset{\sim}{k}_o) \| = 0. \tag{I.3.7}$$

An alternative expression for the dispersion relation can be obtained by dotting $\delta\hat{\underset{\sim}{E}}^*$ into Eq. (I.3.6) to obtain

$$|\underset{\sim}{k}_o \times \delta\hat{\underset{\sim}{E}}|^2 = (\omega_o/c)^2 \; \delta\hat{\underset{\sim}{E}}^* \cdot \underset{\approx}{D}_h \cdot \delta\hat{\underset{\sim}{E}}. \tag{I.3.8}$$

In the next order, $O(\varepsilon)$, we have

$$c \frac{\partial}{\partial \underset{\sim}{x}} \times \delta\hat{\underset{\sim}{E}} = - \frac{\partial}{\partial t} \delta\hat{\underset{\sim}{B}} \tag{I.3.9}$$

$$c \frac{\partial}{\partial \underset{\sim}{x}} \times \delta\hat{\underset{\sim}{B}} = \omega_o \underset{\approx}{D}_a \cdot \delta\hat{\underset{\sim}{E}} + \frac{\partial}{\partial \omega_o} (\omega_o \underset{\approx}{D}_h) \frac{\partial}{\partial t} \delta\hat{\underset{\sim}{E}} - \omega_o \frac{\partial}{\partial \underset{\sim}{k}_o} \underset{\approx}{D}_h \cdot \frac{\partial}{\partial \underset{\sim}{x}} \delta\hat{\underset{\sim}{E}} \;. \tag{I. 3.10}$$

Now we perform the following operations

$$\delta\hat{\underset{\sim}{E}}^* \cdot \text{ Eq. (I.3.10)} + \delta\hat{\underset{\sim}{E}} \cdot \text{ Eq. (I.3.10)}^* - \delta\hat{\underset{\sim}{B}}^* \cdot \text{ Eq. (I.3.9)} - \delta\hat{\underset{\sim}{B}} \cdot \text{ Eq.}^* \text{ (I.3.9)}. \tag{I.3.11}$$

We obtain, noting that $\underset{\approx}{D}(-\omega_o, -\underset{\sim}{k}_o) = \underset{\approx}{D}^*(\omega_o, \underset{\sim}{k}_o)$ ,

$$\frac{\partial}{\partial t} [|\delta\hat{\underset{\sim}{B}}|^2 + \delta E^* \frac{\partial}{\partial \omega_o} (\underset{\approx}{D}_h \omega_o) \cdot \delta\hat{\underset{\sim}{E}}] + 2\omega_o \delta\hat{\underset{\sim}{E}}^* \cdot \underset{\approx}{D}_a \cdot \delta\hat{\underset{\sim}{E}} - \omega_o \frac{\partial}{\partial \underset{\sim}{x}} \cdot \frac{\partial}{\partial \underset{\sim}{k}_o} (\delta\hat{\underset{\sim}{E}}^* \cdot \underset{\approx}{D}_h \cdot \delta\hat{\underset{\sim}{E}})$$

$$= - c \frac{\partial}{\partial \underset{\sim}{x}} \cdot [\delta\hat{\underset{\sim}{E}}^* \times \delta\hat{\underset{\sim}{B}} + \delta\hat{\underset{\sim}{E}} \times \delta\hat{\underset{\sim}{B}}^*]. \tag{I.3.12}$$

From Eq. (I.3.8), we note that

$$|\delta\hat{\underset{\sim}{B}}|^2 = (\frac{c}{\omega_o})^2 \; |\underset{\sim}{k}_o \times \delta\hat{\underset{\sim}{E}}|^2 = \delta\hat{\underset{\sim}{E}}^* \cdot \underset{\approx}{D}_h \cdot \delta\hat{\underset{\sim}{E}}. \tag{I.3.13}$$

Equation (I.3.12) divided by $16\pi$ can then be written as

$$\frac{\partial}{\partial t}\,\delta W - \frac{\omega_o}{16\pi}\frac{\partial}{\partial \underset{\sim}{x}} \cdot \frac{\partial}{\partial \underset{\sim}{k}_o}\,(\delta \hat{\underset{\sim}{E}}^{*}\cdot \underset{\approx}{D}_h \cdot \delta \hat{\underset{\sim}{E}}) + \frac{c}{8\pi}\frac{\partial}{\partial \underset{\sim}{x}} \cdot R_e(\delta \hat{\underset{\sim}{E}}^{*} \times \delta \hat{\underset{\sim}{B}}) +$$

$$\frac{\omega_o}{8\pi}\,\delta \hat{\underset{\sim}{E}}^{*}\cdot \underset{\approx}{D}_a \cdot \delta \hat{\underset{\sim}{E}} = 0, \qquad (I.3.14)$$

where

$$\delta W \equiv \frac{1}{16\pi}\,\delta \hat{\underset{\sim}{E}}^{*} \cdot \frac{\partial}{\partial \omega_o}\,(\omega_o{}^2\, \underset{\approx}{D}_h)\cdot \delta \hat{\underset{\sim}{E}}/\omega_o \;. \qquad (I.3.15)$$

Now, let us define

$$\underset{\sim}{V}_{gr} \equiv - [\; \omega_o \frac{\partial}{\partial \underset{\sim}{k}_o}\,(\delta \hat{\underset{\sim}{E}}^{*}\cdot \underset{\approx}{D}_h \cdot \delta \hat{\underset{\sim}{E}}) - 2c\; Re(\delta \hat{\underset{\sim}{E}}^{*} \times \delta \hat{\underset{\sim}{B}})]/16\pi\delta W$$

$$= - \left\{2c^2[\underset{\sim}{k}_o|\delta \hat{\underset{\sim}{E}}|^2 - Re\;(k_o\cdot \delta \hat{\underset{\sim}{E}}^{*})\delta \hat{\underset{\sim}{E}}] - \omega_o{}^2\,\frac{\partial}{\partial \underset{\sim}{k}_o}\,(\delta \hat{\underset{\sim}{E}}^{*}\cdot \underset{\approx}{D}_h \cdot \delta \hat{\underset{\sim}{E}})\right\}/\left\{\right.$$

$$\left.\frac{\partial}{\partial \omega_o}\,(\omega_o{}^2 \delta \hat{\underset{\sim}{E}}^{*}\cdot \underset{\approx}{D}_h \cdot \delta \hat{\underset{\sim}{E}})\right\}. \qquad (I.3.16)$$

Equation (I.3.14) then becomes the familiar form

$$\frac{\partial}{\partial t}\,\delta W + \frac{\partial}{\partial \underset{\sim}{x}} \cdot (\underset{\sim}{V}_{gr}\;\delta W) = 2\omega_i\;\delta W, \qquad (I.3.17)$$

where

$$\omega_i \equiv - [\frac{\omega_o}{16\pi}\,\delta \hat{\underset{\sim}{E}}^{*}\cdot \underset{\approx}{D}_a \cdot \delta \hat{\underset{\sim}{E}}]/\delta W. \qquad (I.3.18)$$

$\underset{\sim}{V}_{gr}$ as defined by Eq. (I.3.16) can be interpreted as

$$\underset{\sim}{V}_{gr} = \text{total wave energy flux/wave energy}. \qquad (I.3.19)$$

Meanwhile, since

$$\underset{\sim}{V}_{gr} = \partial\omega_o/\partial\underset{\sim}{k}_o, \tag{I.3.20}$$

it shows that the wave energy is carried by the wave packet or, sometimes called, quasi particle which propagates at the velocity $\underset{\sim}{V}_{gr}$. If $\omega_i = 0$; i.e. *the plasma is Hermitian, then the wave energy remains constant along the* propagation path of the wave packet. With $\omega_i \neq 0$ , the wave energy can either increase or decrease along the path.

Homework #2

(1) Using the physical realizability condition, prove that

$$\underset{\approx}{D}\ (-\omega) = \underset{\approx}{D}^*\ (\omega^*) \quad . \tag{H.2.1}$$

(2) The cold plasma Langmuir oscillation with collisional dissaption is governed by the following equations

$$\frac{\partial}{\partial t}\ \delta\underset{\sim}{V} = -\ \frac{e}{m}\ \delta\underset{\sim}{E}\ -\ \nu\delta\underset{\sim}{V} \tag{H.2.2}$$

$$\frac{\partial}{\partial t}\ \delta\underset{\sim}{E} = 4\pi N_o e\ \delta\underset{\sim}{V}\ ; \tag{H.2.3}$$

or, combining Eqs. (H.2.2) and (H.2.3)

$$\frac{\partial^2}{\partial t^2}\ \delta\underset{\sim}{V} = -\ \omega_{pe}{}^2\ \delta\underset{\sim}{V}\ -\ \nu\ \frac{\partial}{\partial t}\ \delta\underset{\sim}{V} \quad . \tag{H.2.4}$$

Here, $\omega_{pe}^2 = 4\pi N_o e^2/m_e$ and we shall assume $|\nu| << |\omega_{pe}|$.

(2.a) Solving Eq. (H.2.4) for $\delta \underset{\sim}{V}$ using the two-time scale approach.

(2.b) Derive $\underset{\approx}{D}$ and indicate the Hermitian and anti-Hermitian parts.

(2.c) Derive an expression for the wave energy $\delta W$.

(2.d) Calculate the damping or growth rate using $\delta W$ and $\underset{\approx}{D}_a$.

(3) Using the disperison relation given by Eq. (I.3.8), prove that the expression of $\underset{\sim}{V}_{gr}$ given by Eq. (I.3.16) agrees with the more familiar one, $\underset{\sim}{V}_{gr} = \partial\omega_o/\partial\underset{\sim}{k}_o$.

## §I.4 Instabilities

Plasmas in both laboratories and space are often far from thermal equilibrium. For example, in mirror machines as well as earth's magnetosphere, the velocity distribution is non-Maxwellian due to the existence of loss cones. Furthermore, in any confined plasmas either inertially or magnetically, there is also nonuniformities in macroscopic thermodynamic quantities such as density, temperature, pressure, etc. In a way, these deviations from thermodynamic equilibrium may be regarded as free energies stored in the plasma. Conventionally, they may be catagorized in the following three types:

(i) Velocity-space free energy: velocity-space anisotropy, beams. Examples are loss-cone and two-stream instabilities.

(ii) Magnetic free energy: current and magnetic field inhomogeneities. Examples are kink and tearing instabilities.

(iii) Expansion free energy: density, temperature, pressure inhomogeneities.
Examples are drift, ballooning/interchange and trapped-particle instabilities.

Instabilities are collective processes which may be triggered (if certain threshold conditions are satisfied) to release the above-mentioned free energies. That is, via the instabilities, these free energies can be converted into either turbulent plasma motions and/or electromagnetic radiations. In this respect, instabilities may be viewed as anomalous (compared to collisional) processes whereby plasmas can relax toward thermal equilibrium and, therefore, instabilities play crucial roles in our understanding of important subjects in both laboratory and space plasmas; such as disruption in tokamaks, anomalous transport processes, collisionless shocks, solar radio bursts, etc.

In an infinite, uniform plasma, we say an instability exists if and only if for some <u>real</u> $\underset{\sim}{k}$, the dispersion relation

$$\varepsilon(\omega,\underset{\sim}{k}) \equiv |\underset{\sim}{\varepsilon}(\omega,\underset{\sim}{k})| = 0 \tag{I.4.1}$$

admits a solution with $\text{Im}\ \omega \equiv \gamma > 0$. Here, we note our convention is

$$\delta\underset{\sim}{E}(\underset{\sim}{x},t) = \delta\hat{\underset{\sim}{E}} \exp(i\underset{\sim}{k} \cdot \underset{\sim}{x} - i\omega t)$$

with $(\omega,\underset{\sim}{k})$ properly understood to be the Laplace-in-t and Fourier-in-$\underset{\sim}{x}$ transform pair. That is, if a linear instability exists, a wave, which is <u>periodic</u> in space with a wave vector $\underset{\sim}{k}$, will exponentiate in time with the growth rate given by $\gamma$.

Based on the instability excitation mechanisms, we may divide instabilities into two types: (i) dissipative-type instabilities and (ii) reactive-type instabilities.

Dissipative-type instabilities are excited via (negative or positive) dissipative processes or, more generally speaking, the anti-Hermitian component of the dielectric tensor; as we have discussed in Sec. §I.2 and Sec. §I.3. Thus, we assume

$$\varepsilon(\omega,\underset{\sim}{k}) = \varepsilon_h(\omega,\underset{\sim}{k}) + i\varepsilon_a(\omega,\underset{\sim}{k}) = 0, \tag{I.4.2}$$

and

$$|\varepsilon_a|/|\varepsilon_h| \sim O(\varepsilon) << 1. \tag{I.4.3}$$

Furthermore, we let

$$\omega = \omega_r + i\gamma \text{ and } |\gamma/\omega_r| \sim O(\varepsilon) << 1.$$

Equation (I.4.2) then becomes, order by order, O(1):

$$\varepsilon_h(\omega_r,\underset{\sim}{k}) = 0 \tag{I.4.4}$$

or

$$\omega_r = \omega_{rj}(\underset{\sim}{k}) \qquad \text{for } j = 1, \ldots \ , N \ , \tag{I.4.5}$$

where N is the number of $\omega$ branches (i.e., normal modes). In the next order, we have, from Taylor expansion of $\varepsilon_h$ about $\omega = \omega_{rj}$ ,

$$\gamma_j = - \varepsilon_a(\omega_{rj},\underset{\sim}{k})/[\frac{\partial}{\partial\omega_{rj}} \varepsilon_h(\omega_{rj},\underset{\sim}{k})] \ . \tag{I.4.6}$$

Thus, for plasmas with negative dissipations ($\varepsilon_a < 0$), $\gamma_j > 0$ if $\partial\varepsilon_h/\partial\omega_{rj} > 0$; i.e., if the ($\omega_{rj}$,$\underset{\sim}{k}$) wave is a positive-energy wave. For a negative-energy wave ($\partial\varepsilon_h/\partial\omega_{rj} < 0$), however, we have $\gamma_j < 0$; the wave is stable or damped. Converse conclusions can be made for positive dissipations. In Homework #3, we shall have an example of a negative-energy wave driven unstable via positive dissipations.

As to reactive-type instabilities, the most well-known example is the beam-plasma instability. First, however, we shall illustrate with the following model dispersion relation

$$\varepsilon_h(\omega,k) = 1 - \frac{\omega_p}{\omega} + \frac{\omega_b}{(\omega-kv_o)} = 0. \qquad (I.4.7)$$

Equation (I.4.7) is a quadratic equation in $\omega$ and, hence, can be readily solved. For simplicity, we shall assume $|\omega_b| << |\omega_p|$. Thus,

$$\omega_{1,2} \simeq \frac{(kv_o+\omega_p-\omega_b) \pm [(kv_o-\omega_p)^2 - 2\omega_b(kv_o+\omega_p)]^{1/2}}{2} \quad . \qquad (I.4.8)$$

Thus, there exists an instability if

$$|kv_o - \omega_p| \lesssim 2(\omega_p\omega_b)^{1/2}, \qquad (I.4.9)$$

and the maximum growth rate $\gamma_{max} \simeq (\omega_p\omega_b)^{1/2}$ occurs at $kv_o = \omega_p$. These results are sketched in terms of the $\omega$-k dispersion curves shown in Fig. (I.4.1).

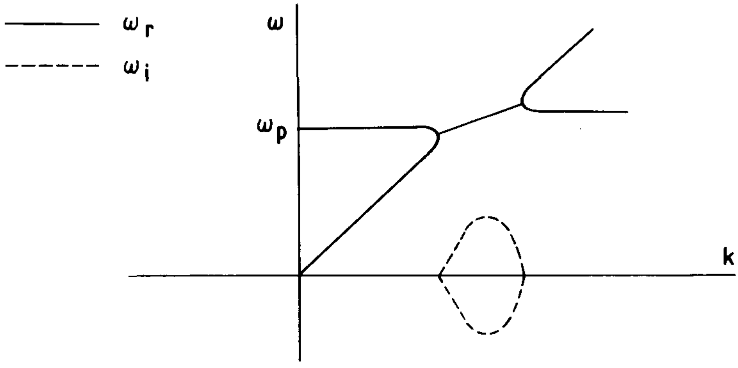


Fig. (I.4.1) Sketch of ω vs. k curves for the model dispersion relation of Eq. (I.4.7).

Since instability occurs at $\omega_{pe} \approx kv_o$, this model reactive-type instability can be understood in terms of coupling between the $\omega_1 \simeq \omega_{pe}$ and $\omega_2 \approx kv_o$ modes. Now, away from coupling, the corresponding dielectric constant for $\omega_1$ and $\omega_2$ modes are, respectively,

$$\varepsilon_{h1} = 1 - \omega_p/\omega_1 , \qquad (I.4.10)$$

and

$$\varepsilon_{h2} = 1 + \omega_b/(\omega_2 - kv_o) . \qquad (I.4.11)$$

since

$$\partial(\omega_1\varepsilon_{h1})/\partial\omega_1 = 1 > 0 , \qquad (I.4.12)$$

the $\omega_1$ mode is a positive-energy wave. On the other hand,

$$\partial(\omega_2\epsilon_{h2})/\partial\omega_2 \simeq 1 - \frac{kv_o\omega_b}{(\omega_2-kv_o)^2} \simeq -\frac{kv_o}{\omega_b} < 0 \ , \qquad (I.4.13)$$

i.e., the $\omega_2$ mode is a negative-energy wave. Thus, the above model reactive-type instability can be explained in terms of coupling between a negative-energy and a positive-energy wave. It is easy to see that if both modes are positive-energy waves, then the system is stable. For example, the following model dispersion relation

$$\epsilon_h(\omega,k) = 1 - \frac{\omega_p}{\omega} - \frac{\omega_b}{(\omega-kv_o)} = 0 \qquad (I.4.14)$$

has no unstable solutions. The $\omega$-k dispersion curve is given in Fig. (I.4.2).

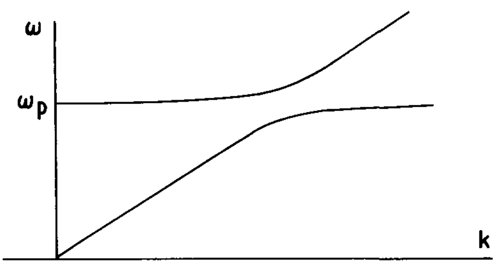


Fig. (I.4.2) Sketch of $\omega$ vs. k curves for the model dispersion relation of Eq. (I.4.14).

Similarly, we can show that the coupling between two negative-energy waves introduces no instability.

We now briefly examine the beam-plasma instability. Here, we let a cold electron beam streaming through a background cold plasma. Let the beam velocity be $v_b$ and density $n_b$. For the background plasma, we have density $n_o$ and $m_i \to \infty$, i.e., immobile ions. For simplicity of analysis, we assume the beam is weak; $n_b \ll n_o$. The corresponding electrostatic dispersion relation can then be shown to be

$$\varepsilon_h = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pb}^2}{(\omega - kv_b)^2} = 0. \qquad (I.4.15)$$

Equation (I.4.15) can also be written as

$$\omega_1^2 = \omega_{pe}^2/[1 - \omega_{pb}^2/(\omega_1 - kv_b)^2] , \qquad (I.4.16)$$

and

$$(\omega_2 - kv_b)^2 = \omega_{pb}^2/(1 - \omega_{pe}^2/\omega_2^2) . \qquad (I.4.17)$$

For $|kv_b| < \omega_{pe}$ , we have

$$\omega_1 \approx \pm\, \omega_{pe}/[1 - \omega_{pb}^2/\omega_{pe}^2]^{1/2} \simeq \pm\, \omega_{pe} , \qquad (I.4.18)$$

and

$$\omega_2 = kv_b \pm i(\omega_{pb}/\omega_{pe})kv_b . \qquad (I.4.19)$$

Meanwhile, for $|kv_b| > \omega_{pe}$ , we have

$$\omega_1 \simeq \pm\, \omega_{pe}/[1 - \omega_{pb}^2/k^2v_b^2]^{1/2} \simeq \pm\, \omega_{pe} , \qquad (I.4.20)$$

and

$$\omega_2 \simeq kv_b \pm \omega_{pb} . \qquad (I.4.21)$$

The corresponding ω-k dispersion curves are sketched in Fig. (I.4.3).

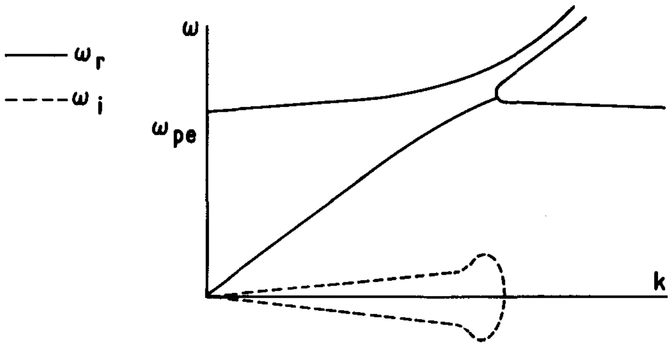


Fig. (I.4.3) Sketch of $\omega$ vs. k cruves for the beam-plasma instability dispersion relation of Eq. (I.4.15).

By the analogy with the model problem, we can understand this instability in terms of the coupling between the positive-energy wave with $\omega \simeq \omega_{pe}$ and the negative-energy wave with $\omega \approx kv_b - \omega_{pb}$. This is clear by noting the respective dielectric constants are

$$\varepsilon_{h1} = 1 - \omega_{pe}^2/\omega_1^2 \quad , \tag{I.4.22}$$

and

$$\varepsilon_{h2} = 1 - \omega_{pb}^2/(\omega_2 - kv_b)^2 \; . \tag{I.4.23}$$

§I.5 Nyquist technique for stability analysis

In many practical applications, the dispersion relation $\varepsilon(\omega,\underset{\sim}{k}) = 0$ is generally too complicated to solve analytically for instability growth rates. In fact, the first thing one would like to know is if the dispersion relation admits solutions of $\omega$ with Im $\omega > 0$. For this purpose, there exists the powerful Nyquist technique.

Let us assume that $\varepsilon$ be analytic in the Im $\omega > 0$ half plane and that it has a finite number of zeroes at $\omega = \omega_m$ for $m = 1,\ldots,N$. That is, $\omega_m$'s are the unstable solutions. We now construct the following function

$$G(\omega) = \frac{1}{\varepsilon(\omega)} \frac{\partial \varepsilon}{\partial \omega} \quad . \tag{I.5.1}$$

It is clear that $G(\omega)$ has poles at $\omega = \omega_m$ . We now define a contour c in the $\mathrm{Im}\,\omega > 0$ plane to be $c = c_1 + c_2$ where $c_1$ lies above the real $\omega$ axis and $c_2$ is an infinite semicircle [c.f., Fig. (I.5.1)].

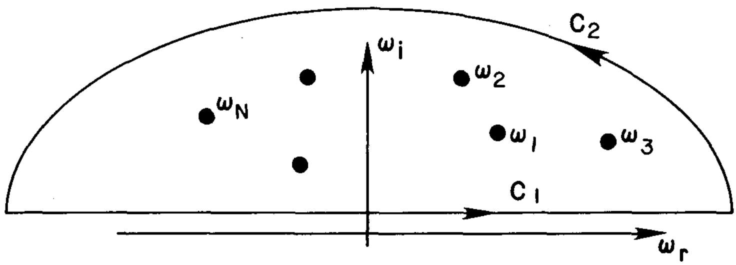


Fig. (I.5.1) Integration contour in the complex ω plane for the Nyquist stability analysis.

From Cauchy theorem, we find

$$\frac{1}{2\pi i} \int_c G(\omega) d\omega = \sum_{m=1}^{N} \mathrm{Res}\, G(\omega = \omega_m) = \sum_{m=1}^{N} \mathrm{Res}\, \left((1/\varepsilon)(\partial\varepsilon/\partial\omega)\right)_{\omega=\omega_m} \quad . \tag{I.5.2}$$

Now, near $\omega = \omega_m$ , we have

$$\varepsilon(\omega) \simeq d_1(\omega-\omega_m)^{P_m} + d_2(\omega-\omega_m)^{P_m+1} + \ldots \quad , \tag{I.5.3}$$

or

$$G(\omega) \simeq \frac{P_m}{(\omega-\omega_m)} \quad . \tag{I.5.4}$$

Equation (I.5.2) then reduces to

$$\frac{1}{2\pi i}\int_c G(\omega)d\omega = \sum_{m=1}^{N} P_m \ . \tag{I.5.5}$$

Equation (I.5.5) shows that given $\varepsilon(\omega,\underset{\sim}{k})$ being analytic in the Im $\omega > 0$ plane, there then exists instabilities if

$$\frac{1}{2\pi i}\int_c \left(\frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial\omega}\right)d\omega = M = \text{a positive integer.} \tag{I.5.6}$$

This is the, sometimes, called Nyquist stability theorem.

We now further explore Eq. (I.5.6). First, we note that

$$\frac{1}{2\pi i}\int_c \left(\frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial\omega}\right)d\omega = \frac{1}{2\pi i}\ \ell n\left[\frac{\varepsilon(c_e)}{\varepsilon(c_s)}\right] \ . \tag{I.5.7}$$

Now, let

$$\varepsilon(c_e) = \varepsilon(c_s)\exp(i2\pi M) \quad , \tag{I.5.8}$$

we, again, recover the result of Eq. (I.5.6). Here, however, M corresponds to the number of times that the $\varepsilon(\omega)$ curve when mapped along the contour c encircles the origin of the $\varepsilon$ plane in the counter-clockwise direction. Thus, we reduce the stability analysis to the problem of mapping $\varepsilon(\omega)$ along the c contour.

Let us illustrate this technique with the beam-plasma instability discussed in Sec. §I.4. The corresponding linear dispersion relation is

$$\varepsilon_h(\omega) = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{(\omega-kv_b)^2} = 0 \ , \tag{I.4.15}$$

with $|\omega_p| \gg |\omega_b|$ assumed for simplicity. First, we consider the $kv_b < \omega_p$ limit which we know is unstable. Now since $\varepsilon_h$ has poles at $\omega = 0$ and $\omega = kv_b$. The $c_1$ contour needs to be deformed about the poles as shown in Fig. (I.5.2).

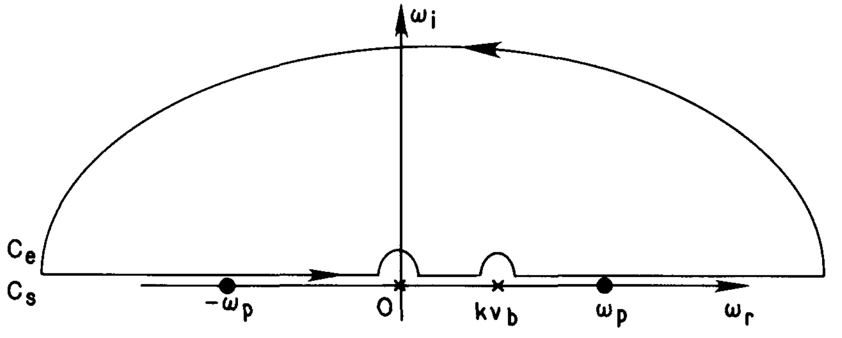


Fig. (I.5.2) ω-plane contour for the Nyquist analysis of the beam-plasma instability dispersion relation of Eq. (I.4.15) in the $kv_b < \omega_p$ limit.

In Fig. (I.5.2), the radius of the two (small) semicircles about the poles, $\delta$, is taken to be infinitesimally small but finite. The mapping of $\varepsilon_h(\omega)$ along $c_\omega = c_1 + c_2$ is shown in Fig. (I.5.3).

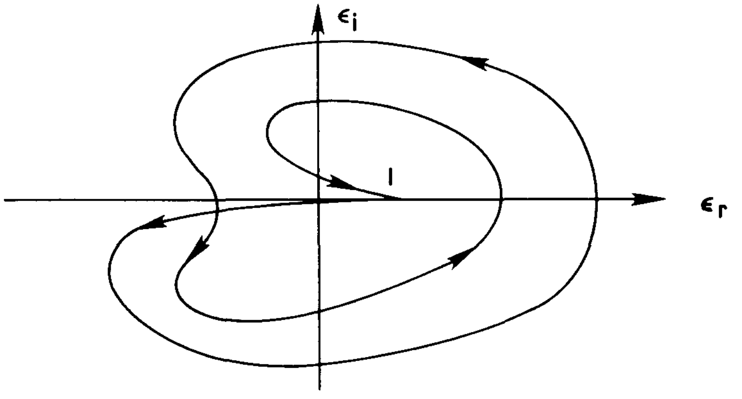


Fig. (I.5.3) Mapping of $\varepsilon_h$ given by Eq. (I.4.15) along the ω-plane contour shown in Fig. (I.5.2).

Thus, $\varepsilon_h(\omega)$ along $c_\omega$ encircles the origin once in the counter-clockwise direction. That is, M = 1 and we have one unstable solution. This, of course, agrees with the results obtained in Sec. §I.4. Next, we consider the $k_o v_b > \omega_p$ limit, which we know is stable. The corresponding $c_\omega$ contour is given in Fig. (I.5.4).

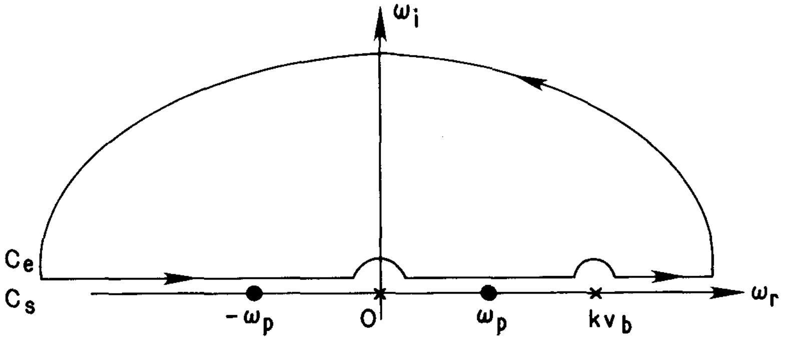


Fig. (I.5.4) ω-plane contour for the Nyquist analysis of the beam-plasma instability dispersion relation of Eq. (I.4.15) in the $kv_b > \omega_p$ limit.

The resultant mapping of $\varepsilon_h(\omega)$ is shown in Fig. (I.5.5).

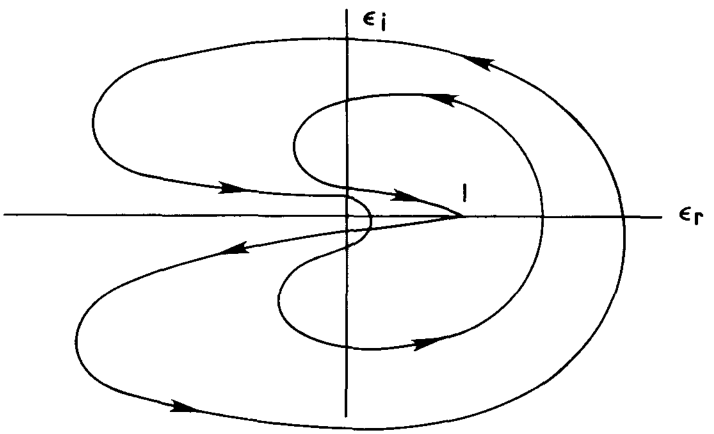


Fig. (I.5.5) Mapping of $\varepsilon_h$ given by Eq. (I.4.15) along the ω-plane contour shown in Fig. (I.5.4).

Thus, in this case $\varepsilon_h(\omega)$ mapped along $c_\omega$ does not encircle the origin and we have M = 0; i.e., we have no instability as it should.

Now the Nyquist technique for stability analyses is not limited to simple dispersion relation. As an example, we shall apply it to a stability analysis governed by a differential equation. This application, which is relevant to eigenmode stability problems to be discussed later, is presented here to further demonstrate the Nyquist technique. Let us consider the following model differential equation.

$$\left[\frac{d^2}{dx^2} + \omega^2 + i\omega\nu - \omega_o^2 - x^2\right]\delta\phi(x) = 0 \ , \tag{I.5.9}$$

where $\nu > 0$ and the boundary conditions are

$$|\delta\phi(x)| \to 0 \quad \text{as} \quad |x| \to \infty \ . \tag{I.5.10}$$

As a physical motivation, Eq. (I.5.9) may be regarded as modelling the wave equation for Bohm - Gross (warm electron plasma) waves in a Gaussian density cavity. In order to apply the Nyquist technique, we need to obtain an equivalent dispersion relation in $\omega$. For that purpose, we apply the operation $\int_{-\infty}^{\infty} dx\ \delta\phi^*(x)$ to Eq. (I.5.9). Noting Eq. (I.5.10), we find

$$\varepsilon(\omega) \equiv \omega^2 + i\ \omega\nu - \omega_o^2 - [\langle|x\delta\phi|^2\rangle + \langle|d\delta\phi/dx|^2\rangle]/\langle|\delta\phi|^2\rangle = 0, \tag{I.5.11}$$

where

$$\langle A\rangle \equiv \int_{-\infty}^{\infty} dx\ A. \tag{I.5.12}$$

Now since $\varepsilon(\omega)$ diverges as $|\omega| \to \infty$, we shall choose the radius of the semi-circle contour, R, to be arbitrary large but finite [refer to Fig. (I.5.6)].

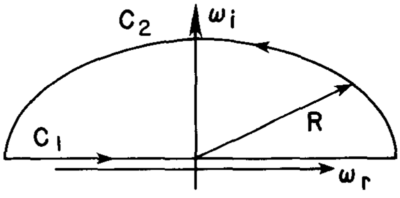


Fig. (I.5.6) ω-plane contour for the Nyquist analysis of the "dispersion relation" of Eq. (I.5.11).

The mapping of $\varepsilon(\omega)$ along $c_\omega$ is shown in Fig. (I.5.7); from which we conclude that Eq. (I.5.9) predicts no instability.

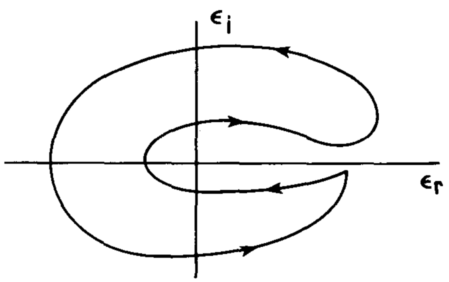


Fig. (I.5.7) Mapping of ε given by Eq. (I.5.11) along the ω-plane contour shown in Fig. (I.5.6).

In fact, we may choose $c_1$ to be infinitesimally below the $\omega_r$ axis (i.e., Im $c_1 \to 0^-$) and prove that no marginally stable solution exists either. Finally, we remark that, while in this model example the above conclusions can also be obtained by either noting that Eq. (I.5.9) is a Weber equation or solving Eq. (I.5.11) which is a quadratic equation in $\omega$, the Nyquist technique is more powerful and applicable to a broader range of problems.

Homework # 3

(1) The following dispersion relation describes beam-plasma interaction including collisional effects on the background electrons.

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega(\omega+i\nu)} - \frac{\omega_b^2}{(\omega-kv_b)^2} = 0 \text{ with } \nu > 0. \tag{H.3.1}$$

Show that in the $|kv_b| >> \omega_p$ limit, which is stable in the reactive limit (i.e., $\nu = 0$), the negative-energy wave, (i.e., the slow beam mode) is unstable due to the positive collisional dissipation.

(2) Consider electrons streaming through background immobile ions with velocity $v_b$. The corresponding electrostatic dispersion relation is

$$\varepsilon = 1 - \frac{\omega_b^2}{(\omega-kv_b)^2} = 0 \tag{H.3.2}$$

Suppose there is an electric field of the plane-wave form

$$\delta \underset{\sim}{E}(x,t) = \frac{1}{2}\left[\delta\hat{E}\exp(-i\omega t + ikx) + c.c.\right],$$

where $\delta\hat{E}$ is independent of t and x. Calculate the phase-averaged particle kinetic energy density for both the slow, $\omega = kv_b - \omega_b$, and the fast modes.

(3) Using Nyquist technique to prove that, as discussed in Sec. §I.4, the model dispersion relation

$$\varepsilon = 1 - \frac{\omega_p}{\omega} + \frac{\omega_b}{\omega-kv_o} = 0 \tag{I.4.7}$$

predicts

(3.a) stability for $kv_o \ll \omega_p$

(3.b) stability for $kv_o \gg \omega_p$

(3.c) instability for $kv_o = \omega_p$.

## §I.6 Absolute and convective instabilities

[Ref. Ch. 2 in R. J. Brigg's "Electron Stream Interaction with Plasmas" M.I.T. Press.]

Up to now, our definition of instabilities (Sec. §I.4) is based on perturbations which have plane-wave spatial dependence; i.e., $\exp(i\underset{\sim}{k}\cdot\underset{\sim}{x})$. In other words, the perturbations are taken to be periodic in $\underset{\sim}{x}$ and, hence, have infinite spatial extent. In reality, however, the perturbations are generally of finite spatial extent. Treating as an initial-value problem, there then exists two types of time-asymptotic ($t \to \infty$) behaviors at a fixed finite spatial point. One is the absolute instability characterizing by perturbations which become unbounded as $t \to \infty$ at every fixed finite point in space. The other is the convective instability characterizing by perturbations which propagate and grow along the system such that, at any fixed finite point in space, the perturbations vanish time-asymptotically. A sketch of these two-types of time-asymptotic behaviors is shown in Fig. (I.6.1).

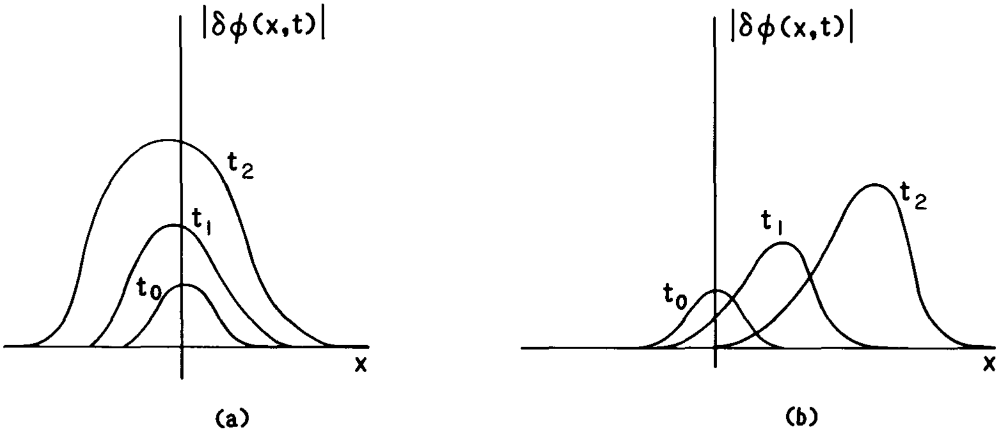


Fig. (I.6.1) Sketches of time-asymptotic perturbations of (a) an absolute instability where $|\delta\phi(x,t)| \to \infty$ as $t \to \infty$ and (b) a convective instability where $|\delta\phi(x,t)| \to 0$ as $t \to \infty$.

We emphasize that the differentiation of these two types of instability is not simply an academic exercise but carries important practical implications. This is because (i) instabilities are grown out of low-level thermal fluctuations and (ii) the unstable region is, in practice, of a limited spatial extent. For convective instabilities, the perturbations are spatially amplified inside the unstable region. Thus, their amplitudes as well as their nonlinear consequences can be estimated. For example, if the unstable region is small, the instabilities may be practically insignificant. On the other hand, absolute instabilities can grow to large amplitudes such that their saturation can only be due to nonlinear effects.

In terms of operators, let us assume the perturbation, $\delta\phi(t,x)$ is governed by the following wave equation

$$\varepsilon(i \frac{\partial}{\partial t} , -i \frac{\partial}{\partial x})\delta\phi(t,x) = 0, \qquad (I.6.1)$$

subject to an initial perturbation, $\delta\phi(t=0,x)$. Here, we reiterate that the system is linearly unstable; i.e., the linear dispersion relation

$$\varepsilon(\omega,k) = 0 \ , \qquad (I.6.2)$$

has unstable roots, Im $\omega > 0$, for real k. We, furthermore, consider the one-dimensional case only. Generalization to higher dimensionalities is straightforward.

Laplace transforming Eq. (I.6.1) in time, we obtain

$$\varepsilon(\omega, -i \frac{\partial}{\partial x})\delta\phi(\omega,x) = s(x), \qquad (I.6.2)$$

where s(x) represents the effective initial perturbations. Let us denote the Green's function response be $\delta G(\omega,x)$ such that

$$\varepsilon(\omega, -i \frac{\partial}{\partial x})\delta G(\omega,x) = \delta(x) \quad . \qquad (I.6.3)$$

We then have

$$\delta\phi(\omega,x) = \int dx's(x')\delta G(\omega,x-x'). \qquad (I.6.4)$$

Equation (I.6.4) shows that it is sufficient to examine Eq. (I.6.3) in order to understand the time-asymptotic behaviors.

Formulating in terms of the Green's function, we have

$$\delta G(t,x) = \int_{C_\omega} \frac{d\omega}{2\pi} \delta G(\omega,x) \ e^{-i\omega t}, \qquad (I.6.5)$$

and

$$\delta G(\omega,x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{\varepsilon(\omega,k)} e^{ikx} . \tag{I.6.6}$$

Here, the $\omega$ integration contour, $c_\omega$, must lie above the zeroes of $\varepsilon(\omega,k)$ for a given real k in order to ensure the causality condition is satisfied. Furthermore, as being physically resonable, we assume that $\varepsilon(\omega,k)$ is sufficiently well-behaved as $|\omega|$ or $|k|$ approaches infinity. The $\omega$ and k integration contours for, respectively, $t > 0$ and $x > 0$ can then be closed as shown in Fig. (I.6.2).

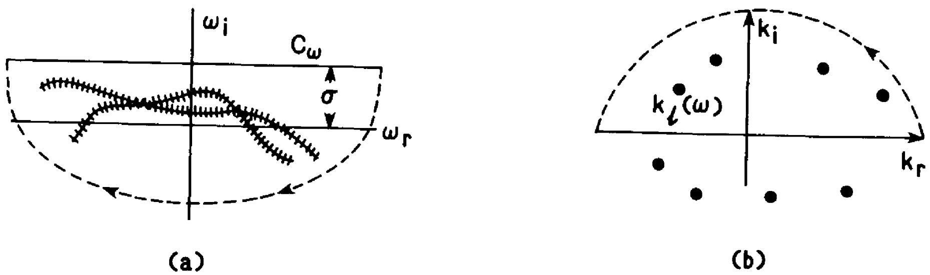


Fig. (I.6.2) Sketches of contours for (a) $\omega$-plane integration of Eq. (I.6.5) and (b) k-plane integrations of Eq. (I.6.6).

In Fig. (I.6.2a), ┼┼┼┼┼ corresponds to the solutions of the dispersion relation $\varepsilon(\omega_j, k) = 0$ for real k and $j = 1,\ldots, N$ corresponds to the jth branch of the normal modes. That Im $\omega_j > 0$ for some real k, of course, is consistent with the fact that the system is unstable. Further, as discussed above, Im $c_\omega = \sigma >$ Max Im $(\omega_j)$ and, hence, $\delta G(\omega,x)$ is analytic for $\omega$ on and above $c_\omega$. Meanwhile, in Fig. (I.6.2b), • corresponds to solutions of the dispersion relation for a <u>given $\omega$ on $c_\omega$</u>; i.e., $\varepsilon[\omega,k_\ell(\omega)] = 0$, $\ell = 1,\ldots M$ and $\omega$ on $c_\omega$.

Since $\delta G(\omega,x)$ is analytic only for $\omega_i \geq \sigma$, we need to analytically continuate it as we close the $\omega$-integration contour in the $\omega_i < 0$ half plane. Referring to Fig. (I.6.3a), let us take one point on $c_\omega$, $\omega = \omega_o$ and continuate it downward.

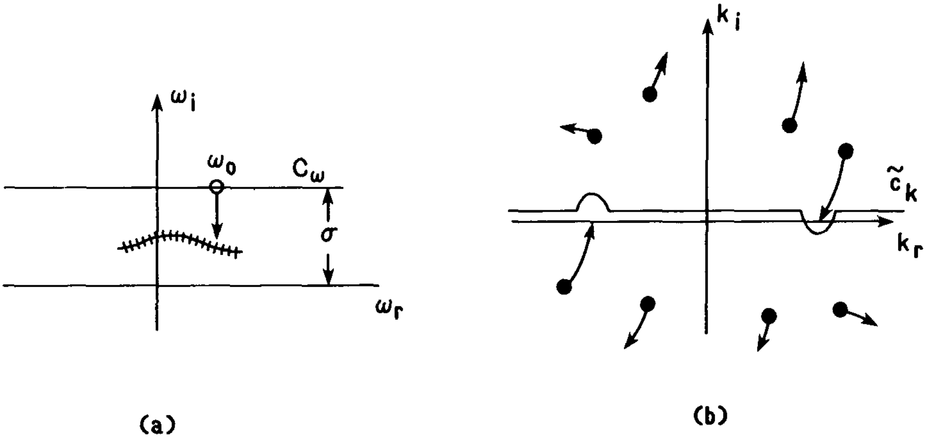


Fig. (I.6.3) Deformed contours for analytic continuations of integrations in (a) the ω plane and (b) the k plane.

Since $k_\ell = k_\ell(\omega)$ depends on ω, the poles in k plane will move around as $\omega_o$ moves downward. In particular, we know that, as $\omega_o \to \omega_j$, some pole(s) in k plane must move toward the $k_r$-axis [c.f. Fig. (I.6.3b)]. This is because $\omega_j$ is defined by $\varepsilon(\omega_j,\ k = \text{real}) = 0$. As, say, $k_\ell(\omega_s)$ becomes real, $\delta G(\omega,x)$ as defined by Eq. (I.6.6) becomes non-analytic since $k_\ell(\omega_j)$ lies on the k-integration contour $c_k$ which is the $k_r$ axis. In this respect, $\omega_j(k)$'s are the branch lines of $\delta G(\omega,x)$ in the ω plane. In order to analytically continuate $\delta G(\omega,x)$ below $\omega_j(k)$, it is then necessary to deform the k-integration contour around those poles which approach or cross the $k_r$ axis; i.e., we must choose a new contour, $\tilde{c}_k$ , as illustrated in Fig. (I.6.3b). The analytic continuation of δG as defined by Eq. (I.6.6) is then given by

$$\delta\hat{G}(\omega,x) = \int_{\tilde{c}_k} \frac{dk}{2\pi}\, \frac{1}{\varepsilon(\omega,k)} \cdot e^{ikx} \quad . \tag{I.6.7}$$

Thus, with $\tilde{c}_k$ so chosen, the k plane poles, $k_\ell(\omega)$, never cross the $\tilde{c}_k$ as ω moves downward. That is, when we close the k-plane integration in the $k_i > 0$ half plane, the <u>identities</u> of those poles which contribute to the integral do <u>not</u> change. We then have

$$\hat{\delta G}(\omega,x) = i \sum_{\ell} \frac{e^{ik_{\ell}(\omega)}}{\partial\varepsilon(\omega,k_{\ell})/\partial k_{\ell}} , \qquad (I.6.8)$$

with the understanding that only those $k_{\ell}$ which have Im $k_{\ell} > 0$ when Im $\omega \geq \sigma$ are included.

Substituting Eq. (I.6.8) into Eq. (I.6.5), it is clear that poles of $\delta\hat{G}$ will contribute to the $\omega$ integration. In particular, in the time-asymptotic limit ($t \to \infty$), the pole with maximum $\omega_i$ dominates. Let this $\omega$-plane pole be $\omega_s$. Thus, $\partial\varepsilon[\omega_s, k_s(\omega_s)]/\partial k_s = 0$; i.e., at $\omega = \omega_s$, $\varepsilon(\omega_s, k) = 0$ has a double root at $k = k_s$. In other words, as $\omega \to \omega_s$, we have two k-plane poles converging at $k = k_s$. Expanding $\varepsilon(\omega,k)$ about $\omega_s$ and $k_s$, respectively, we find

$$\varepsilon(\omega,k) \simeq \left(\frac{\partial\varepsilon}{\partial\omega_s}\right)(\omega-\omega_s) + \frac{1}{2}\left(\frac{\partial^2\varepsilon}{\partial k_s^2}\right)(k-k_s)^2. \qquad (I.6.9)$$

Substituting Eq. (I.6.9) into Eq. (I.6.7), it is fairly straightforward to show that $\delta\hat{G}(\omega_s,x)$ is singular (regular) if the two converging poles lie on the opposite (same) side of $\tilde{c}_k$. In particular, we have, for the singular case, that, noting

$$\lim_{|\alpha| \to 0^+} \int_{-\infty}^{\infty} \frac{dx}{(x+i\alpha)(x-i\alpha)} = \frac{\pi}{\alpha} \text{ for Re } \alpha > 0 , \qquad (I.6.10)$$

$$\delta\hat{G}(\omega,x) \simeq \left[2 \left(\frac{\partial\varepsilon}{\partial\omega_s}\right)\left(\frac{\partial^2\varepsilon}{\partial k_s^2}\right)\right]^{-1/2} (\omega-\omega_s)^{-1/2} e^{ik_s x} \qquad (I.6.11)$$

for $\omega \simeq \omega_s$. Equation (I.6.11) into Eq. (I.6.5), we find that

$$\delta G(t,x) \simeq \left[2\left(\frac{\partial\varepsilon}{\partial\omega_s}\right)\left(\frac{\partial^2\varepsilon}{\partial k_s^2}\right)\right]^{-1/2} e^{-i\omega_s t+ik_s x} \int_{c_\omega} \frac{d\omega}{2\pi} \cdot \frac{e^{-i(\omega-\omega_s)t}}{(\omega-\omega_s)^{1/2}} , \qquad (I.6.12)$$

where $\tilde{c}_\omega$ as sketched in Fig. (I.6.4) must go around the branch cut.

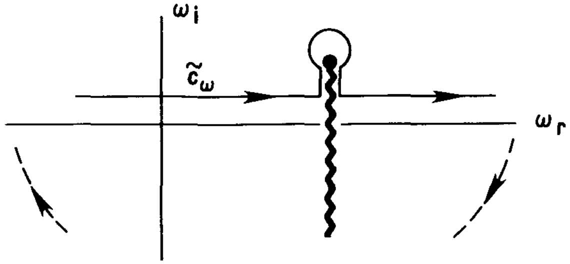


Fig. (I.6.4) Sketch of the integration contour around the branch cut defined by Eq. (I.6.12).

It is easy to show that

$$\lim_{t \to \infty} \left| \int_{\tilde{c}_\omega} \frac{d\omega}{2\pi} \frac{e^{-i(\omega-\omega_s)t}}{(\omega-\omega_s)^{1/2}} \right| \propto \frac{1}{t^{1/2}} . \tag{I.6.13}$$

Thus, we obtain, in the time asymptotic limit,

$$\lim_{t \to \infty} \delta G(t,x) \propto t^{-1/2} \exp[i(k_s x - i\omega_s t)]. \tag{I.6.14}$$

Equation (I.6.14) clearly shows that if Im $\omega_s > 0$ we then have $|\delta G(t,x)|$ or $|\delta\phi(t,x)| \to \infty$ as $t \to \infty$, i.e., an absolute instability. Otherwise, for Im $\omega_s \leq 0$, we have $|\delta G|$ or $|\delta\phi| \to 0$ as $t \to \infty$; i.e., a convective instability.

Let us summarize the results obtained in this section. First, we assume the system is linearly unstable, i.e., the linear dispersion relation $\varepsilon(\omega,k) = 0$ has unstable (Im $\omega > 0$) roots for real k. We then show that, if (i) there exists two k solutions of $\varepsilon(\omega,k) = 0$ which lie, respectively, on the upper and lower half k plane for Im $\omega \geq \sigma \equiv \text{Im}(c_\omega)$, and (ii) the two k solutions coalesce ($\partial\varepsilon/\partial k_s = 0$) at $\omega = \omega_s$ with Im $\omega_s > 0$, we then have an absolute instability. Otherwise, the system only exhibits convective instability. We remark that the condition $\partial\varepsilon/\partial k_s = 0$ at $\omega = \omega_s$ is equivalent to the condition that the group velocity $\partial\omega_s/\partial k_s = 0$. Thus, the above requirements for absolute instabilities are also consistent with our physical intuition that absolute instabilities correspond to standing-still wavepackets which, while growing in time, also tend to spread out in both positive and negative directions. Finally, we emphasize that the nature of the instability depends crucially on the reference frame. This is intuitively expected, because for an observer travelling with the group velocity of the wave packet a convective instability in the laboratory frame will appear to be an absolute instability to him/her.

Let us illustrate the application of the above stability criteria with the following three examples:

(i) Consider the model dispersion relation

$$(k-\omega/v_1)(k-\omega/v_2) = -\,k_o^2 , \tag{I.6.15}$$

where $v_1v_2 > 0$. By simply considering the $k \to 0$ limit, it is clear that Eq. (I.6.15) predicts instability. Meanwhile, in the limit Im $c_\omega \to \infty$ we find $k_1 \to \omega/v_1$ and $k_2 \to \omega/v_2$; i.e., the two k-plane poles lie on the same side of k-integration contour $c_k$ and, hence, the instability is convective.

(ii) We now consider an another model dispersion relation

$$(k-\omega/v_1)(k+\omega/v_2) = k_o^2 , \tag{I.6.16}$$

here, again, $v_1v_2 > 0$ and the system is unstable. Now as Im $c_\omega \to \infty$, $k_1 \to \omega/v_1$ and $k_2 \to -\omega/v_2$ and, thus, two k-plane poles are on the opposite side of $c_k$. We now calculate $\omega_s$ where $k_1$ and $k_2$ coalesce. From, $\partial\omega_s/\partial k_s = 0$, we find

$$k_s = \frac{\omega_s}{2} (1/v_1 - 1/v_2). \qquad \text{(I.6.17)}$$

For simplicity, we further assume $v_1 = v_2$. Then $k_s = 0$ and, from Eq. (I.6.16), we find

$$\omega_s^2 = - k_o^2 v_1^2 \quad \text{or} \quad \omega_s = \pm i k_o v_1 . \qquad \text{(I.6.18)}$$

Thus, Im $\omega_s > 0$ and we have an absolute instability.

(iii) Finally, we consider the beam-plasma instability given by the dispersion relation

$$1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{(\omega - k v_b)^2} = 0. \qquad \text{(I.6.19)}$$

Now, as in Example (i), we have, in the limit $c_\omega \to \infty$, $k_{1,2} \to \omega/v_b$ and, thus, this instability is convective.