PREFACE

Mathematicians and physicists share the common aim of finding unified explanations of the phenomena with which they deal. In the late 1970's each community found a theory holding the promise of such a unification for a wide class of phenomena. Physicists found the "dual resonance" model which soon developed into "string theory", which could encompass as special cases the gauge theories of particle interactions as well as Einstein's theory of gravity and introduced a new principle, supersymmetry, leading later to supergravity. Mathematicians discovered the existence of a class of infinite-dimensional Lie algebras, called affine Kac-Moody algebras, with properties which generalised in a relatively straightforward way those of the familiar finite-dimensional simple algebras. The root systems of these algebras, although infinite, were finite-dimensional and possessed geometric properties leading to a systematic understanding of certain combinatorial identities involving infinite products and sums, known in some cases from the nineteenth century, such as Jacobi's triple product identity.

About ten years later it emerged that certain of the irreducible representations of these algebras could be constructed rather explicitly using the concept of a "vertex operator", one of the key ingredients in string theory. The forging of this vital link initiated a unification of ideas from mathematics and physics which is still in full spate and enriching each subject. It is these developments which provide the occasion for the present collection of reprints.

Before describing other applications in physics, we shall briefly explain why, in retrospect, it is natural for infinite-dimensional Lie algebras to be relevant. Physics deals with the regularities or symmetries of nature and the symmetries of any given physical system form a group. Finite or discrete groups are relevant to objects such as crystals but, because space and time are continuous, we should expect continuous groups, i.e. Lie groups, to enter physics. These are infinite groups but the original examples were finite-dimensional (i.e. their elements could be thought of as depending on a finite number of continuous parameters), such as the group of rotations in three dimensions, $SO(3)$, used in the study of atomic spectra and elsewhere. Quantum mechanics furnished the vector spaces on which these groups act naturally and, since that theory has traditionally been formulated in terms of hermitian observables, physicists have been led naturally to consider the Lie algebra of generators of the Lie group. For $SO(3)$ this consists of the angular momenta.

The theory of elementary particles combines quantum mechanics and special relativity as well as other ideas. In the early 1960's the theory of "current algebras", motivated by data on radio-active (i.e. "weak") decays held sway, later to be supplanted by two apparently different theories, string theory and the unified theory of gauge interactions in the late 1960's. In the latter theory, forces were associated with the generators of the "gauge group", a compact Lie group chosen on the basis of experimental data, and the plethora of "elementary" particles was fitted into families, i.e. irreducible representations, of the gauge group. The weak and electromagnetic forces were successfully described by the theory proposed by Weinberg and Salam, which took the group $U(2)$, consisting of 2×2 matrices, as gauge group. The strong forces, responsible indirectly for the forces that hold nuclei together, were described by an $SU(3)$ gauge group.

The desire to fit these groups together, as subgroups of a larger simple group, in as economical a way as possible, led in the 1970's to an interest in larger groups, such as $SU(5)$, $SO(10)$, and even the exceptional group E_6. This programme was called "grand unification". These and other developments persuaded many theoretical physicists that a general understanding of the Lie algebras was necessary, not only as a working tool but, more significantly, also as a guide and language for the formulation of concepts.

The affine Kac-Moody algebras constitute, in a well-defined sense, the simplest kind of infinite-dimensional algebra, with properties naturally generalising those of finite-dimensional Lie algebras. Automatically associated with such an algebra is a Virasoro algebra, another infinite-dimensional algebra which was first identified and studied in string theory. During the last twenty years there has been a growing number of situations in which one or both of these two types of infinite-dimensional Lie algebras (i.e. affine Kac-Moody or Virasoro) have played a rôle at least comparably important to those assumed by finite-dimensional Lie algebras.

These situations are ones in which the physical systems involved are in some essential respect two-dimensional. For example, it is the string world-sheet, the surface swept out by the string in its motion in space-time, which is two-dimensional. The elementary particles are quantum excitations of the string. The success of string theory in unifying all known basic symmetry principles, including gauge invariance, explains why it is currently regarded as the most promising candidate for the unified theory of all the interactions, including gravity.

A surprising and rewarding arena for the theory concerns the behaviour of statistical physics models consisting of, say, spin variables at the sites of a two-dimensional lattice interacting with their nearest neighbours. At a certain critical temperature, the system can make a "second order" phase transition whereby it changes its state (as, for example, ice melts into water). The scale given by the lattice spacing effectively diverges and therefore becomes irrelevant, leaving a scale invariant theory controlled by the Virasoro algebra, whose representation theory determines the "critical exponents" of the transition. These exponents specify certain power law behaviours in spatial separations which are measurable in the laboratory, where two-dimensional systems can indeed be constructed and are of considerable current technological interest. This result, that the possible critical behaviour of such models is thus largely independent of the details of the model, gives explicit examples of the concept of "universality" introduced in solid state physics in the 1960's.

Yet another manifestation of these algebras is in the theory of solitons and integrable systems. Often integrable non-linear equations of physical or engineering significance can be regarded as "zero curvature" conditions for connections taking values in an affine Kac-Moody algebra. The sine-Gordon equation is a prime example and the vertex operator construction was first proposed for its soliton by Skyrme at the beginning of the 1960's. He had in mind applications to a unified theory of elementary particles but the concept of soliton also has applications in nerve fibres, optical fibres and many other areas of science and engineering.

A framework which can encompass so many studies at the frontiers of research in physics and engineering is obviously worthy of study and must be expected to be of importance in mathematics, as had indeed been realised independently as we mentioned earlier.

The fact that so many areas are interrelated ought to stimulate new developments in each of the constituent subjects and indeed this has already happened. For example, recent developments in incorporating gauge symmetries into string theory are in part due to mathematical stimulus (see chapter 2). There seems every reason to believe that this fruitful interrelation will continue and that, in particular, future developments in string theory will exploit and stimulate progress in the theory of infinite-dimensional Lie algebras.

Prompted by the increasing interest in this area amongst physicists, Dr. K.K. Phua suggested to us that we might produce a volume of reprints, including our recent review article published in International Journal of Modern Physics A. In selecting papers for this volume, we have tried to choose those which form good sources for the current developments in physics and those likely to be relevant in the future. Such criteria are, of course, subjective and result in the exclusion of many papers of crucial importance historically. Additionally, we have had it in mind that we wished to produce a book of no more than 500 to 600 pages.