Chapter 1

MANY PARTICLE SYSTEMS AND FIELD THEORIES

In this chapter, we begin with a nonrelativistic (NR) quantum mechanical system with n degrees of freedom. We then generalize it to the continuum limit. This will lead naturally to the field quantization.

1. An NR Quantum System with n Degrees of Freedom

Consider an n degrees of freedom quantum system

$$H(q, p) = \Sigma\, p_i^2/2m + V(q). \tag{1.1}$$

The coordinates q_i and the momenta p_i obey the equal-time quantization relations

$$[q_i, q_j] = [p_i, p_j] = 0 \tag{1.2a}$$

$$[q_i, p_j] = i\delta_{ij}\hbar. \tag{1.2b}$$

In the following, we shall choose the units such that $\hbar = 1$. Equations (1.1) and (1.2) determine the system completely. We can describe the system either in the Schrödinger picture, or in the Heisenberg picture. In the Schrödinger picture, the time evolution of the physical system is in the wave function $\psi(q, t)$ which satisfies the Schrödinger equation,

$$i\frac{\partial\psi(q, t)}{\partial t} = \left[-\frac{1}{2m}\frac{\partial^2}{\partial q^2} + V(q)\right]\psi(q, t). \tag{1.3}$$

In the Heisenberg picture, the time evolution is associated with the operators $p_i(t)$ and $q_i(t)$ which obey the Heisenberg equations of motion,

$$\dot{q}_i = \frac{1}{i}[q_i, H] = p_i/m \tag{1.4a}$$

$$\dot{p}_i = \frac{1}{i}[p_i, H] = -\frac{\partial V}{\partial q_i}. \tag{1.4b}$$

We can establish easily that these two representations are equivalent.

2. Continuum Limit

Next we consider a string of density ρ, under tension T, and lying near the bottom of a cylinder, as shown in Fig. 1.1a.

Classically, we may describe the dynamics of the string by its transverse displacement $q(x, t)$ from its equilibrium position at (x, t) (See Fig. 1.1b). Note that the continuous variable x is the analog of the discrete index i discussed earlier. Thus, x is an index parameter, and not a dynamical variable. The dynamical variables are $q(x, t)$ and $\dot{q}(x, t)$ which are the generalization of q and p/m. Quantities $q(x, t)$ and $\dot{q}(x, t)$ are simple examples of field variables.

Classically, the generalization of a discrete system to a continuum system is straightforward. The Lagrangian and Hamiltonian of this string system are

$$L = \int_0^L dx\, \mathscr{L}(q, \dot{q}) \tag{1.5}$$

$$H = \int_0^L dx\, \mathscr{H}(q, \pi) \tag{1.6}$$

where

$$\mathscr{L} = \frac{1}{2}\left[\rho\dot{q}^2 - T\left(\frac{\partial q}{\partial x}\right)^2\right] - V(q) \tag{1.7}$$

$$\mathscr{H} = \pi^2/2\rho + \frac{T}{2}\left(\frac{\partial q}{\partial x}\right)^2 + V(q) \tag{1.8}$$

$$\pi \equiv \frac{\partial \mathscr{L}}{\partial \dot{q}} = \rho\dot{q}\,. \tag{1.9}$$

In Eqs. (1.7) and (1.8), $V(q)$ is the potential energy density of the string as it moves along the cylinder. (See Fig. 1.1c.)

When we treat the string as a quantum system, we need to impose the equal-time quantization rules

$$[q(x, t), q(y, t)] = [\pi(x, t), \pi(y, t)] = 0 \tag{1.10}$$

$$[q(x, t), \pi(y, t)] = i\delta(x - y). \quad (1.11)$$

In short, to change from a discrete system to a continuum system, we need to replace discrete summations by integrations, and Kronecker δ's by Dirac δ-functions.

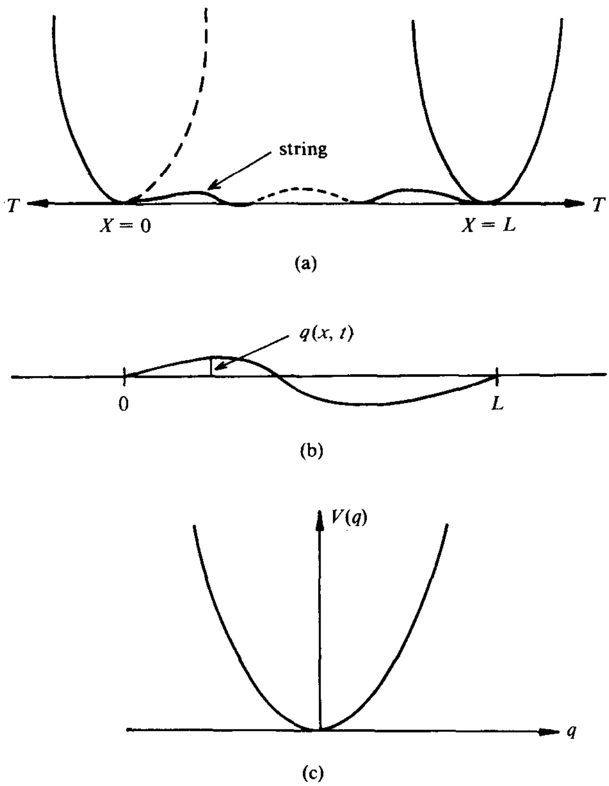

Fig. 1.1. (a) A string lies near the bottom of a cylinder. (b) $q(x, t)$ denotes the displacement of the string at (x, t) from its equilibrium position. (c) $V(q)$ describes the potential energy density as a function of q.

Just as in the discrete case, we may study the system in the Schrödinger representation. We need to replace the wave function $\psi(q, t)$ by a wave functional $\psi(q(x), t)$. In function $q(x)$ space, the momentum operator becomes a functional differentiation operator

$$\pi(x) \to \frac{1}{i}\frac{\delta}{\delta q(x)}. \quad (1.12)$$

The functional Schrödinger equation takes the form

$$\int_0^L dx\left[-\frac{1}{2\rho}\left(\frac{\delta}{\delta q(x)}\right)^2+\frac{T}{2}\left(\frac{\partial q}{\partial x}\right)^2+V(q)\right]\psi(q(x),t)=i\frac{\partial}{\partial t}\psi(q(x),t). \tag{1.13}$$

In Eq. (1.13), the first term is the generalization of Laplace operator to functional space. Obviously, this is not a simple equation to solve.

In the Heisenberg representation, we encounter t-dependent $q(x, t)$ and $\pi(x, t)$. The Heisenberg equations of motion are

$$\dot{q}(x,t)=\frac{1}{i}[q(x,t),H]=\pi(x,t)/\rho \tag{1.14}$$

$$\dot{\pi}(x,t)=\frac{1}{i}[\pi(x,t),H]. \tag{1.15}$$

Using the canonical quantization relations, we can reduce the second equation to

$$\begin{aligned}\dot{\pi}(x,t)&=\frac{1}{i}\left[\pi(x,t),\int_0^L dy\left(\frac{\pi(y,t)^2}{2\rho}+\frac{T}{2}\left(\frac{\partial q}{\partial y}\right)^2+V(q(y))\right)\right]\\&=\frac{1}{i}\int_0^L dy\left\{T\left(\frac{\partial q}{\partial y}\right)\left[\pi(x,t),\frac{\partial q(y,t)}{\partial y}\right]+[\pi(x,t),V(q(y))]\right\}\\&=\frac{1}{i}\int \mathrm{dy}\left\{T\left(\frac{\partial q}{\partial y}\right)\left(-i\frac{\partial}{\partial y}\delta(x-y)\right)+V'(q(y))(-i\delta(x-y))\right\}\\&=T\frac{\partial^2 q(x,t)}{\partial x^2}-V'(q(x,t)).\end{aligned} \tag{1.16}$$

Except for the noncommutativity relations, these equations are the same as the classical equations of motion.

3. A Free Field

We consider the simple case in which $V(q)$ is quadratic in q. By proper scale transformations, we can absorb ρ and T into the new fields ϕ and $\dot{\phi}$, giving

$$\mathscr{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{m^2}{2}\phi^2 \tag{1.17a}$$

$$\mathscr{H} = \frac{1}{2}\pi^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{m^2}{2}\phi^2 \tag{1.17b}$$

where ϕ and $\dot{\phi}$ obey canonical quantization relation. The Heisenberg equations of motion are

$$\pi(x, t) = \dot{\phi}\,(x, t) \tag{1.18}$$

$$\dot{\pi}(x, t) = \frac{1}{i}\,[\pi(x, t), H]$$

$$= \frac{\partial^2\phi}{\partial x^2} - m^2\,\phi\,(x, t). \tag{1.19}$$

Combining (1.18) and (1.19), we have

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + m^2\phi = 0\,. \tag{1.20}$$

Equation (1.20) is known as the Klein-Gordon equation. This equation together with equal-time commutation relations

$$[\phi, \phi] = [\pi, \pi] = 0$$

$$[\phi\,(x, t), \pi\,(y, t)] = i\,\delta\,(x - y)$$

determine the system completely.

Since (1.20) is linear in ϕ, it is easy to find its eigenfunctions. We begin with a finite string with the periodic boundary condition $\phi\,(0, t) = \phi\,(L, t)$. The eigenfunctions are of the form

$$e^{ikx - i\omega_k t} \quad \text{and} \quad e^{-ikx + i\omega_k t}. \tag{1.21}$$

The periodic boundary condition requires

$$kL = 2\pi n \quad \text{or} \quad k = 2\pi n/L \tag{1.22}$$

with

$$n = 0, \pm 1, \pm 2, \ldots\,.$$

Equation (1.20) determines the frequency ω_k :

$$- \omega_k^2 + k^2 + m^2 = 0 \tag{1.23}$$

or

$$\omega_k = \sqrt{k^2 + m^2} . \tag{1.24}$$

Note that we have taken both $e^{ikx-i\omega t}$ and its complex conjugate as eigenfunctions. Hence we only need to keep positive ω_k in (1.23). The notation $kx - \omega t$ is for future convenience. It will be identified as the relativistic scalar product.

We can expand $\phi(x, t)$ and $\dot{\phi}(x, t)$ in terms of the eigenfunctions

$$\phi(x, t) = \sum_k \frac{1}{\sqrt{2\omega_k L}} \left(a_k \, e^{ikx - i\omega_k t} + a_k^\dagger \, e^{-ikx + i\omega_k t}\right). \tag{1.25}$$

$$\begin{aligned} \pi(x, t) &= \dot{\phi}(x, t) \\ &= -i \sum_k \sqrt{\frac{\omega_k}{2L}} \left(a_k \, e^{ikx - i\omega_k t} - a_k^\dagger \, e^{-ikx + i\omega_k t}\right). \end{aligned} \tag{1.26}$$

Their coefficients of expansion a_k and $a_k^\dagger$ are still operators, but are independent of x and t. The choice of normalization is to make a_k and $a_k^\dagger$ have simple physical interpretation.

To find the commutation relations between a_k and $a_k^\dagger$, we need to express a_k and $a_k^\dagger$ in terms of ϕ and π. We can accomplish this by taking fourier transforms. We note that both a_k and $a_k^\dagger$ appear in the fourier transforms. To obtain a_k alone, we need to fourier transform the combination $\omega_k \phi(x, t) + i\pi(x, t)$, giving

$$a_k = \frac{1}{\sqrt{2L\omega_k}} \int_0^L dx \, e^{-ikx + i\omega_k t} \left(\omega_k \phi(x, t) + i\pi(x, t)\right). \tag{1.27}$$

Similarly, we obtain

$$a_k^\dagger = \frac{1}{\sqrt{2L\omega_k}} \int_0^L dx \, e^{ikx - i\omega_k t} \left(\omega_k \phi(x, t) - i\pi(x, t)\right). \tag{1.28}$$

It is straightforward to verify that

$$[a_k, a_l] = [a_k^\dagger, a_l^\dagger] = 0. \tag{1.29}$$

The commutation between a_k and $a_l^\dagger$ gives

$$[a_k, a_l^\dagger] = \frac{1}{2L\sqrt{\omega_k\omega_l}} \int dxdy\, e^{-ikx + i\omega_k t} e^{ily - i\omega_l t}$$

$$\times [\omega_k\phi(x, t) + i\pi(x, t), \omega_l\phi(y, t) - i\pi(y, t)]$$

$$= \frac{1}{2L\sqrt{\omega_k\omega_l}} \int dxdy\, e^{-ikx + i\omega_k t} e^{ily - i\omega_l t}$$

$$\times (\omega_k\delta(x - y) + \omega_l\delta(x - y))$$

$$= \frac{\omega_k + \omega_l}{2L\sqrt{\omega_k\omega_l}} \int dx\, e^{-i(k-l)x + i(\omega_k - \omega_l)t}. \tag{1.30}$$

The x-integration leads to a Kronecker δ:

$$\int_0^L dx\, e^{-i(k-l)x} = L\delta_{kl}. \tag{1.31}$$

Under the condition $k = l$, the t-dependence drops out completely. We then have

$$[a_k, a_l^\dagger] = \delta_{kl}. \tag{1.32}$$

Equations (1.29) and (1.32) imply that a_k and $a_k^\dagger$ have the interpretation of annihilation and creation operators. The modes associated with different k's are decoupled.

Substituting (1.25) and (1.26) into the Hamiltonian (1.6) and (1.17), we have

$$H = \int dx\, \mathscr{H}$$

$$= \int_0^L dx \left[\frac{1}{2}\pi(x, t)^2 + \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{1}{2}m^2\phi^2\right]$$

$$= \int_0^L dx \left\{-\frac{1}{2}\sum_k\sum_l \frac{\sqrt{\omega_k\omega_l}}{2L}\left(a_k e^{ikx - i\omega_k t} - a_k^\dagger e^{-ikx + i\omega_k t}\right)\right.$$

$$\times\left(a_l e^{ilx-i\omega_l t}-a_l^\dagger e^{-ilx+i\omega_l t}\right)$$

$$+\frac{1}{2}\sum_k\sum_l\frac{1}{2L\sqrt{\omega_k\omega_l}}\left(ika_k e^{ikx-i\omega_k t}-ika_k^\dagger e^{-ikx+i\omega_k t}\right)$$

$$\times\left(ila_l e^{ilx-i\omega_l t}-ila_l^\dagger e^{-ilx+i\omega_l t}\right)$$

$$+\frac{1}{2}m^2\sum_k\sum_l\frac{1}{2L\sqrt{\omega_k\omega_l}}\left(a_k e^{ikx-i\omega_k t}+a_k^\dagger e^{-ikx+i\omega_k t}\right)$$

$$\times\left(a_l e^{ilx-i\omega_l t}+a_l^\dagger e^{-ilx-i\omega_l t}\right)\Bigg\}. \tag{1.33}$$

First, look at the $a_k a_l$ terms. The x-integrations always lead to the Kronecker delta, $\delta_{k+l,0}$. This Kronecker delta implies that $l=-k$ and $\omega_k=\omega_l$. The coefficient of $a_k a_l$ is proportional to

$$-\omega_k+\frac{k^2}{\omega_k}+\frac{m^2}{\omega_k}=0. \tag{1.34}$$

Hence, the $a_k a_l$ terms cancel each other. This cancellation also occurs for $a_k^\dagger a_l^\dagger$ terms. For the $a_k^\dagger a_l$ and $a_k a_l^\dagger$ terms, instead of cancelling each other, these terms add. After some algebra, we have

$$H=\sum_k\omega_k\frac{a_k^\dagger a_k+a_k a_k^\dagger}{2}. \tag{1.35}$$

Making use of (1.32), we can write (1.35) as

$$H=\sum_k\omega_k\left(a_k^\dagger a_k+\tfrac{1}{2}\right). \tag{1.36}$$

The expression $a_k^\dagger a_k$ may be identified as the number operator for mode k.

The ground state of this system, $|0\rangle$, is annihilated by all a_k

$$a_k|0\rangle=0, \qquad \text{all } k. \tag{1.37}$$

In the present system, the ground state is a nondegenerate state with the lowest energy. It may be properly normalized to $\langle 0|0\rangle = 1$. We can describe an excited state by its occupation numbers $\{n_k\}$ in the k-modes:

$$|\{n_k\}\rangle = \prod_k \frac{(a_k^\dagger)^{n_k}}{\sqrt{n_k!}}|0\rangle . \tag{1.38}$$

The energy of this excited state is

$$E = \sum_k (n_k\omega_k + \omega_k/2) . \tag{1.39}$$

The ground state corresponds to $n_k = 0$ for all k. It is important to note that the ground state energy is not only nonvanishing, but actually diverges. The divergence of the zero-point energy is a general feature of a quantum field theory. Since we usually measure the energy of a system from and above its ground state energy, this zero-point energy does not have any observable effect. We can substract the unobserved infinite zero-point energy to give

$$E' \equiv E - E\,(\text{ground state}) = \sum_k n_k\omega_k . \tag{1.40}$$

For a state with finite occupation numbers, E' is finite.

We can also introduce the total momentum for the string. It is

$$\begin{aligned} P &= -\int_0^L dx\, \pi \frac{\partial\phi}{\partial x} \\ &= i\sum_k\sum_l \sqrt{\frac{\omega_k}{2L}}\sqrt{\frac{1}{2\omega_l L}}\int_0^L dx\left(a_k\, e^{ikx-i\omega_k t} - a_k^\dagger\, e^{-ikx+i\omega_k t}\right) \\ &\qquad \times \left(il a_l\, e^{ilx-i\omega_l t} - il a_l^\dagger\, e^{-ilx+i\omega_l t}\right). \end{aligned} \tag{1.41}$$

We look at the $a_k\, a_l$ terms. After x-integration, we find that only modes obeying $k + l = 0$ contribute. Then the summation over k and l vanishes identically,

$$\sum_k\sum_l l\, a_k\, a_l = \sum_k\sum_l \frac{1}{2}(k+l)\, a_k\, a_l = 0 .$$

Similarly, the $a_k^\dagger a_l^\dagger$ terms also cancel. The only remaining terms are $a_k a_l^\dagger$ and $a_k^\dagger a_l$ terms. After x-integration, we have

$$P = i \sum_k \sum_l \frac{1}{2L} \sqrt{\frac{\omega_k}{\omega_l}} \left(-ila_k a_l^\dagger - ila_k^\dagger a_l\right) L\delta_{kl}$$

$$= \sum_k k \frac{a_k^\dagger a_k + a_k a_k^\dagger}{2}$$

$$= \sum_k k \, a_k^\dagger a_k. \tag{1.42}$$

By using the symmetry of $k \leftrightarrow -k$, we have dropped the constant $\sum k/2$.

Thus for a state with occupation numbers $\{n_k\}$, the total momentum is

$$P = \sum n_k k. \tag{1.43}$$

We now concentrate on the energy and momentum associated with the excited modes. We identify them as the energy and momentum of the associated "quanta". We shall refer to these quanta as particles.

We call the ground state as the vacuum, the empty state. It contains no quanta. Its energy E' and momentum P are zero.

The state with $n_k = 1$ and all other occupation numbers being zeros is referred to as possessing a single quantum of momentum k and energy ω_k. We interpret this quantum as a particle with momentum k and energy

$$\omega_k = \sqrt{k^2 + m^2}. \tag{1.44}$$

The state with $n_k = n_l = 1$ and all other occupation numbers being zeros is a two-quantum state with momentum and energy (k, ω_k) and (l, ω_l) respectively. The total energy E' and momentum P of this 2-quantum state are

$$E' = \omega_k + \omega_l, \qquad P = k + l. \tag{1.45}$$

We may interpret this two-quantum state as a noninteracting two-particle state. In the present case, all multiparticle states are noninteracting.

The noninteracting field theory is a consequence of a quadratic potential $V(\phi) = m^2\phi^2/2$. If the potential $V(\phi)$ contains higher order terms such as a ϕ^4 term, then we have an interacting field theory. We shall learn how to handle the interacting case in the next chapter.

4. Nonrelativistic Many-Particle Systems

We have seen that a quantum field theory is intrinsically a many-particle system. It is capable of describing the interaction of an indefinite number of particles. On the other hand, the nonrelativistic quantum mechanics that we have studied earlier is a one-particle system. It describes the interaction of a single particle with a space-dependent potential $V(x)$. In the following, we shall use the quantum field theory language to describe a one-particle system. In a sense, we are introducing unnecessary complications into a simple system. As we shall see, we can gain some insight from this approach.

One way to turn a single particle theory into a field theory is to create many copies of the one-particle system. We also make an assumption that the particles so created do not interact among themselves. The one-particle system obeys the Schrödinger equation

$$i\frac{\partial\psi(x,t)}{\partial t}=\left[-\frac{1}{2m}\nabla^2+V(x)\right]\psi(x,t). \tag{1.46}$$

For ease of explanation, we assume that all eigenstates are discrete, and that their wave functions $\psi(x, t)$ obey the normalization and completeness conditions

$$\int dx\,\psi^*_m(x,t)\,\psi_n(x,t)=\delta_{mn} \tag{1.47}$$

$$\sum_n \psi_n(x,t)\,\psi^*_n(y,t)=\delta(x-y). \tag{1.48}$$

We introduce the annihilation and the creation operators for the eigenmode n as a_n, $a_n^\dagger$ obeying the usual commutation relations,

$$[a_n, a_m]=[a_n^\dagger, a_m^\dagger]=0,$$

$$[a_n, a_m^\dagger]=\delta_{nm}. \tag{1.49}$$

We can now introduce the field variables $\phi(x, t)$ and $\phi^*(x, t)$ as

$$\phi(x,t)=\sum_n \psi_n(x,t)\,a_n, \tag{1.50}$$

$$\phi^*(x,t)=\sum_n \psi^*_n(x,t)\,a_n^\dagger. \tag{1.51}$$

The field $\phi(x, t)$ so constructed obeys the linear (Schrödinger) equation

$$i\frac{\partial}{\partial t}\phi(x, t) = \left[-\frac{1}{2m}\nabla^2 + V(x)\right]\phi(x, t). \tag{1.52}$$

It is straightforward to establish the equal-time quantization relations among the field variables:

$$[\phi(x, t), \phi(y, t)] = [\phi^*(x, t), \phi^*(y, t)] = 0 \tag{1.53}$$

and

$$[\phi(x, t), \phi^*(y, t)] = \sum_n \psi_n(x, t)\,\psi_n^*(y, t) = \delta(x-y). \tag{1.54}$$

Equations (1.52)–(1.54) are the field theoretical description of the noninteracting one-particle system. We can derive these equations from a Lagrange function,

$$\mathscr{L} = i\phi^*(x, t)\,\dot{\phi}(x, t) - \frac{1}{2m}\nabla\phi^*\cdot\nabla\phi$$

$$- V(x)\,\phi^*(x, t)\,\phi(x, t). \tag{1.55}$$

It is easy to see that the momentum conjugate to $\phi(x, t)$ is

$$\pi(x, t) \equiv \frac{\partial\mathscr{L}}{\partial\dot{\phi}} = i\phi^*(x, t) \tag{1.56}$$

and the Hamiltonian density is

$$\mathscr{H} = \pi\dot{\phi} - \mathscr{L} = \frac{1}{2m}\nabla\phi^*\cdot\nabla\phi + V(x)\,\phi^*\phi. \tag{1.57}$$

In terms of ϕ and π, Eqs. (1.53) and (1.54) are indeed the usual quantization relations. The total Hamiltonian of the many-particle system is

$$H = \int dx\,\mathscr{H} = \sum_n \omega_n a_n^\dagger a_n \tag{1.58}$$

where ω_n is the one-particle energy of state n. Note that in (1.55) ϕ^* is always on the left of ϕ. There is no zero-point energy in (1.58). The operators $\phi(x, t)$ and $\phi^*(x, t)$ may be interpreted as the annihilation and creation operators of a particle at x and t.

Just as in the string case, we can introduce the vacuum, $|0\rangle$, as the no-particle state. The 1-particle state, 2-particle state, . . . are

$$|1_n\rangle \equiv a_n^\dagger |0\rangle$$

$$|2_{m,n}\rangle \equiv a_m^\dagger a_n^\dagger |0\rangle, \ldots,$$

respectively. We can recover the one-particle Schrödinger wave function $\psi(x, t)$ from the field variable $\phi(x, t)$ and the one-particle state $|1\rangle$ via

$$\psi(x, t) = \langle 0 | \phi(x, t) | 1 \rangle. \tag{1.59}$$

In addition, we can define a two-particle wave function as

$$\psi(x, y, t) \equiv \langle 0 | \phi(x, t)\, \phi(y, t) \mid 2 \rangle = \psi(y, x, t). \tag{1.60}$$

From the canonical quantization rule, we find that the two-particle wave function is symmetric with respect to the interchange of x and y. Particles with this symmetry property are said to obey Bose-Einstein statistics. We wish to point out that we did not derive this property: we assumed it. In nature, there are particles which do not obey Bose-Einstein statistics. We shall discuss this point in more detail in the future.

In reality, particles do interact with each other. We consider a situation in which the particles not only interact with the external potential $V_1(x)$, but also interact with each other via a potential $V_2(x, y)$. Using the field theory language, we can describe the interaction easily as

$$H = \int dx \left[\frac{1}{2m} \nabla\phi^*(x, t) \cdot \nabla\phi(x, t) + V_1(x)\, \phi^*(x, t)\, \phi(x, t) \right]$$

$$+ \frac{1}{2} \int dxdy\, \phi^*(x, t)\, \phi(x, t)\, V_2(x, y)\, \phi^*(y, t)\, \phi(y, t). \tag{1.61}$$

In (1.61), $\phi^*(x, t)\phi(x, t)$ describes the particle density operator at position (x, t). The two-particle interaction energy is proportional to the product of particle densities at x and y. The factor 1/2 is introduced to take care of the double counting. This is one of the systems that we shall study in detail.