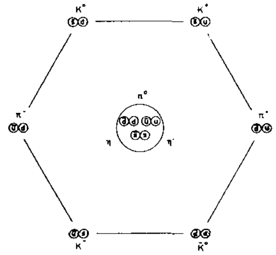
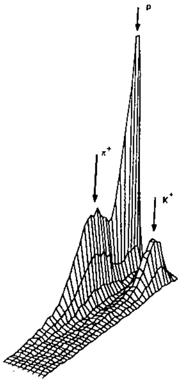
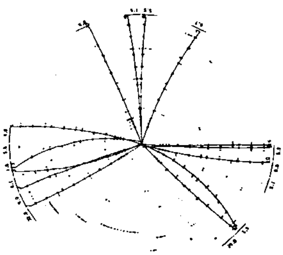
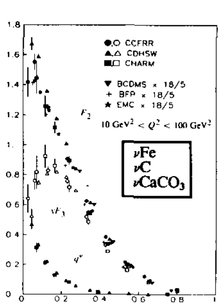
CHAPTER 1

CHIRAL SYMMETRY - SPECTRAL SUM RULES - MS-SCHEME AND PERTURBATIVE QCD









Prof. Murray Gell-Mann contemplating his meson classification, some meson peaks, the QCD jets and the nucleon structure function

1. PERTURBATIVE QCD AND CHIRAL SYMMETRY

Since the pioneering work of Gell-Mann, Fritzsch and Leutwyler[1] and after the discovery of the asymptotic freedom properties of QCD[2], there are various reasons for believing that QCD is the best candidate theory of strong interactions though the confinement problem is still unsolved owing to the peculiar infrared behaviour of the theory. The asymptotic freedom property of QCD at high momentum allows perturbative calculations in a series-expansion of the strong interaction coupling constant to give a nice description of various hard processes[3] (deep inelastic scattering, Drell-Yan, jets, high P_T...)

a) The QCD Lagrangian and its Symmetry

The QCD Lagragian density is :

$$\mathcal{L}_{QCD}(x) = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + i \sum_{j=1}^{n} \bar{\psi}^\alpha_j \gamma^\mu (D_\mu)_{\alpha\beta}\psi^\beta_j - \sum_{j=1}^{n} m_j \bar{\psi}^\alpha_j \psi_j$$

$$- \frac{1}{2\alpha_G} \partial_\mu A^\mu_a \partial^\mu A_\mu - \partial_\mu \bar{\varphi}_a D^\mu \varphi_a , \qquad (1.1)$$

where $G^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu$ $(a \equiv 1,2,\ldots,8)$ are Yang-Mills [4] field strengths constructed from the gluon fields $A^a_\mu(x)$. $(D_\mu)_{\alpha\beta} \equiv \delta_{\alpha\beta} \partial_\mu - ig \sum_a \frac{1}{2} \lambda^a_{\alpha\beta} A^a_\mu$ are covariant derivatives acting on the quark colour component α,β ≡ red, blue and yellow. $\lambda^a_{\alpha\beta}$ are the eight 3×3 colour matrices and f_{abc} the structure constants which close the SU(3) Lie algebra :

$$[T_a, T_b] = i f_{abc} T_c \quad , \qquad (1.2)$$

where $(T^a)_{\alpha\beta} = \frac{1}{2} \lambda^a_{\alpha\beta}$ in the fundamental colour $\underline{3}$ representation, whilst in the adjoint $\underline{8}$ representation of gluon basis $(T^a)_{bc} = -i f^a_{bc}$. The last two terms in (1.1) are respectively the gauge-fixing term necessary for a covariant quantization in the gluon sector [$\alpha_G = O(1)$ in the Landau (Feynman) gauge] and the Fadeev-Popov[5] ghost term necessary to eliminate unphysical particles from the theory ($\varphi^a(x)$ are eight anticommuting scalar fields in the $\underline{8}$ of SU(3)).

$\mathcal{L}_{QCD}(x)$ is locally invariant under the BRS[6] transformations:

$$
\begin{aligned}
A_\mu(x) &\longrightarrow A_\mu(x) + \omega\, D_\mu\, \varphi \quad , \\
\psi_i(x) &\longrightarrow \exp(-\, ig\omega\, \vec{T}.\vec{\varphi})\, \psi_i \quad , \\
\bar{\varphi} &\longrightarrow \bar{\varphi} + \frac{\omega}{\alpha_G}\, \partial_\mu A^\mu \quad , \\
\varphi &\longrightarrow \varphi - \frac{1}{2}\, g\, \omega\, \vec{\varphi} \times \vec{\varphi} \quad ,
\end{aligned} \tag{1.3}
$$

where $\omega(x)$ is an arbitrary parameter.

$\mathcal{L}_{QCD}(x)$ is invariant under the $U(1)_B$ global transformation:

$$\psi_i(x) \longrightarrow \exp(-\, i\, \theta\, \mathbb{1})\, \psi_i(x) \quad , \tag{1.4}$$

to which corresponds the conserved baryonic current :

$$J^\mu(x) = \sum_i \bar{\psi}_i\, \gamma^\mu\, \psi_i \tag{1.5}$$

and the baryonic charge generator of the $U(1)_B$ group :

$$B = \int d^3x\, J^0(\vec{x}, t) \quad . \tag{1.6}$$

For massless quarks $\mathcal{L}_{QCD}$ is also invariant under the axial $U(1)_A$ transformation :

$$\psi_i \longrightarrow (-i\, \theta\, \mathbb{1}\, \gamma_5)\, \psi_i \quad , \tag{1.7}$$

acting on quark-flavour components. The corresponding current

$$J_5^\mu(x) = \sum_i \bar{\psi}_i \gamma^\mu \gamma^5 \psi_i \qquad (1.8)$$

has an anomalous divergence

$$\partial_\mu J_5^\mu(x) = \frac{g^2}{4\pi^2} \frac{n}{8} \epsilon_{\mu\nu\rho\sigma} G_a^{\mu\nu} G_a^{\rho\sigma} , \qquad (1.9)$$

where the rate of the change of the associated axial charge

$$\dot{Q}_5 = \int d^3x \, \partial_0 J_5^0 (\vec{x}, t) , \qquad (1.10)$$

is zero in the absence of instanton-type solutions[7].

In the massless quark limit ($m_j = 0$), $\mathcal{L}_{QCD}$ also possesses a $SU(n)_L \times SU(n)_R$ global chiral symmetry and is invariant under the global transformation :

$$\begin{aligned} \psi_i &\longrightarrow \exp\left(-i\ \theta^A T_A\right) \psi_i , \\ \psi_i &\longrightarrow \exp\left(-i\ \theta^A T_A \gamma_5\right) \psi_i , \end{aligned} \qquad (1.11)$$

where T^A $A \equiv 1,\ldots,n^2-1$ are the infinitesimal generators of the SU(n) group acting on the quark-flavour components. The associated Noether currents are the vector and axial-vector currents :

$$\begin{aligned} V_\mu^A(x) &= \bar{\psi}_i \gamma_\mu T_{ij}^A \psi_j , \\ A_\mu^A(x) &= \bar{\psi}_i \gamma_\mu \gamma_5 T_{ij}^A \psi_j , \end{aligned} \qquad (1.12)$$

which are the currents of the algebra of the currents of Gell-Mann[1,8].

The corresponding charges which are the generators of $SU(n)_L \times SU(n)_R$

are :

$$Q_L^A = \int d^3x \left(V_o^A - A_o^A\right) ,$$
$$Q_R^A = \int d^3x \left(V_o^A + A_o^A\right) . \qquad (1.13)$$

b) **Chiral Symmetry Breaking and PCAC**

The charges in 1.13 are conserved in the massless quark limit. In the Nambu-Goldstone[9] realization of chiral symmetry, the axial charge does not annihilate the vacuum. This is the basis of the successes of current algebra and pion PCAC [8]. In this scheme, the chiral flavour group $G \equiv SU(n)_L \times SU(n)_R$ is broken spontaneously by the light (u,d,s) quark vacuum condensates down to a subgroup $H \equiv SU(n)_{L+R}$ where the vacuums are symmetrical :

$$\left\langle \bar{\psi}_u \psi_u \right\rangle = \left\langle \bar{\psi}_d \psi_d \right\rangle = \left\langle \bar{\psi}_s \psi_s \right\rangle . \qquad (1.14)$$

This spontaneous breaking mechanism is accompanied by n^2-1 massless Goldstone P (pion-like) bosons which are associated with each unbroken generator of the coset space G/H. On the other hand, the vector charge is assumed to annihilate the vacuum and the corresponding symmetry is achieved à la Wigner-Weyl[10]. In this case, the particles are classified in irreducible representations of $SU(n)_{L+R}$ and form parity doublets. In addition to the electromagnetic mass which the Goldstone boson can aquire[11], they get a mass mainly from an explicit breaking ($m_i \neq 0$) of the $SU(n)_L \times SU(n)_R$ global symmetry. In this case, the divergence of the axial-vector current reads :

$$\partial_\mu A^\mu(x)^i_j = (m_i + m_j)\, \bar{\psi}_i (i\,\gamma_5)\, \psi_j , \qquad (1.15a)$$

to which are associated the quasi-Goldstone parameters defined as :

$$\left\langle 0 \left| \partial_\mu A^\mu \right| \pi \right\rangle = \sqrt{2}\, f_\pi\, m_\pi^2\, \vec{\pi} \quad , \tag{1.15b}$$

where $\vec{\pi}$ is the pion field and f_π = 93.3 MeV controls the $\pi \to \mu\nu$ decay. Current algebra also tells us that the two-point correlator associated with (1.15) is related to the axial-current one via a Ward identity[8] (up to equal-time commutators) :

$$q^\mu q^\nu\, \Pi_5^{\mu\nu} = \psi_5(q^2) - \int d^4 x\, e^{iqx} \delta(x_o)\, q^\nu \left\langle 0 \left| \left[A^0(x),\, A^{\nu +}(o)\right]\right| 0 \right\rangle$$

$$+\, i \int d^4x\, e^{iqx}\, \delta(x_o)\, . \left\langle 0 \left| \left[\partial_\mu A^\mu,\, (A^0(x))^+\right] \right| 0 \right\rangle \quad , \tag{1.16a}$$

with :

$$\psi_5(q^2) = i \int d^4x\, e^{iqx} \left\langle 0 \left| \mathbb{T}\, \partial_\mu A^\mu(x) \left(\partial_\nu A^\nu(o)\right)^+ \right| 0 \right\rangle \quad ,$$

$$\Pi_5^{\mu\nu}(q^2) = i \int d^4 x\, e^{iqx} \left\langle 0 \left| \mathbb{T}\, A^\mu(x)\, (A^\nu(o))^+ \right| 0 \right\rangle \quad . \tag{1.16b}$$

At $q = 0$, the identity (1.16) reduces to :

$$\psi_5(0) = -i\, (m_u + m_d) \left\langle 0 \left| \left[\bar{\psi}_d(o)\, i\, \gamma_5\, \psi_u(o)\; Q_5^+\right] \right| 0 \right\rangle , \tag{1.17a}$$

where Q_5 is the axial-charge generator. In the Nambu-Goldstone realization of chiral symmetry $Q_5 \mid 0 \rangle \neq 0$. Then we get

$$\psi_5(0) = -(m_u + m_d) \left\langle \bar{\psi}_d\, \psi_d + \bar{\psi}_u\, \psi_u \right\rangle . \tag{1.17b}$$

Using (1.15b) in the definition of $\psi_5(q^2)$ and equating this with (1.17b), we have the pion PCAC relation at $q = 0$,

$$2\, m_\pi^2\, f_\pi^2 = -(m_u + m_d) \left\langle \bar{\psi}_u \psi_u + \bar{\psi}_d \psi_d \right\rangle \ . \qquad (1.18)$$

We also know from experiments that the spectrum of the pseudo-scalar bosons octet (π, K, η) does not show any degeneracy in the masses. This suggests a large explicit breaking of the $SU(3)_L \times SU(3)_R$ chiral flavour group à la Gell-Mann, Oakes and Renner[12] :

$$\mathcal{L}_{GOR} = -\epsilon_0 U_0(x) - \epsilon_8 U_8(x) - \epsilon_3 U_3(x) \quad , \qquad (1.19a)$$

where the Hermitian scalar densities $U_a(x)$ expressed in terms of quark bilinears read :

$$U_a(x) = \mathrm{Tr}\, \bar{\psi} \lambda_a \psi \quad , \qquad a = 0,3,8$$

with (1.19b)

$$\lambda_0 = \sqrt{\frac{2}{3}} \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -2 \end{pmatrix} ,$$

and where the symmetry-breaking parameters ϵ_a are combinations of the quark masses :

$$\epsilon_0 = \frac{1}{\sqrt{2}} (m_u + m_d + m_s) \quad ,$$

$$\epsilon_3 = \frac{1}{2} (m_u - m_d) \quad ,$$

$$\epsilon_8 = \frac{1}{\sqrt{3}} \cdot \frac{1}{2} (m_u + m_d - 2\, m_s) \ . \qquad (1.19c)$$

Therefore, it is essential to have good control of the quark mass values and a deviation from the SU(3) symmetrical relation in (1.14). In fact, there are estimates of the quark-mass ratios from current algebra [13,14], which are reliable (with perhaps the exception of the up quark) as the mass ratio is not renormalized. On the other hand, it is much more difficult to estimate the absolute values of the quark masses and similarly of the vacuum condensate which is correlated to it via the PCAC relation in (1.18). This reliability needs a consistent renormalization framework rendered possible only with the advent of QCD as we shall see later on.

2. PRE-QCD CURRENT ALGEBRA WEINBERG SUM RULES

Spectral function sum rules were used long before the advent of QCD. They are usually known as dispersion sum rules in current algebra[8] and most of them are based on the assumed asymptotic behaviour of the absorptive amplitudes. Let us discuss in detail two types of superconvergent sum rules :

a) Asymptotic realizations of $SU(2)_L \times SU(2)_R$ chiral symmetry

Weinberg has proposed two sum rules[15] (WSR) based on the belief that the $SU(2)_L \times SU(2)_R$ chiral symmetry is realized asymptotically in nature. In modern QCD language, his discussion is based on the fact that the axial-vector current has two QCD realizations: the first one is the short-distance realization in terms of the quark fields :

$$\left\langle 0 \,\middle|\, \bar{d}\, \gamma^\mu \gamma^5 u \,\middle|\, \pi \right\rangle = \sqrt{2}\, f_\pi\, p^\mu \quad , \qquad (1.20)$$

where f_π = 93.3 MeV is the pion decay constant and p^μ is its momentum. The long-distance realization of the axial current is obtained from the chiral Lagrangian of the non-linear σ model :

$$\mathcal{L}_\sigma = -\frac{1}{4} f_\pi^2 \,\mathrm{Tr}\left\{\partial_\mu U\, \partial^\mu U^+\right\} \quad , \qquad (1.21)$$

where $U = \exp\left(i\, \vec{\tau}.\vec{\pi}\, | f_\pi\right)$ is the pion rotation matrix, $\vec{\pi}$ the pion field and τ the Pauli matrices. Now one can show how the WSR connect these two realizations. It is appropriate to study the two-point correlator :

$$W^{\mu\nu}_{LR} = i \int d^4 x\, e^{iqx} \left\langle 0 \,\middle|\, \mathbb{T}\, J^{\mu}_{L}(x) \left(J^{\nu}_{R}(o)\right)^{+} \middle|\, 0 \right\rangle$$

$$= -\left(g^{\mu\nu} q^2 - q^{\mu}q^{\nu}\right) \Pi^{(1)}_{LR}(q^2) + q^{\mu}q^{\nu}\, \Pi^{(o)}_{LR}(q^2) \,, \qquad (1.22)$$

where J^{μ}_{L} and J^{ν}_{R} are left- and right-handed currents which read in terms of the quark fields :

$$J^{\mu}_{L} \equiv \bar{u}\, \gamma^{\mu}(1-\gamma_5)\, d \quad , \quad J^{\mu}_{R} \equiv \bar{u}\, \gamma^{\mu}(1+\gamma_5)\, d \quad . \qquad (1.23)$$

$\Pi^{(1)}$ and $\Pi^{(o)}$ are the transverse and longitudinal parts of the correlator. In the asymptotic ($q^2 \longrightarrow \infty$) or chiral limit ($m_{u,d} = 0$) where $SU(2)_L \times SU(2)_R$ chiral symmetry is realized, the asymptotic expression of $W^{\mu\nu}_{LR}$ is zero. This vanishing of $W^{\mu\nu}_{LR}$ can be expressed in terms of two WSR of the absorptive parts simply using the well-known Hilbert representation issued from the analyticity properties of Green's function :

$$\mathrm{Re}\,\Pi_{LR}(q^2) = \frac{1}{\pi}\int_0^{\infty} \frac{dt}{t-q^2-i\epsilon}\, \frac{1}{\pi}\, \mathrm{Im}\,\Pi_{LR}(t) + \text{"subtraction..."} \,. \qquad (1.24)$$

Then, the two famous WSR read :

$$\int_0^{\infty} dt\; \mathrm{Im}\left(\Pi^{(1)}_{LR} + \Pi^{(o)}_{LR}\right)(t) = 0 \quad , \qquad (1.25)$$

$$\int_0^{\infty} dt\; t\; \mathrm{Im}\,\Pi^{(1)}_{LR}(t) = 0 \quad , \qquad (1.26)$$

where the first is the $q^{\mu}q^{\nu}$ component of $W^{\mu\nu}_{LR}$ and the second its $g^{\mu\nu}$ parts. Eqs (1.25) and (1.26) express a duality between the long-range (spectral function) and high-energy (theory) parts of hadrons. The spectral function appearing in the sum rules can be studied using the long-distance behaviour of the axial and vector currents :

$$A^{\mu}(x) = -\sqrt{2}\, f_{\pi}\, \partial_{\mu} \vec{\pi} + \left(\sqrt{\frac{2}{3}}\right) [\vec{\pi}, \vec{\pi}\, \partial^{\mu} \vec{\pi}] \Big/ f_{\pi} + \dots ,$$

$$V^{\mu}(x) = i\, \vec{\pi}\, \partial_{\mu} \vec{\pi} \qquad . \qquad (1.27)$$

Possible final-state interactions between pseudoscalar particles can lead to the formation of resonances having the quantum numbers 1^{--}, 1^{++}, 0^{-+} and 0^{++}. Using a narrow-width approximation and assuming that the π, A_1 and ρ dominate the spectral functions, Weinberg has derived from (1.25) and (1.26) the constraints :

$$\frac{M_{\rho}^2}{2\gamma_{\rho}^2} - \frac{M_{A_1}^2}{2\gamma_{A_1}^2} - 2\, f_{\pi}^2 = 0 \quad ,$$

$$\frac{M_{\rho}^4}{2\gamma_{\rho}^2} - \frac{M_{A_1}^4}{2\gamma_{A_1}^2} = 0 \quad , \qquad (1.28)$$

where γ_V is the V-meson coupling to the corresponding current :

$$\left\langle 0\, | V^{\mu} |\, \rho \right\rangle = \sqrt{2}\, \frac{M_{\rho}^2}{2\gamma_{\rho}}\, \epsilon^{\mu} \quad , \qquad (1.29a)$$

with the normalization :

$$\Gamma_{\rho \to e^+ e^-} \simeq \frac{2}{3}\, \pi\, \alpha^2\, \frac{M_{\rho}}{2\gamma_{\rho}^2} \quad . \qquad (1.29b)$$

From the above crude assumptions, one can already deduce from (1.28) a prediction of the A_1 mass by giving $M_{\rho} = 0.77$ GeV, $\gamma_{\rho} \simeq 2.55$ and f_{π} :

$$M_{A_1} \simeq 1.1 \text{ GeV} \quad . \tag{1.30}$$

However, if one adds to (1.28) a relation between f_π, γ_ρ and M_ρ deduced from the use of soft-pion techniques plus ρ-universality for the ρ into ππ decay (the approximate KSFR relation[16]) :

$$f_\pi^2 \simeq \frac{M_\rho^2}{16\gamma_\rho^2} \quad , \tag{1.31}$$

one arrives at the Weinberg mass formula for the A_1 meson :

$$M_{A_1} \simeq \sqrt{2}\, M_\rho \quad , \tag{1.32}$$

which, within the crude approximation used, is very successful compared to the data. Possible QCD improvements of the WSR will be discussed later on.

b) Asymptotic Realizations of $SU(3)_F$ Symmetry

Weinberg-inspired sum rules have been also derived from the asymptotic realization of flavour symmetry. These are the so-called Das-Mathur-Okubo (DMO) sum rules[17]. The DMO sum rules can be studied from the two-point correlator :

$$\Pi_i^{\mu\nu}(q) = i \int d^4 x\, e^{iqx} \left\langle 0 \left| \mathbb{T}\, V_i^\mu(x) \left(V_i^\nu(o)\right)^+ \right| 0 \right\rangle$$

$$\equiv -\,(g^{\mu\nu} q^2 - q^\mu q^\nu)\, \Pi_i(q^2) \quad , \tag{1.33a}$$

where $V_i^\mu \equiv \bar{\psi}_i \gamma^\mu \psi_i$ $(i \equiv u,d,s \ldots)$ are the flavour components of the electromagnetic current :

$$J^\mu_{EM}(x) = \frac{2}{3} V_u^\mu - \frac{1}{3} V_d^\mu + \frac{2}{3} V_c^\mu - \frac{1}{3} V_s^\mu + \ldots \quad . \tag{1.34b}$$

Within the asymptotic ($q^2 \longrightarrow \infty$) or massless ($m_i = 0$) limit, we can derive the DMO sum rule[17]:

$$\int_0^\infty dt\ (\mathrm{Im}\ \Pi_3(t) - \mathrm{Im}\ \Pi_8(t)) \equiv \int_0^\infty dt\ \mathrm{Im}\ (\Pi_u + \Pi_d - 2\ \Pi_s) = 0, \qquad (1.35)$$

which corresponds to the difference between the isovector and isoscalar spectral functions associated with $SU(3)_F$. Saturating (1.35) by the lowest resonance masses, we obtain the well-known successful phenomenological relation among vector mesons :

$$M_\rho\ \Gamma_{\rho \to e^+e^-} - 3 \left(M_\omega\ \Gamma_{\omega \to e^+e^-} + M_\varphi\ \Gamma_{\varphi \to e^+e^-}\right) \simeq 0. \qquad (1.36)$$

An alternative way of writing the sum rules in (1.35) is in terms of the $e^+e^- \longrightarrow$ Hadrons total cross-section which follows from the optical theorem :

$$\sigma_H(t) = \frac{4\pi^2\alpha}{t}\ e^2\ \frac{1}{\pi}\ \mathrm{Im}\ \Pi(t) \qquad (1.37)$$

and which is useful as we have complete data for the total cross-section. Eqs (1.25),(1.26) and (1.35) have shown constraints for low-energy data which follow from the asymptotic behaviour of the spectral functions. These are the prototype sum rules which will be refined and extended within QCD.

3. A SURVEY OF QCD SPECTRAL SUM RULES

Spectral sum rules are different versions and/or improvements of the Hilbert representation in (1.24). For the purposes of more general discussion, let us forget QCD for the moment, i.e. the theoretical side Re $\Pi(q^2)$, and we shall concentrate on the RHS spectral integral.

a) Laplace Transform Sum Rule

This type of sum rule is derived from (1.24) by applying to both sides the inverse Laplace operator[18]: ($Q^2 \equiv -q^2 > 0$)

$$\hat{\mathbb{L}} \equiv \lim_{\substack{Q^2, N\to\infty \\ N/Q^2 \equiv \tau \text{ fixed}}} (-1)^N \frac{(Q^2)^N}{(N-1)!} \frac{\partial^N}{(\partial Q^2)^N} . \tag{1.38}$$

In this case, one gets the exponential form :

$$\hat{\mathbb{L}}\, \Pi = \tau \int_0^\infty dt\; e^{-t\tau} \frac{1}{\pi} \operatorname{Im} \Pi(t) \quad , \tag{1.39}$$

from which one can derive the ratio of moments[19]:

$$R(\tau) = -\frac{d}{d\tau} \log \int_0^\infty dt\; e^{-t\tau} \frac{1}{\pi} \operatorname{Im} \Pi(t) \tag{1.40}$$

or the finite energy like[18]:

$$R_c(\tau) = \frac{\int_0^{t_c} t\; dt\; e^{-t\tau} \frac{1}{\pi} \operatorname{Im} \Pi(t)}{\int_0^{t_c} dt\; e^{-t\tau} \frac{1}{\pi} \operatorname{Im} \Pi(t)} . \tag{1.41}$$

As can be seen in the derivation of the Laplace sum rule, one has to assume that various derivatives exist. For an approximate truncated series like in QCD[20], this existence is satisfied. The advantages of $\hat{\mathbb{L}}\Pi$ are two-fold. Firstly, the use of various derivatives helps to eliminate the subtraction terms in (1.24) which are often polynomials in q^2. Secondly, the exponential factor increases the role of the ground state into the spectral integral if the QSSR variable τ is not too small. This fact is welcome for low-energy physics. The advantage of (1.40) and (1.41) can explicitly be seen if one uses the simple duality ansatz "one resonance" plus "continuum" for parametrizing the spectral function. One can see in this parametrization that these two sum rules give an expression of the mass squared of the ground state. The accuracy of this simple duality ansatz will be tested later on. One can illustrate the sum rule by taking the example of the three-dimensional harmonic oscillator in quantum mechanics[19]. In this case, the RHS of the sum rule in (1.39) reads :

$$F(\tau) = \sum_{n=0,2,4...} (R_n)^2 e^{-E_n \tau} , \quad (1.42)$$

where R_n is the radial wave function for zero angular momentum and E_n the corresponding eigenvalue. τ is the parameter which regulates the energy resolution of the sum rule and plays the role of an "imaginary time" variable. As one has an exact solution of the LHS for the harmonic oscillator potential $V(r) = \frac{1}{2} m \omega^2 r^2$, one can see that in the limit $\tau \to \infty$ the exact expression $R(\tau \to \infty) \equiv -\frac{d}{d\tau} \text{Log } F(\tau)$ tends to the lowest eigenvalues $E_o = \frac{3}{2} \omega$. At finite τ and for a truncated series in τ, one can observe (see Fig. 1.1) that $R_{approx}(\tau)$ stays above the eigenvalue E_o as a consequence of the positivity of R. The agreement between R_{approx} and R_{exact} increases if one adds more and more terms in the τ-expansion. The minimum of R_{approx} provides an upper bound to the value of E_o while the distance between R_{approx} and E_o controls the strength of the continuum to the sum rule. However, by working with a truncated series as in QCD, we do not often have a nice minimum for R_{approx}. This minimum is replaced in some cases by an inflexion point where the optimal information on the resonance properties is obtained. We shall see later on that this previous example mimics the case of QCD quite well.

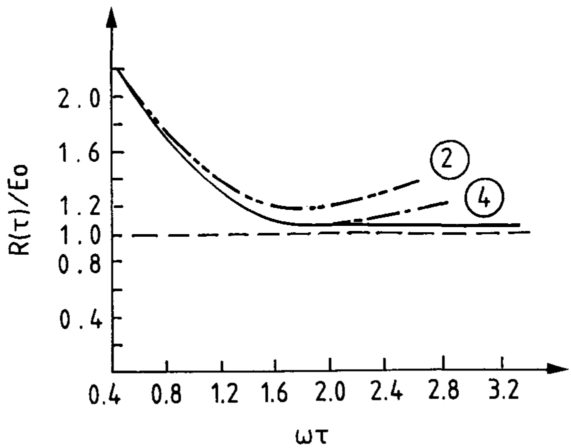


Fig. 1.1 : The ratio of moments normalized to the ground-state energy versus the imaginary time variable for the case of the harmonic oscillator potential. (2) and (4) : approximate series including the second and fourth order terms; ___ exact solution.

b) Finite Energy Sum Rule (FESR)

Another version of QSSR is the FESR :

$$\int_0^{Q^2} dt\ t^n\ \frac{1}{\pi}\ \mathrm{Im}\ \Pi_{Theor}(t) \simeq \int_0^{Q^2} dt\ t^n\ \frac{1}{\pi}\ \mathrm{Im}\ \Pi_{EXP}\ (t) \quad n = 0,1,.. \quad , \quad (1.43)$$

which was known a long time before QCD[21]. Eq.(1.43) can be derived in many ways. A use of the Cauchy theorem (Fig. 1.2) on a finite radius contour in the complex q^2 plane is one way[22] :

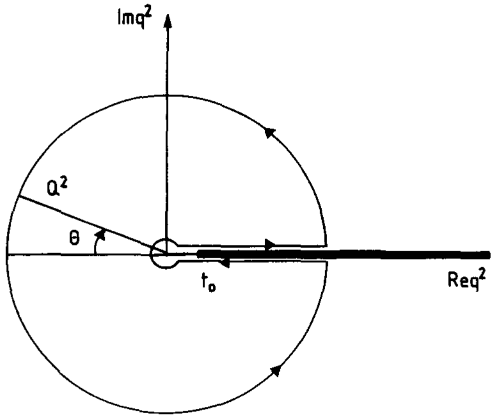


Fig. 1.2 : Cauchy contour in the complex q^2 plane.

$$\frac{1}{2\pi i} \oint dz\ z^n\ \Pi(z) = 0 \quad . \tag{1.44}$$

If one neglects the contribution of the little circle around the origin which is safer if $\Pi(0) = 0$, one deduces the moments :

$$\mathcal{M}^{(n)}(Q^2) = \int_0^{Q^2} dt\ t^n\ \frac{1}{\pi}\ \mathrm{Im}\ \Pi(t) = (-1)^{n+1}\ \frac{(Q^2)^{n-1}}{2\pi} \ .$$

$$\cdot \int_{-\pi}^{+\pi} d\theta\ e^{i(n+1)\theta}\ \Pi(Q^2\ e^{i\theta}) \ , \tag{1.45}$$

where the *LHS* is known from the data and the *RHS* from the theory. However, as the FESR diverges for increasing n, the real axis is dominated by the high Q^2 region. For the RHS to reproduce this correctly, more information on the behaviour of the two-point correlator in the region of the big circle near the cut is needed. This means that more and more non-leading terms in the series expansion become important at large n and can destroy the convergences of the series.

Another way of deriving the FESR which casts light upon the meaning of local duality is the Gaussian sum rule[23] which reads:

$$G(\hat{t},\sigma) = \frac{1}{\sqrt{4\pi\sigma}} \int_0^\infty dt\ e^{-\frac{(t+\hat{t})^2}{4\sigma}} \frac{1}{\pi} \mathrm{Im}\,\Pi(t) \ , \tag{1.46}$$

for a Gaussian centred at $\hat{t}$ with a finite width resolution $\sqrt{4\pi\sigma}$. Eq.(1.46) can be derived by applying the inverse Laplace operator :

$$\hat{\mathbb{L}} \equiv \lim_{\substack{N,\tau^2 \to \infty \\ \frac{N}{\tau^2} \equiv \sigma}} \frac{(-\tau^2)^N}{(N-1)!} \frac{d^N}{(d\tau^2)^N} \ , \tag{1.47}$$

to the already Laplace-transformed quantity :

$$F(\tau) = e^{-\hat{t}\tau}\ \tau^{-1} \int_0^\infty dt\ e^{-t\tau} \frac{1}{\pi} \mathrm{Im}\,\Pi(t) \quad . \tag{1.48}$$

One can already note from (1.46) that in the limit $\sigma = 0$, one has the strict local duality :

$$G(\hat{t},0) = \frac{1}{\pi} \mathrm{Im}\,\Pi(\hat{t}) \quad . \tag{1.49}$$

Also, (1.46) obeys the heat-evolution equation :

$$\left(\frac{\partial^2}{\partial \hat{t}^2} - \frac{\partial}{\partial\sigma}\right) G(\hat{t},\ \sigma) = 0 \quad , \tag{1.50a}$$

with the initial condition in (1.49), where now $\hat{t}$ is the position, σ the time evolution and $\frac{1}{\pi}$ Im $\Pi(t)$ the temperature distribution in the region $0 \leq \hat{t} \leq \infty$. The two boundary conditions for $\sigma > 0$:

$$G(\hat{t} = 0, \sigma) = 0 \quad ,$$
$$\frac{\partial G}{\partial \hat{t}} (\hat{t}, \sigma)\Big|_{\hat{t}=0} = 0 \quad , \qquad (1.50b)$$

lead to two independent solutions $U^-(\hat{t}, \sigma)$ and $U^+(\hat{t}, \sigma)$ where $G(\hat{t}, \sigma) = \frac{1}{2} (U^+ + U^-) (\hat{t}, \sigma)$. These solutions can be expressed in terms of Hermite polynomials. The conservation of the total heat implies the duality relation :

$$\int_{-\infty}^{+\infty} d\hat{t}\; G(\hat{t}, \sigma) = \int_0^{\infty} dt\; \frac{1}{\pi}\, \mathrm{Im}\, \Pi(t) = \int_0^{\infty} d\hat{t}\; U^+(\hat{t}, \sigma) \quad , \qquad (1.51)$$

where the last equality comes from the symmetry properties of $U^+(\hat{t}, \sigma)$. A relation involving higher moments of the spectral function can also be deduced using the generating function of Hermite polynomials and leads to the sum rules :

$$\sigma^n \int_0^{\infty} d\hat{t}\; H_{2n} \left(\hat{t}/_{2\sqrt{\sigma}}\right) U^+(\hat{t}, \sigma) = \int_0^{\infty} dt\; t^{2n}\, \frac{1}{\pi}\, \mathrm{Im}\, \Pi(t) \quad , \qquad (1.52a)$$

$$\sigma^{n+1/2} \int_0^{\infty} d\hat{t}\; H_{2n+1} \left(\hat{t}/_{2\sqrt{\sigma}}\right) U^-(\hat{t}, \sigma) = \int_0^{\infty} dt\; t^{2n+1}\, \frac{1}{\pi}\, \mathrm{Im}\, \Pi(t) \quad , \qquad (1.52b)$$

which only become useful once statements about the restriction to finite intervals can be made. In this case, (1.52) leads to the FESR in (1.43). Finally, the last (but not the least) way of deriving (1.43) is simply to take the coefficient of the τ variable in the two sides of Laplace sum rule in (1.39) [24] . This latter method can be formalized by using the zeta function prescription inspired from the non-relativistic approach[25] . In fact, if H is a Hamilton operator, the associated zeta-function can be written as :

$$\zeta(n) = \frac{1}{\Gamma(n)} \int_0^\infty dt\ \tau^{n-1}\ \mathrm{tr}\ e^{-Ht} \quad , \qquad (1.53a)$$

i.e. in field theory :

$$\zeta(n) = \frac{1}{\Gamma(n)} \int_0^\infty d\tau\ \tau^{n-1} \int_0^\infty dt\ e^{-t\tau}\ \frac{1}{\pi}\ \mathrm{Im}\ \Pi(t) \quad , \qquad (1.53b)$$

where the last integral is the familiar Laplace transform of Im $\Pi(t)$. If this Laplace-transform and its successive derivatives are a series in τ, then, one can easily derive (1.43) by comparing the exact expression of $\zeta(n = 0)$ with its approximate form.

Now, let us return to the FESR in (1.43). Contrary to the Laplace transform (1.39), where the role of the lowest ground state is important, the FESR is governed by the effects of high-mass resonances, i.e. it needs good control of the continuum contributions to the sum rule. In some cases, where a stability in t_c (continuum threshold) does not occur, this is a great disadvantage.

c) Analytical Continuation

Various versions of this method have been discussed in the literature [26]. In most cases, the problem is formulated in terms of norm problems for the input errors and is quite similar to the standard χ^2- minimization used in numerical analysis. More explicitly let us take a simple example. A polynomial in t is used for approximating the $\frac{1}{t-q^2}$ term of (1.24) in the real axis[26a]. Then, applying the Cauchy theorem to the finite Q^2 contour in the complex Q^2 plane, one arrives at the sum rule :

$$\Pi(q^2) = \frac{1}{2i\pi} \oint_C dt \left(\frac{1}{t-q^2} - \sum a_n\ t^n \right) \Pi(t) +$$

$$+ \left[\Delta_n \equiv \frac{1}{\pi} \int_0^{Q^2} dt \left(\frac{1}{t-q^2} - \sum_n a_n t^n \right) \mathrm{Im}\, \Pi(t) \right] , \qquad (1.54)$$

where Δ_n is the "fit error" which should tend to zero, if the result is optimal. An important difference with previous sum rules is that in the RHS the data enters only in Δ_n whilst the main part of $\Pi(q^2)$ is given by its theoretical side. However, it is difficult to appreciate the reliability of the results coming from the method due to the ad hoc uses of the polynomial parametrization (or in general of the kernels in the integrals) and to the strong dependence of the results on the values of the input errors. Moreover, the sum rule in its form (1.54) might also depend on the arbitrary subtraction scale and would be less appropriate for the study of the resonance parameters. Moreover, the way of extrapolating the QCD information up to small q^2 is doubtful. From these weak points all mathematical bagages used to formulate the sum rules might loose their efficiency in the physical uses of the sum rule. More refinements and phenomenological tests of this approach are needed before a definite claim about its advantage can be made.

d) Moment Sum Rules

Sum rules of the type :

$$\mathcal{M}^{(N)} \equiv \frac{(-1)^N}{N!} \frac{d^N}{(dQ^2)^N} \Pi(Q^2) \Big|_{Q^2=Q_0^2} = \int_0^{\infty} \frac{dt}{\left(t+Q_0^2\right)^{N+1}} \frac{1}{\pi} \mathrm{Im}\, \Pi(t) \qquad (1.55)$$

for finite N are often used in the literature. To our knowledge, these sum rules were first discussed by Yndurain[27)] in connection with the study of $e^+e^- \rightarrow$ Hadrons data and used later for heavy-quark systems [28,29)] . As in the case of the Laplace sum rule, one needs to assume that various derivatives of $\Pi(Q^2)$ exist. Also, one can see that for high moments, the role of the ground state is enhanced in the sum rule. Therefore (1.55) is a good candidate for studying the low-energy properties of hadrons as we shall see later on.

We have given a brief general survey of spectral function sum-rule methods which we believe can be applied for a general class of QCD-like theories. As one can see all the methods presented here have their own advantages and disadvantages. For the particular case of QCD where the theory is not yet solved exactly, some questions, though important, such as the existence of high derivatives at high Q^2 as well as a correct and convincing way of estimating the true theoretical systematic errors in the sum rules analysis remain academic. We have checked in a QCD-like model*) such as the non-linear σ model in two dimensions that the high derivatives for a two-point correlator exist unambiguously. Also, one can always test a posteriori whether the assumptions used for the analysis make sense.

In this review, we shall mainly concentrate on the use of the Laplace-transform (1.39-41) and moments (1.55) owing to their sensitivity to the low-energy behaviour of the spectral functions. However, in some cases, we shall also discuss for comparison constraints from FESR (1.43) which complement in many cases the Laplace Sum Rule (LSR) results.

Now, let us come to a study of the perturbative and non-perturbative aspects of QCD, which are the basis of the sum rules approach discussed in this book.

4. $\overline{MS}$ SCHEME FOR PERTURBATIVE QCD**)

a) Renormalization Constants

As in QED, the evaluation of QCD Feynman diagrams leads (in many cases) to divergent results. Finite physical answers need a renormalization of the QCD parameters (vertices, coupling, masses...). However, the renormalization programme of QED[30] cannot be extended trivially to QCD. Here quarks are off-shell and the standard on-shell

*) I wish to thank G. Veneziano for this suggestion.

**)This discussion is mainly based on the Physics Report quoted in Ref. 34).

renormalization and a Pauli-Villars[31)] regularization, which are successful in QED, cannot often be used. In QCD, one uses instead the method of dimensional regularization and renormalization (so-called $\overline{\rm MS}$-scheme[32 to 35)]) which is proven to preserve gauge invariance to all orders of perturbation theory. Its most important feature is the concept of analytical continuation of the dimension of space-time to complex n (n=4 for low-energy space-time). In this approach, the infrared and ultraviolet divergences are transformed into poles in $\epsilon \equiv$ n-4. They are of the form :

$$\sum_{p=1} \frac{Z^{(p)}}{\epsilon^{p}} \tag{1.56}$$

and will appear as counterterms in the initial Lagrangian constrained by the Slavnov-Taylor identities[36)]. For renormalizable theories the $Z^{(p)}$ are constants or polynomials in inverse of the square of some momentum. In QCD, the counterterms of the Lagrangian are :

$$\begin{aligned}
\Delta \mathcal{L}_{QCD} = \; & \Delta_{3YM} \frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\mu A^\nu) \; + \\
& \Delta_{1YM} \frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) \, g \, \vec{A}^\nu \times \vec{A}^\mu \; + \\
& \Delta_5 \frac{1}{4} g^2 \left(\vec{A}_\mu \times \vec{A}_\nu\right)(\vec{A}^\mu \times \vec{A}^\nu) \; - \\
& \Delta_{2F} \, i \sum_j \bar{\psi}_j \gamma^\mu \partial_\mu \psi_j + \Delta_4 \sum_j m_j \bar{\psi}_j \psi_j \\
& - \Delta_{1F} \, g \, \bar{\psi} \frac{\lambda}{2} \gamma^\mu \psi \vec{A}_\mu \; + \\
& \Delta_6 \frac{1}{2\alpha_G} \left(\partial_\mu \vec{A}^\mu\right)^2 + \tilde{\Delta}_3 \left(\partial_\mu \bar{\varphi}\right)^2 + \tilde{\Delta}_1 \, g \, \partial_\mu \vec{\varphi} A^\mu \times \psi
\end{aligned} \tag{1.57}$$

where $\vec{A}_\mu \times \vec{A}_\nu \equiv f_{abc} A^b_\mu A^c_\nu$. *It is possible to rescale the fields in* such a way that $\mathcal{L}_{QCD} + \Delta \mathcal{L}_{QCD}$ has the form in (1.1) but in terms of "bare" quantities. This manipulation is correlated to the introduction of renormalization constants.

Table 1.1

Dimensions of Couplings and fields

Name	Notation	Dimension
gauge coupling	g	$\frac{1}{2}(4-n)$
quark mass	m_i	1
covariant gauge parameter	α_G	0
fermion field	$\psi_j(x)$	$\frac{1}{2}(n-1)$
gluon field	$A^a_\mu(x)$	$\frac{1}{2}(n-2)$
Fadeev-Popov field	$\varphi^a(x)$	$\frac{1}{2}(n-2)$

Taking into account the dimension obtained in the 4-ϵ world (see Table 1.1) via the mass scale ν, one has relations between renormalized and bare parameters :

$$g^R = \nu^{-\epsilon/2} g^B Z_\alpha^{-1/2} \quad : \quad g^2/4\pi \equiv \alpha_s \quad ,$$
$$m^R_j = m^B_j Z^{-1}_m \quad ,$$
$$\alpha^R_G = \alpha^B_G Z^{-1}_G \quad ,$$
$$\left(\psi^\alpha_j\right)^R = \nu^{\epsilon/2} \left(\psi^j_\alpha\right)^B (Z_{2F})^{-1/2} \quad ,$$
$$A^\mu_R = \nu^{\epsilon/2} \left(A^a_\mu\right)_B (Z_{3YM})^{-1/2} \quad ,$$

$$(\varphi^a)_R = \nu^{\epsilon/2} (\varphi^a)_B \left(\tilde{Z}_3\right)^{-1/2} , \tag{1.58}$$

where $Z_i \equiv 1-\Delta_i$. Introducing renormalization constants for the quark-gluon-quark vertex as

$$\left(g \bar{\psi} A \psi\right)_R = \left(g_B \bar{\psi}_B A_B \psi_B\right) \nu^{\epsilon} Z_{1F}^{-1} , \tag{1.59}$$

and analogously for the three gluon (Z_{1YM}), ghost-gluon-ghost $\left(\tilde{Z}_1\right)$ and four-gluon (Z_5) vertices one can deduce :

$$\begin{aligned} g_B^{YM} &= Z_{1YM} Z_{3YM}^{-3/2} g_R , \\ \tilde{g}_B &= \tilde{Z}_1^{-1} Z_{3YM}^{-1/2} g_R , \\ g_B^F &= Z_{1F} Z_{3YM}^{-1/2} Z_{2F}^{-1} g_R , \\ \left(g_B^{(5)}\right)^2 &= Z_5 Z_{3YM}^{-2} g_R^2 , \end{aligned} \tag{1.60}$$

which are related to each over by BRS[6] invariance :

$$g_{YM}^B = \ldots\ldots = g_B^{(5)} , \tag{1.61a}$$

leading to the Slavnov-Taylor[36] identities :

$$Z_{3YM}/Z_{1YM} = \tilde{Z}_3/\tilde{Z}_1 = Z_{2F}/Z_{1F} , \tag{1.61b}$$

and

$$Z_5 = Z_{1YM}^2/Z_{3YM} . \tag{1.62}$$

The mass renormalization constant is :

$$m_B = \left(Z_m \equiv Z_4 Z_{2F}^{-1}\right) m_R , \tag{1.63}$$

and the gauge one is :

$$\alpha_G^B = \alpha_G^R \; Z_G^{-1} \; Z_{3YM} \quad . \tag{1.64}$$

More generally, for a Green's function with n_{YM}, $\tilde{n}$ and n_F external gluons, ghost and fermion fields, one can associate the renormalization constants :

$$Z_\Gamma = \left(Z_{3YM}^{1/2}\right)^{-n_{YM}} \left(\tilde{Z}_3^{1/2}\right)^{-\tilde{n}} \left(Z_{2F}^{1/2}\right)^{-n_F} \quad . \tag{1.65}$$

Expressions of these renormalization constants and the corresponding anomalous dimensions are known from standard diagram techniques (see Table 1.2).

Table 1.2 : Anomalous dimension $\gamma_i = \dfrac{\nu}{Z_i} \dfrac{dZ_i}{d\nu}$ *in the t'Hooft scheme for* $SU(N)_c \times SU(n)_F$

Fermion field $\quad \gamma_{2F} = \left(\dfrac{\alpha_s}{\pi}\right) \dfrac{N^2-1}{2N} \dfrac{\alpha_G}{2} + \mathcal{O}\left(\dfrac{\alpha_s}{\pi}\right)^2$

Gluon field $\quad \gamma_{3YM} = -\left(\dfrac{\alpha_s}{\pi}\right) \left\{\dfrac{N}{4}\left(\dfrac{13}{3} - \alpha_G\right) - \dfrac{2}{3}\left(\dfrac{1}{2}\right)n\right\}$

Ghost field $\quad \tilde{\gamma}_3 = -\left(\dfrac{\alpha_s}{\pi}\right) \dfrac{N}{8} (3 - \alpha_G)$

Mass $\quad \gamma_m = (\gamma_1 \equiv 2)\left(\dfrac{\alpha_s}{\pi}\right) + \left(\gamma_2 \equiv \dfrac{1}{6}\left(\dfrac{101}{2} - \dfrac{5n}{3}\right)\right)\left(\dfrac{\alpha_s}{\pi}\right)^2$

$$+ \left(\gamma_3 \equiv \frac{1}{128}\left[1249 + \left(\frac{160}{3}\,\zeta(3) - \frac{2216}{27}\right)n - \frac{140}{81}\,n^2\right]\right)\left(\frac{\alpha_s}{\pi}\right)^3 \quad \text{for } N=3$$

Coupling constant $\beta(\alpha_s) = \frac{\nu}{\alpha_s}\frac{d\alpha_s}{d\nu} = -\frac{\nu}{Z_\alpha}\frac{dZ_\alpha}{d\nu}$

$$= \left[\beta_1 \equiv -\frac{1}{2}b_1 = \frac{1}{2}\left(-11 + \frac{2}{3}n\right)\right]\left(\frac{\alpha_s}{\pi}\right) + \left[\beta_2 \equiv -\frac{1}{8}b_2 = -\frac{1}{4}\left(51 - \frac{19}{3}n\right)\right]\left(\frac{\alpha_s}{\pi}\right)^2$$

$$+ \left[\beta_3 = -\frac{1}{32}\left(\frac{2857}{2} + \frac{325}{54}n^2 - \frac{5033}{18}n\right)\right]\left(\frac{\alpha_s}{\pi}\right)^3 \quad \text{for } N=3$$

Gauge $\beta_G = \nu\frac{d\alpha_G}{d\nu} = -\alpha_G\gamma_{3YM}$

Three-gluon : $\gamma_{1YM} = -\left(\frac{\alpha_s}{\pi}\right)\left\{\left(\frac{17}{6} - \frac{3}{2}\alpha_G\right)\frac{N}{4} - \frac{2}{3}\left(\frac{1}{2}\right)n\right\}$

Ghost-gluon-ghost : $\tilde{\gamma}_1 = \left(\frac{\alpha_s}{\pi}\right)\alpha_G\frac{N}{4}$

Fermion-gluon-fermion : $\gamma_{1F} = \left(\frac{\alpha_s}{\pi}\right)\frac{1}{2}\left\{\frac{N}{4}(3 + \alpha_G) - \alpha_G\frac{N^2-1}{2N}\right\}$

---

i) Z_{2F} and Z_m come from the evaluation of the quark self-energy diagram parametrized as :

$$\Sigma = m_B\,\Sigma_1 + \left(\hat{p} - m_B\right)\Sigma_2 \equiv P \qquad + \ldots, \qquad (1.66a)$$

where the full unrenormalized propagator reads :

$$S_F = \left(\frac{1}{1-\Sigma_2}\right) \frac{1}{\hat{p} - m_B(1 + \Sigma_1/(1-\Sigma_2))} \; . \qquad (1.66b)$$

Then

$$Z_{2F} = \frac{1}{(1-\Sigma_2)\text{pole}} \qquad Z_m = 1 - \Sigma_1 \Big|_{\text{pole}} \; . \qquad (1.66c)$$

Expressions of Σ_1^B and Σ_2^B derived from the rules and properties in appendix A read :

$$\Sigma_1^B = \left(g_B \cdot \nu^{-\epsilon/2}\right)^2 \frac{N^2-1}{2N} \frac{1}{(16\pi^2)^{1-\epsilon/4}} \int_0^1 dx \left\{\Gamma\left(\frac{\epsilon}{2}\right)\left(\frac{\mathbb{R}^2}{\nu^2}\right)^{-\epsilon/2}\right.$$

$$\left. \times \left[2(2-x) - \epsilon(1-x) + (1-\alpha_G)(1-2x)\right] + (1-\alpha_G)\; 2x(1-x) \frac{p^2}{m_B^2 - p^2 x}\right\} . \quad (1.67a)$$

with $\mathbb{R}^2 = (1-x)\left(m_B^2 - p^2 x\right) - i\epsilon'$.
Then, (see e.g. Ref. 33) :

$$\Sigma_1^B = \left(\frac{\alpha_s}{\pi}\right) \frac{N^2-1}{2N} \frac{1}{2} \left\{ \frac{3}{2}\left(\frac{2}{\epsilon}\right) + \frac{3}{2}(\log 4\pi - \gamma) + \frac{5}{2}\right.$$

$$+ \frac{3}{2} \log \frac{\nu^2}{m_B^2 - p^2} + \left(\frac{1}{2}\right)\frac{m_B^2}{-p^2} - \left(\frac{1}{2}\right)\frac{m_B^2}{-p^2}\left(4 + \frac{m_B^2}{-p^2}\right) \log\left(1 - \frac{p^2}{m_B^2}\right)$$

$$\left. + (1-\alpha_G)\left[-\frac{1}{2} - \left(\frac{1}{2}\right)\frac{m_B^2}{-p^2} + \left(\frac{1}{2}\right)\frac{m_B^2}{-p^2}\left(1 + \frac{m_B^2}{-p^2}\right)\log\left(1 - \frac{p^2}{m_B^2}\right)\right]\right\}, \qquad (1.67b)$$

where here and in the following $\gamma \equiv \gamma_E = 0.5772...$ denotes the Euler constant.

Also

$$\Sigma_2^B = \left(g_B . \nu^{-\epsilon/2}\right)^2 \frac{N^2-1}{2N} \frac{1}{(16\pi^2)^{1-\epsilon/4}} \int_0^1 dx \left\{ \Gamma\left(\frac{\epsilon}{2}\right) \left(\frac{\mathbb{R}^2}{\nu^2}\right)^{-\epsilon/2} \right.$$

$$\left. \times \left[-2x + \epsilon x + (1-\alpha_G)\, 2(1-x)\right] + (1-\alpha_G)\, 2x(1-x) \frac{p^2}{m_B^2 - p^2 x} \right\} \quad (1.67c)$$

and then

$$\Sigma_2^B = \left(\frac{\alpha_s}{\pi}\right) \frac{N^2-1}{2N} \left(\frac{1}{4}\right) \left[-1 + (1-\alpha_G)\right] \left\{ \frac{2}{\epsilon} + \log 4\pi - \gamma + 1 - \log \frac{m_B^2 - p^2}{\nu^2} \right.$$

$$\left. + \left(\frac{m_B^2}{-p^2}\right)^2 \log\left(1 - \frac{p^2}{m_B^2}\right) - \frac{m_B^2}{-p^2} \right\}, \quad (1.67d)$$

which shows that Σ_2 vanishes, in the Landau gauge ($\alpha_G = 0$), at order α_s/π. For completeness, we give the asymptotic expressions of Σ_1^B and Σ_2^B.

For Σ_1, one has

$$\Sigma_1^B \Big|_{-p^2 \gg m^2} = \left(\frac{\alpha_s}{\pi}\right) \frac{N^2-1}{2N} \left(\frac{1}{2}\right) \left\{ \frac{3}{\epsilon} + \frac{3}{2} (\log 4\pi - \gamma) + \frac{5}{2} - \frac{3}{2} \log \frac{-p^2}{\nu^2} \right.$$

$$+ (1 - \alpha_G)\left(-\frac{1}{2}\right) + \mathcal{O}\left(\frac{m^2}{-p^2}\log\frac{-p^2}{m^1}\right)\Bigg\}, \qquad (1.67e)$$

$$\Sigma_1^B\Big|_{-p^2 \ll m^2} = \left(\frac{\alpha_s}{\pi}\right)\frac{N^2-1}{2N}\left(\frac{1}{2}\right)\Bigg\{\frac{3}{\epsilon} + \frac{3}{2}(\log 4\pi - \gamma) + \frac{3}{2}\log\frac{\nu^2}{m_B^2} + \frac{3}{4}$$

$$+ \frac{5}{6}\left(\frac{-p^2}{m_B^2}\right) + (1-\alpha_G)\left(-\frac{1}{4} - \frac{1}{12}\left(\frac{-p^2}{m_B^2}\right)\right)\Bigg\}. \qquad (1.67f)$$

In the time-like region, one has :

$$\Sigma_1^B\Big|_{p^2=m^2=\nu^2} = \left(\frac{\alpha_s}{\pi}\right)\frac{N^2-1}{2N}\left(\frac{1}{2}\right)\left\{\frac{3}{\epsilon} + \frac{3}{2}(\log 4\pi - \gamma) + 2\right\}, \qquad (1.67g)$$

where $\Sigma_1^B\Big|_{p^2=m^2=\nu^2}$ is gauge-independent and is related to the mass defined at the pole of the fermion propagator (see section 2). For Σ_2 one gets

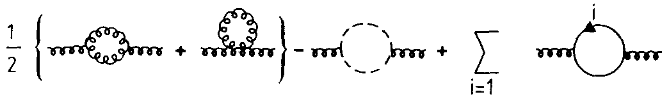
$$\Sigma_2^B\Big|_{-p^2 \gg m^2} = \left(\frac{\alpha_s}{\pi}\right)\frac{N^2-1}{2N}\left(\frac{1}{4}\right)[-1+(1-\alpha_G)]\Bigg\{\frac{2}{\epsilon} + \log 4\pi - \gamma + 1 - \log\frac{-p^2}{\nu^2}$$

$$+ \mathcal{O}\left(\frac{m^2}{p^2}\right)^2\log\frac{-p^2}{m^2}\Bigg\}, \qquad (1.67h)$$

$$\Sigma_2^B\Big|_{-p^2 \ll m^2} = \left(\frac{\alpha_s}{\pi}\right) \frac{N^2-1}{2N} \left(\frac{1}{4}\right) [-1 + (1-\alpha_G)] \left\{\frac{2}{\epsilon} + \log 4\pi - \gamma + \log \frac{\nu^2}{m^2} + \frac{1}{2} - \frac{2}{3}\left(\frac{-p^2}{m_B^2}\right) + \mathcal{O}\left(\frac{-p^2}{m_B^2}\right)^2 \right\}. \quad (1.67i)$$

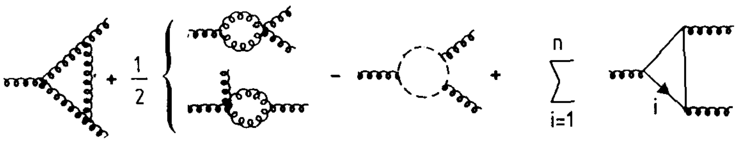
Note that the pole in $1/\epsilon$ is constant according to the theorem given in Ref.32) for a renormalizable theory.

ii) Z_{3YM} comes from the gluon propagator :



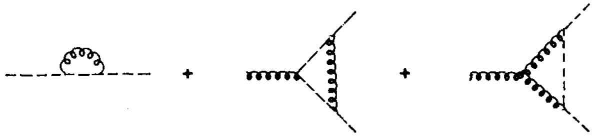
(1.68)

iii) Z_{1YM} comes from the three-gluon vertex :



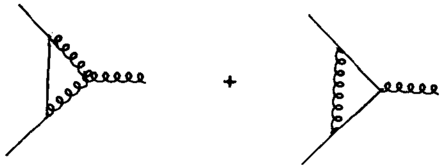
(1.69)

iv) $\tilde{Z}_1$ and $\tilde{Z}_3$ come respectively from the ghost self-energy and ghost--gluon-ghost vertex



(1.70)

v) Z_{1F} is obtained from the fermion-gluon-fermion vertex



(1.71)

Once we have these previous renormalization constants, we can for instance deduce that of the coupling constant :

$$Z_\alpha = Z_{1YM}^2 \; Z_{3YM}^{-3} = 1 - \left(\frac{\alpha_s}{\pi}\right)\left(\frac{11}{3}\frac{N}{2} - \frac{2}{3}\frac{n}{2}\right)\frac{1}{\epsilon} + \ldots \qquad (1.72)$$

b) **Renormalization Group Equation (RGE) :**

Now, we are ready to study the renormalization group equation introduced by Stueckelberg and Peterman in the contex of QED[37].
Let the renormalized Green's function be:

$$\Gamma_R(\nu, p_1 \ldots p_N, g, \alpha_G, m_j) = Z_\Gamma \; \Gamma_B(p_1 \ldots p_N, g, \alpha_G, m_j) \quad . \qquad (1.73)$$

The fact that Γ_B is independent of ν implies the disappearance of the total derivative $\nu \dfrac{d\Gamma_B}{d\nu} = 0$ which is equivalent to :

$$\left\{ \nu \frac{\partial}{\partial \nu} + \nu \frac{d\alpha_s}{d\nu}\frac{\partial}{\partial \alpha_s} + \sum_j \frac{\nu}{m_j}\frac{dm_j}{d\nu} m_j \frac{\partial}{\partial m_j} + \nu \frac{d\alpha_s}{d\nu}\frac{\partial}{\partial \alpha_G} - \right.$$

$$- \frac{1}{Z_\Gamma} \nu \frac{dZ_\Gamma}{d\nu} \Bigg\} \Gamma_R \; (\nu, \; p_1 \ldots p_N; g \ldots) = 0. \qquad (1.74)$$

One can introduce the universal parameters β function β_i and anomalous dimension γ_i defined as :

$$\alpha_s \beta(\alpha_s) = \nu \frac{d\alpha_s}{d\nu}\bigg|_{g_B,\, m_B \text{ fixed}} ; \qquad \beta_G(\alpha_s) = \nu \frac{d\alpha_G}{d\nu}\bigg|_{g_B,\, m_B \text{ fixed}}$$

$$\gamma_m = - \frac{\nu}{m^R} \frac{dm^R}{d\nu}\bigg|_{g_B,\, m_B \text{ fixed}} ; \qquad \gamma_{2F} = \frac{\nu}{Z_{2F}} \frac{dZ_{2F}}{d\nu} ,$$

$$\gamma_{3YM} = \frac{\nu}{Z_{3YM}} \frac{d\, Z_{3YM}}{d\nu} ; \qquad \tilde{\gamma}_3 = \frac{\nu}{\tilde{Z}_3} \frac{d\tilde{Z}_3}{d\nu} ,$$

$$\gamma_\Gamma = \frac{\nu}{Z_\Gamma} \frac{dZ_\Gamma}{d\nu} = - \frac{1}{2} \left(n_{YM} \gamma_{3YM} + n_F \gamma_{2F} + \tilde{n}\, \tilde{\gamma}_3 \right) , \qquad (1.75)$$

which transforms (1.74) into the renormalization group equation (RGE) :

$$\left\{ \nu \frac{\partial}{\partial \nu} + \beta(\alpha_s)\, \alpha_s \frac{\partial}{\partial \alpha_s} - \sum_j \gamma_m(\alpha_s)\, m_j \frac{\partial}{\partial m_j} + \beta_G \frac{\partial}{\partial \alpha_G} - \gamma_\Gamma \right\} \Gamma^R = 0 \quad . \qquad (1.76)$$

The expressions of the above universal parameters can be easily obtained from (1.75) and the values of Z_i in Table 1.2. However, noting that $\beta(\alpha_s)$ is independent of the fermion mass in the $\overline{MS}$-scheme, one can write :

$$\alpha_s \;\beta(\alpha_s, \epsilon) = \nu \frac{d\, \alpha_s^R}{d\nu} = \nu \frac{d}{d\nu} \left(\alpha_s^B \, \nu^{-\epsilon} \, Z_\alpha^{-1} \right)$$

$$= - \epsilon \, \alpha_s - \alpha_s^R \frac{1}{Z_\alpha} \nu \frac{d \, Z_\alpha}{d\nu} \, . \tag{1.77}$$

The fact that Z_α in (1.72) is ν-independent allows one to write

$$\left\{ \alpha_s^R \, \beta(\alpha_s, \epsilon) + \epsilon \, \alpha_s^R + \left(\alpha_s^R\right)^2 \beta(\alpha_s, \epsilon) \frac{\partial}{\partial \alpha_s^R} \right\} Z_\alpha = 0 \quad , \tag{1.78}$$

which inserted into the expression of Z_α in terms of $\frac{1}{\epsilon}$ poles (Eq. 1.56) gives the differential equation :

$$\alpha_s^R \, \beta(\alpha_s, \epsilon) = - \epsilon \, \alpha_s^R + \left(\text{finite term} \equiv \alpha_s^R \, \beta(\alpha_s) \right) \tag{1.79}$$

and then :

$$\beta(\alpha_s) = \alpha_s \frac{\partial \, Z_\alpha^{(1)}}{\partial \, \alpha_s} \quad , \tag{1.80}$$

i.e. $\beta(\alpha_s)$ is just the coefficient of the $\frac{1}{\epsilon}$ term of Z_α.
The same reasoning applies to the anomalous dimensions, i.e with the sign convention used in (1.75), they are the opposite of the $\frac{1}{\epsilon}$ coefficient of the corresponding renormalization constant. Gauge and scheme invariance of the β and γ functions can also be proven [33)34)].

Let us now solve the RGE in (1.76). If D is the dimension of Γ in units of mass and we scale the momenta $p_1 \ldots p_N$ by a dimensionless factor λ, the Euler theorem on homogeneous function gives :

$$\left\{ \lambda \frac{\partial}{\partial \lambda} + \sum_j m_j \frac{\partial}{\partial m_j} + \nu \frac{\partial}{\partial \nu} - D \right\} \Gamma_R \left(\lambda p_1 \ldots \lambda p_N \; ; \; \alpha_s, \alpha_G, \, m_j, \nu \right) = 0. \tag{1.81a}$$

Introducing for convenience the dimensionless variables

$$t \equiv \log \lambda \qquad x_j \equiv m_j/\nu \quad , \tag{1.81b}$$

we arrive at the desired form of the RGE :

$$\left\{ - \frac{\partial}{\partial t} + \beta(\alpha_s)\ \alpha_s \frac{\partial}{\partial \alpha_s} + \beta_G \frac{\partial}{\partial \alpha_G} - \sum_j (1+\gamma_m)\ x_j \frac{\partial}{\partial x_j} + D - \gamma_\Gamma \right\} .$$

$$. \ \Gamma_R \left(e^t p_1 , \ldots , e^t p_B \ ; \ \alpha_s , \ \alpha_G , \ x_j , \ \nu \right) = 0 \quad , \tag{1.82}$$

with the solution :

$$\Gamma^R \left(e^t p_1 \ldots , \ e^t p_N \ ; \ \alpha_s , \ \alpha_G , \ x_j , \ \nu \right) = \lambda^D \ . \ \Gamma \ (p_1 , \ldots , p_N ; t = 0,$$

$$\bar{\alpha}_s , \ \bar{\alpha}_G , \ \bar{x}_j) \ \exp \left\{ - \int_0^t dt' \ \gamma_\Gamma \left(\bar{\alpha}_s \ (t' , \ \alpha_s) \right) \right\} . \tag{1.83}$$

One should note that the Green's function has acquired an extra dimension induced by the exponential factor. This is why it is called anomalous dimension.

c) **Running Parameters**

$\bar{\alpha}_s(t)$, $\bar{x}_i(t)$ and $\bar{\alpha}_G(t)$ of (1.83) are respectively the running-coupling, mass and gauge solutions of the differential equations :

$$\frac{d\bar{\alpha}_s}{dt} = \bar{\alpha}_s \ \beta\left(\bar{\alpha}_s\right) \quad : \quad \bar{\alpha}_s(0, \ \alpha_s) = \alpha_s^R(\nu) \quad , \tag{1.84}$$

$$\frac{d\bar{x}_i}{dt} = - \left[1 + \gamma_m\left(\bar{\alpha}_s\right) \right] \bar{x}_i(t) \quad : \quad \bar{x}_i(0, \ \alpha_s) = x_i^R(\nu) \quad , \tag{1.85}$$

$$\frac{d\bar{\alpha}_G}{dt} = \beta_G\left(\bar{\alpha}_s\right) \quad : \quad \bar{\alpha}_G(0,\ \alpha_s) = \alpha_G(\nu)\ . \tag{1.86}$$

The solutions of the two first equations read to two loops[34]:

$$\bar{\alpha}_s = \bar{\alpha}_s^{(1)} \left\{1 - \frac{\bar{\alpha}_s^{(1)}}{\pi} \frac{\beta_2}{\beta_1} \log \log \frac{-q^2}{\Lambda^2}\right\}\ , \tag{1.87}$$

$$\bar{m}_1 = \bar{m}_1^{(1)} \left\{1 + \left(\gamma_1 \frac{\beta_2}{\beta_1^2} \log \log \frac{-q^2}{\Lambda^2} - \frac{1}{\beta_1}\left(\gamma_2 - \gamma_1 \frac{\beta_2}{\beta_1}\right)\right) \frac{\bar{\alpha}_s^{(1)}}{\pi}\right\}\ , \tag{1.88}$$

with : $\bar{\alpha}_s^{(1)} = \pi/-\left(\beta_1 \log \sqrt{-q^2}/\Lambda\right)$ the one loop solution of (1.84) ;

$\gamma_m(\alpha_s) \equiv \gamma_1 \dfrac{\alpha_s}{\pi} + \gamma_2 \left(\dfrac{\alpha_s}{\pi}\right)^2$ and $\beta(\alpha_s) \equiv \beta_1 \left(\dfrac{\alpha_s}{\pi}\right) + \beta_2 \left(\dfrac{\alpha_s}{\pi}\right)^2$. Λ is a renormalization group invariant (RGI) but scheme-dependent defined as :

$$\frac{1}{2} \log \nu^2 + \frac{\pi}{\beta_1 \alpha_\sigma(\nu)} - \frac{\beta_2}{\beta_1^2} \log \left(\frac{1 + \beta_2/\beta_1 \dfrac{\alpha_s(\nu)}{\pi}}{\alpha_s(\nu)}\right)$$

$$= \frac{1}{2} \log \Lambda^2 - \frac{\beta_2}{\beta_1^2} \log \left(-\frac{\beta_1}{2\pi}\right)\ . \tag{1.89}$$

Analogously to Λ, one can also introduce the RGI mass $\hat{m}_1$ defined to one loop as[22b)] :

$$\hat{m}_i = m_i(\nu)\left(\frac{\pi}{-\beta_1\ \alpha_s(\nu)}\right)^{\gamma_1/-\beta_1}\quad , \tag{1.90}$$

which is related to the one-loop running mass as :

$$\bar{m}_i^{(1)} = \hat{m}_i \Big/ \left(\log\ \sqrt{-q^2}/\Lambda\right)^{\gamma_1/-\beta_1}\quad . \tag{1.91}$$

One can also introduce a RGI spontaneous mass μ_i associated with the quark-vacuum condensate using the fact that the product $m\langle\bar{\psi}\ \psi\rangle$ entering into the PCAC relation (1.19) does not get renormalized. Then :

$$\langle\bar{\psi}_i\ \psi_i\rangle = -\ \mu_i^3\ \left(\log\ \sqrt{-q^2}/\Lambda\right)^{\gamma_1/-\beta_1}\quad . \tag{1.92}$$

For completeness, we also give $\bar{\alpha}_s(t)$ and $\bar{m}_i(t)$ to three loops and for $SU(3)_F$ where γ_3 and β_3 have been calculated by Tarasov. This calculation is quoted in Ref. 38) :

$$\frac{\bar{\alpha}_s}{\pi} = \frac{4}{9L}\left\{1 - 0.79\ \frac{\log\ L}{L} + 0.62\ \frac{\log^2\ L}{L^2} - 0.62\ \frac{\log\ L}{L^2}\right\}\quad ,$$

$$\bar{m}_i = \hat{m}_i\ \left(\frac{9}{2}\ \frac{\bar{\alpha}_s}{\pi}\right)^{4/9}\ \left\{1 + 0.895\ \frac{\bar{\alpha}_s}{\pi} + 2.707\ \left(\frac{\bar{\alpha}_s}{\pi}\right)^2\right\}\quad , \tag{1.93}$$

with $L \equiv \log - q^2/\Lambda^2$.

Within the previous generalities, we are now ready to study the specific example of the two-point correlator useful for the QSSR analysis.

d) The RGE for the Two-Point Correlator

Let $\Pi\left(q^2, \alpha_s, m_i, \nu\right)$ be a generic notation for the two-point correlator :

$$\Pi(q^2) = i \int d^4x \, e^{iqx} \left\langle 0 \mid \mathbb{T} \, J_H(x) \, (J_H(0))^{+} \mid 0 \right\rangle , \qquad (1.94)$$

built from the hadronic current of quarks and/or gluon bilinear fields. In $n = 4-\epsilon$ dimension, $\Pi(q^2)$ gets an extra $\nu^{-\epsilon}$ dimension. The renormalized two-point correlator is[22b,33,34]:

$$\Pi_R\left(q^2, \alpha_s, m_i, \nu\right) \equiv \Pi_B\left(q^2, \alpha_s^B, m_i^B, \epsilon\right) - \nu^{-\epsilon} \, C\left(q^2, \alpha_s^B, m_i^B, \epsilon\right) , \qquad (1.95)$$

where in the $\overline{MS}$ scheme, C are the ϵ-pole terms :

$$C\left(q^2, \alpha_s^B, m_i^B, \epsilon\right) = \sum_k \frac{1}{\epsilon^k} C_k\left(q^2, \alpha_s, m_i\right) , \qquad (1.96)$$

where as usual C_k are constants or polynomials in m^2/q^2. The fact that Π_B is independent of ν, implies the relation :

$$\left\{\nu \frac{\partial}{\partial \nu} + \beta(\alpha_s) \, \alpha_s \frac{\partial}{\partial \alpha_s} - \sum \gamma_m \, m_i \frac{\partial}{\partial m_i}\right\} \Pi_R\left(q^2, \alpha_s, m_i, \nu\right)$$
$$= - \nu \frac{d}{d\nu} \left(\nu^{-\epsilon} \sum_k \frac{1}{\epsilon^k} C_k\right) . \qquad (1.97)$$

Rewriting :

$$\nu \frac{d}{d\nu} \nu^{-\epsilon} \sum_k \frac{1}{\epsilon^k} C_k = \left\{\nu \frac{\partial}{\partial \nu} + \nu \frac{d\alpha_s}{d\nu} \frac{\partial}{\partial \alpha_s} - \sum \gamma_m \, m_i \frac{\partial}{\partial m_i}\right\} . \, \nu^{-\epsilon} \sum_k \frac{1}{\epsilon^k} C_k \qquad (1.98)$$

and using :

$$\nu \frac{d\alpha_s}{d\nu} = - \epsilon \, \alpha_s + \alpha_s \, \beta(\alpha_s) \quad , \tag{1.99}$$

and the fact that the equation is finite for $\epsilon \to 0$, one gets :

$$\lim_{\epsilon\to 0} \nu \frac{d}{d\nu} \left(\nu^{-\epsilon} \sum_k \frac{1}{\epsilon^k} C_k \right) = - \frac{\partial}{\partial\alpha_s} (\alpha_s \, C_1) \quad , \tag{1.101a}$$

and the set of recursive equations for $k > 1$:

$$\left(\alpha_s \, \beta(\alpha_s) - \sum_i \gamma_m \, m_i \frac{\partial}{\partial m_i} \right) C_k = \frac{\partial}{\partial\alpha_s} (\alpha_s \, C_{k+1}) \quad . \tag{1.101b}$$

The dimensionless condition of Π reads :

$$\left\{ \nu \frac{\partial}{\partial\nu} + \lambda \frac{\partial}{\partial\lambda} + \sum_i m_i \frac{\partial}{\partial m_i} \right\} \Pi \left(\lambda^2 \nu^2 , \, \alpha_s , m , \nu \right) = 0 \quad , \tag{1.102}$$

where $t \equiv \log \lambda$. Therefore, one arrives at the RGE for the two-point correlator :

$$\left\{ - \frac{\partial}{\partial t} + \beta(\alpha_s) \, \alpha_s \frac{\partial}{\partial\alpha_s} - \sum_i (1+\gamma_m) \, x_i \frac{\partial}{\partial x_i} \right\} \Pi(t, \, \alpha_s , x_i) =$$

$$\frac{\partial}{\partial\alpha_s} (\alpha_s \, C_1) \equiv D_s \quad , \tag{1.103}$$

with the solution :

$$\Pi(t, \alpha_s , \, x_i) = \Pi \left(t=0, \, \bar{\alpha}_s(t) , \, \bar{x}_i(t) \right) -$$

$$\int_0^t dt' \, D_s \left[t-t' , \, \bar{\alpha}_s(t') , \, \bar{x}(t') \right] . \tag{1.104}$$

e) Renormalization of composite operators

We have seen from the example of WSR in paragraph 2) that we are dealing with two-point correlators associated with the local current colourless operators $J_H(x)$ built from quarks and/or gluon fields. So, before any QSSR analysis, one should control the renormalization of such operators. This problem can be conveniently studied using background field techniques[39]. Operators can be classified into three classes : class I are gauge invariants which do not vanish after use of the classical equation of motions. Class II are gauge-invariant but vanish after use of the classical equation of motion. Class III are gauge-dependent operators. Therefore any composite renormalized operators can be written as :

$$\mathcal{O} = Z_I\ O^B_I + Z_{II}\ O^B_{II} + Z_{III}\ O^B_{III} \quad . \tag{1.105a}$$

The great advantage of the background field techniques is that for graphs with external quark and background fields, one only needs gauge invariant counterterms, i.e :

$$Z_{III} = 0 \quad , \tag{1.105b}$$

which is a consequence of the background field gauge invariance under quantization and renormalization. Let us illustrate the approach by studying the renormalization of the $G_{\mu\nu}\ G^{\mu\nu}$ gluon operator[40] in the presence of massive quarks. We have three types of dimension-four operators :

$$O_1 = -\frac{1}{4}\, i\ GG \quad ; \quad O_2 = -\ \bar\psi\left(\hat{D}+im\right)\psi \quad ; \quad O_3 = im\ \bar\psi\psi \quad . \tag{1.106}$$

The renormalized O_1 operator is in general a combination of these three "bare" operators :

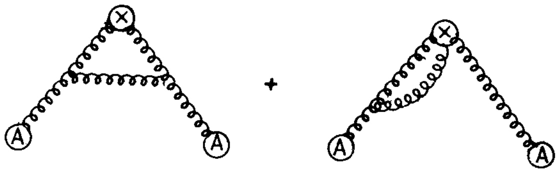
$$O_1 = Z_{11} O_1^B + Z_{12}\ O_2^B + Z_{13}\ O_3^B \qquad (1.107)$$

The renormalization constants Z_{1j} are mass-independent in the $\overline{MS}$ scheme. In particular, one can already get Z_{11} and Z_{12} in the massless limit. In any case, we insert at zero momentum the O_1 operator into the gluon and quark propagators :

$$\left\langle A_a^\mu\ O_1\ A_b^\nu \right\rangle = Z_\alpha^{-1}\ Z_{11} \left\langle A_a^\mu\ O_1^B\ A_b^\nu \right\rangle + Z_{12} \left\langle A_a^\mu\ O_2^B\ A_b^\nu \right\rangle \ ,$$

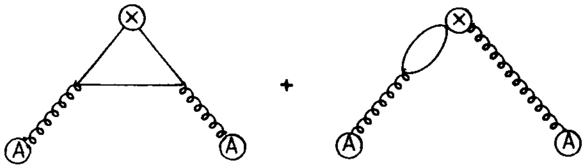
$$\left\langle \bar\psi\ O_1\ \psi \right\rangle = Z_{11} \left\langle \bar\psi\ O_1^B\ \psi \right\rangle + Z_{2F}\ Z_{12} \left\langle \bar\psi\ O_2^B\ \psi \right\rangle + Z_{2F} Z_{13} \left\langle \bar\psi\ O_3^B\ \psi \right\rangle. \quad (1.108)$$

The insertion of O_1^B into the gluon propagator corresponds to the Feynman rule $-i\ \delta_{ab}\left(p^2 g_{\mu\nu} - p_\mu p_\nu\right)$ and one has to evaluate :



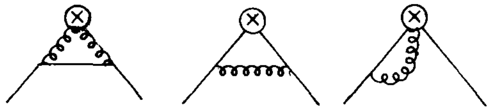
(1.109)

The zero momentum insertion of O_2^B into the gluon propagator corresponds to the rule $i\hat{p}$ and we have to calculate :



(1.110)

The insertion into the quark propagator corresponds respectively to $ig\,\frac{\lambda a}{2}\,\gamma^{\mu}$, $i\left(\hat{p} - m_B\right)$ and im_B for the O_1, O_2 and O_3 operators.



(1.111)

Evaluations of the above diagrams give in the Landau gauge[40] :

$$Z_{11}^{(2)} = Z_{\alpha}^{(2)} \quad ; \quad Z_{12}^{(2)} = 0 \quad ; \quad Z_{13}^{(2)} = -\frac{\gamma_1}{\epsilon}\frac{\alpha_s}{\pi} , \qquad (1.112)$$

as $Z_{2F}^{(2)} = 1$ in the Landau gauge. The index (2) means second order in (α_s/π) . Therefore one has :

$$GG = \left(1 + \frac{\alpha_s}{\pi}\frac{B_1}{\epsilon}\right)(GG)_B + 4\,\frac{\gamma_1}{\epsilon}\frac{\alpha_s}{\pi}\,m_B\left(\bar{\psi}\psi\right)_B , \qquad (1.113)$$

i.e., GG is not multiplicatively renormalizable. In fact, one can deduce from (1.113) the renormalization group invariant combination:

$$\frac{1}{4}\,\beta(\alpha_s)\;GG + \gamma_m\;m\;\bar{\psi}\psi \quad , \qquad (1.114a)$$

which appears in the trace of the energy-momentum tensor :

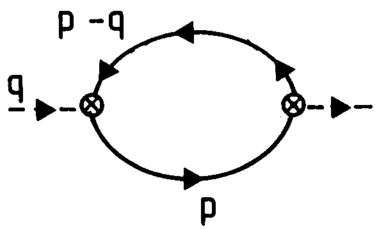
$$\Theta_{\mu}^{\mu} = \frac{1}{4}\,\beta(\alpha_s)\;GG + (1+\gamma_m)\;\bar{\psi}\psi , \qquad (1.114b)$$

where $\beta(\alpha_s)$ and $\gamma_m(\alpha_s)$ are the β function and the mass anomalous dimension respectively. Proofs of the RGI of quantities like the vector current or the (pseudo) scalar currents are straightforward. Much more involved operators can be found in Refs 41), 42) and 48).

f) Pseudoscalar Correlator and Current Algebra Ward Identity

In order to illustrate our discussion of this section, let us show explicitly the evaluation within dimensional regularization of the two-point correlator built from the pseudoscalar current defined in (1.25).

i) <u>To lowest order of QCD</u>, we shall compute the following diagram :



Using usual Feynman rules, it reads :

$$\nu^{\epsilon}\ \psi_5(q^2) = (m_i+m_j)^2\ (-i)\ N \int \frac{d^n p}{(2\pi)^n}$$

$$\mathrm{Tr}\left\{ (i\gamma_5)\ \frac{1}{\hat{p}-m+i\epsilon'}\ (i\gamma_5)\ \frac{1}{\hat{p}-\hat{q}-m_j+i\epsilon'} \right\} . \qquad (1.115)$$

Parametrizing the quark propagators and using the properties of Dirac matrices in n dimensions[34] (see Appendix A), we obtain :

$$\psi_5^B(q^2) = (m_i+m_j)^2\ \frac{N}{4\pi^2} \int_0^1 dx \left(\frac{2}{\epsilon} + \log\ 4\pi-\gamma\right) \left(\frac{\mathbb{R}^2-i\epsilon'}{\nu^2}\right)^{-\epsilon/2} .$$

$$\cdot \left\{ \left(3 + \frac{\epsilon}{2}\right) q^2 x(1-x) - 2\left(1 + \frac{\epsilon}{4}\right) \left(m_i^2 x + m_j^2 (1-x) + m_i m_j\right)\right\} , \quad (1.116)$$

where $\mathbb{R}^2 \equiv -q^2 x\ (1-x) + m_i^2 x + m_j^2\ (1-x)$ and $\gamma = 0.5772...$ the Euler constant.

Two limits of (1.116) are particularly interesting :

$$\psi_5^B\left(-q^2 \gg m_{i,j}^2\right) \simeq \frac{3}{8\pi^2} (m_i+m_j)^2\ q^2 \left\{\left(\frac{2}{\epsilon} + \log 4\pi - \gamma - \log -\frac{q^2}{\nu^2}\right) \cdot \right.$$

$$\cdot \left(1 + \frac{2\left(m_i^2 + m_j^2 - m_i m_j\right)}{-q^2}\right) + 2 + \frac{\epsilon}{4} \log^2 \left(-\frac{q^2}{\nu^2}\right) -$$

$$\left. \frac{\epsilon}{2} (\log 4\pi - \gamma + 2) \log -\frac{q^2}{\nu^2}\right\} , \quad (1.117)$$

$$\psi_5^B(q^2 \to 0) \simeq \frac{3}{4\pi^2} (m_i+m_j) \left\{\left(m_i^3 \log \frac{m_i^2}{\nu^2} + m_j^3 \log \frac{m_j^2}{\nu^2}\right)\right.$$

$$\left. - \left(\frac{2}{\epsilon} + \log 4\pi - \gamma - 1\right) \left(m_i^3 + m_j^3\right)\right\} . \quad (1.118)$$

For $q \to 0$, we know from the current algebra Ward identity in (1.16) that $\psi_5(q^2)$ is related to $\Pi_5^{\mu\nu}(q^2)$. A perturbative evaluation of the axial two-point correlator gives :

$$q_\mu q_\nu \Pi_5^{\mu\nu} = \frac{N}{8\pi^2} \int_0^1 dx \, \frac{m_i^2 x + m_j^2 (1-x) + m_i m_j}{q^2} \left(\frac{\mathbb{R}^2 - i\epsilon'}{\nu^2} \right)^{-\epsilon/2} \Gamma(\epsilon/2) . \quad (1.119)$$

Using $\log \mathbb{R}^2 = \log|R^2| - i\pi\,\theta\,(-R^2)$, one gets from (1.116) and (1.117) the well-known result :

$$\mathrm{Im}\,\psi_5(t) = \mathrm{Im}\left(q_\mu q_\nu \Pi_5^{\mu\nu}(t)\right) =$$

$$= \frac{3}{8\pi^2} (m_i + m_j)^2 \left(1 - \frac{(m_i - m_j)^2}{t}\right) \lambda^{1/2}\left(1, \frac{m_i^2}{t}, \frac{m_j^2}{t}\right) .$$

$$. \, \theta\left(t - (m_i + m_j)^2\right) \quad (1.120)$$

$$q_\mu q_\nu \Pi_5^{\mu\nu} = \psi_5(q^2) - \frac{3}{4\pi^2} (m_i + m_j) \left(m_i^3 \log \frac{m_i^2}{\nu^2} + m_j^3 \log \frac{m_j^2}{\nu^2} \right) . \quad (1.121)$$

(1.121) suggests that at zero momentum $\psi_5(0)$ possesses a small perturbative contribution or alternatively the "true" quark condensate $\langle \bar\psi \psi \rangle$ can be a combination of the one in (1.17b) and the perturbative piece in (1.121). We shall come back to this point later.

ii) $\psi_5(q^2)$ to two loops in the $\overline{MS}$ scheme

In order to simplify life for the reader, we shall ignore the quark mass in the internal loops and quote the bare result of Becchi et al to two loops[43]:

$$\psi_5^B(q^2) = \nu^{-\epsilon} \left(m_i^B + m_j^B\right)^2 . \left(\frac{3q^2}{8\pi^2}\right) \left\{ \frac{2}{\epsilon} + \log 4\pi - \gamma + 2 - \log \frac{-q^2}{\nu^2} \right.$$

$$- \frac{\epsilon}{2} (\log 4\pi - \gamma + 2) \log \frac{-q^2}{\nu^2} + \frac{\epsilon}{4} \log^2 \frac{-q^2}{\nu^2} +$$

$$\left.\left(\frac{g^B \nu^{-\epsilon/2}}{4\pi^2}\right)^2 \left[\frac{4}{\epsilon^2} + \frac{4}{\epsilon} (\log 4\pi - \gamma) + \frac{29}{3\epsilon} + \mathcal{O}(1)\right]\left(\frac{- q^2}{\nu^2}\right)^{-\epsilon}\right\} \quad . \qquad (1.122)$$

We introduce the renormalized parameter :

$$g^B \nu^{-\epsilon/2} = g\left\{1 + \mathcal{O}\left(\frac{\alpha_s}{\pi}\right)\right\} \quad ,$$

$$m_i^B = m_i \left(1 - \frac{2}{\epsilon}\frac{\alpha_s}{\pi}\right) \quad , \qquad (1.123)$$

and we obtain :

$$\psi_5^B(q^2) = \frac{3}{8\pi^2}(m_i + m_j)^2 q^2 \left\{\frac{2}{\epsilon} + \log 4\pi - \gamma + 2 - \log \frac{- q^2}{\nu^2} + \left(\frac{\alpha_s}{\pi}\right)\right.$$

$$\left.\left[- \frac{4}{\epsilon^2} + \frac{5}{3\epsilon} + \log^2 \frac{-q^2}{\nu^2} - \left(\frac{17}{3} + 2 (\log 4\pi - \gamma)\right) \log \frac{-q^2}{\nu^2}\right]\right\} . \qquad (1.124)$$

First we learn that the lowest order terms which vanish for $\epsilon \to 0$ induce terms in the two-loop results via the mass renormalization. We also see that the non-local $\frac{1}{\epsilon} \log \frac{- q^2}{\nu^2}$ pole has disappeared. This consists of a double check of our results. A last check can be done with the RGE[34] . $\psi_5(q^2)$ obeys the RGE of the type in (1.103) where D_s is the coefficient of the $\frac{1}{\epsilon}$ term :

$$D_s \equiv D_0 + D_1 \left(\frac{\alpha_s}{\pi}\right) \quad , \tag{1.125a}$$

with :

$$D_0 = - \frac{3}{8\pi^2} (x_i + x_j)^2 \; 2 \; e^{-2t} \quad ,$$

$$D_1 - \frac{3}{8\pi^2} (x_i + x_j)^2 \; \frac{10}{3} \; e^{-2t} \; . \tag{1.125b}$$

Let us now write $\psi_5(q^2)$ in terms of dimensionless variables :

$$\psi_5(t, \alpha_s, x_i) = - \frac{3q^4}{8\pi^2} (x_i + x_j)^2 \; e^{-2t} \left\{ -2t + \log 4\pi - \gamma + 2 + \left(\frac{\alpha_s}{\pi}\right) (4at^2 + 2bt + c) \right\} \quad , \tag{1.126}$$

where a,b,c have to be determined. Using the RGE, one obtains the constraint :

$$D_0 = - \left(\frac{3}{8\pi^2}\right) (x_i + x_j)^2 \; e^{-2t} \; . \; 2 \quad ,$$

$$D_1 = - \left(\frac{3}{8\pi^2}\right) (x_i + x_j)^2 e^{-2t} \left\{ -8at - 2b - 2\gamma_1 (\log 4\pi - \gamma + 2) + 2\gamma_1 . 2t \right\} \; , \tag{1.127}$$

where $\gamma_1 = 2$ is the mass anomalous dimension. The fact that D_1 cannot depend on t implies :

$$- 4a + 2\,\gamma_1 = 0 \quad \Longrightarrow \quad a = 1 \quad . \tag{1.128}$$

The relation between C_1 and D in (1.103) implies :

$$C_1^{(0)} = D_0 \quad . \tag{1.129}$$

$C_1^{(1)}$ is not fixed by the RGE but we know it from the calculation in (1.124). It is :

$$C_1^{(1)} = \left(\frac{5}{3}\right)\left(-\frac{3}{8\pi^2}\right)(x_i + x_j)^2\, e^{-2t} \quad , \tag{1.130}$$

We can now deduce from (1.103)

$$2\,C_1^{(1)} = D_1 \tag{1.131a}$$

and the recursive relation implies :

$$C_2^{(1)} = -\frac{3}{8\pi^2}(x_i + x_j)^2\, e^{-2t}\; 2\gamma_1 \quad . \tag{1.131b}$$

Eq. (1.131) inserted into (1.127) implies :

$$- 2b - 2\,\gamma_1(\log 4\pi - \gamma + 2) = \frac{10}{3} \; . \tag{1.132}$$

As one can see, the RGE and the explicit calculation of the $\frac{1}{\epsilon}$ coefficient at $\left(\frac{\alpha_s}{\pi}\right)$ allows one to fix the coefficients of $\frac{1}{\epsilon^2}$, $\log^2 - q^2/\nu^2$ and $\log - q^2/\nu^2$ at that order. This is a really impressive result !

A complete two-loop expression of $\psi_5(q^2)$ including quark masses has been obtained by Broadhurst[44]. The use of his result at $q = 0$ implies :

$$\psi_5^R(0) = \frac{3}{4\pi^2}(m_i + m_j)\left(m_i^3 Z_i + m_j^3 Z_j\right) , \qquad (1.133a)$$

with :

$$Z_i = 1 - \log\frac{m_i^2}{\nu^2} + \left(\frac{2\alpha_s}{3\pi}\right)\left(5 - 5\log\frac{m_i^2}{\nu^2} + 3\log^2\frac{m_i^2}{\nu^2}\right) , \qquad (1.133b)$$

and improves the Ward identity in (1.121).

Three-loop expressions of $\psi_5(q^2)$ also exist in the chiral limit $m_i=0$. In the $\overline{MS}$ scheme and for $SU(n)_F$, it reads[24c] :

$$\psi_5(q^2) = \frac{3}{8\pi^2}(m_i + m_j)^2\left\{-q^2\log -\frac{q^2}{\nu^2}\left[1 + \frac{17}{3}\frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{11089}{144} - \frac{611}{24}\xi(3) + n\left(\frac{2}{3}\xi(3) - \frac{65}{24}\right)\right)\right]\right.$$

$$+\left(\frac{\alpha_s}{\pi}\right)\left(q^2\log^2 -\frac{q^2}{\nu^2}\right)\left[1 - \frac{\alpha_s}{\pi}\left(-\frac{53}{3} + \frac{11}{18}n\right)\right]$$

$$\left. - \left(\frac{\alpha_s}{\pi}\right)^2 q^2\log^3 -\frac{q^2}{\nu^2}\left(\frac{19}{12} - \frac{n}{18}\right)\right\} , \qquad (1.134)$$

where $\xi(3) = 1.202$ is the Riemann's zeta function.

The complete two-loop result for the spectral function is[46]:

$$\frac{1}{\pi}\ \mathrm{Im}\ \psi_5(t) = \frac{3(m_i+m_j)^2}{8\pi^2\ q^2}\ \bar{q}^4\ v \left\{1 + \frac{4\alpha_s}{3\pi} \left\{\frac{3}{8}(7-v^2)\right.\right.$$

$$+ \sum_i (v+v^{-1})\ (Li_2(\alpha_1\alpha_2) - Li_2\ (-\alpha_i) - \log\alpha_i\ \log\beta_i)$$

$$\left.\left. + A_i\ \log\alpha_i + B_i\ \log\beta_i\right\}\ .\ \mathcal{O}\left(\alpha_s^2\right)\right\}\ , \qquad (1.135a)$$

where $Li_2(x) = -\int_0^x dx/x\ \log(1-x)$ and :

$$A_i = \frac{3}{4}\left(\frac{3m_i+m_j}{m_i+m_j}\right) - \frac{19+2v^2+3v^4}{32v} - \frac{m_i(m_i-m_j)}{\bar{q}^2\ v(1+v)}\left(1+v+\frac{2v}{1+\alpha_i}\right)\ ;$$

$$B_i = 2 + 2\left(m_i^2 - m_j^2\right)\Big/\bar{q}^2\ v\ ;$$

$$\alpha_i = \frac{m_i}{m_j}\left(\frac{1-v}{1+v}\right)\ ;\quad \beta_i = (1+\alpha_i)^{1/2}\ (1+v)^2\ /4v$$

$$\bar{q}^2 = q^2 - (m_i - m_j)^2\ ;\quad v = \left(1 - 4\ \frac{m_i m_j}{\bar{q}^2}\right)^{1/2}\ . \qquad (1.135b)$$

In this case where $m_j = 0$ this expression simplifies as :

$$\frac{1}{\pi}\ \mathrm{Im}\ \psi_5(t) = \frac{3}{8\pi^2}\ xt^2(1-x)^2 \left\{1 + \frac{4}{3}\frac{\alpha_s}{\pi}\left[\frac{9}{4} + 2\ Li_2(x)\right.\right.$$

$$+ \log x\ \log(1-x) - \frac{3}{2}\log\left(\frac{1}{x}-1\right) - \log(1-x)$$

$$\left. + x\log\left(\frac{1}{x}-1\right) - (x/(1-x))\ \log x\right\}\ ,$$

where $$x \equiv m_i^2 / t. \tag{1.135c}$$

The expression of the two-point correlator of the scalar current $\partial_\mu \left(\bar{\psi}_i \ \gamma_\mu \ \psi_j\right)$ can be obtained from (1.135) by changing m_j into $- m_j$ in all terms.

g) Vector Correlator to Two Loops

We shall be concerned with :

$$\Pi_V^{\mu\nu} = - (g^{\mu\nu} q^2 - q^\mu q^\nu) \Pi_V^{(1)}(q^2) + q^\mu q^\nu \Pi^{(o)}(q^2)$$

$$\equiv i \int d^4x \ e^{iqx} \left\langle 0 \mid \mathbb{T} \ V_{ij}^\mu(x) \ \left(V_{ij}^\nu(0)\right)^+ \mid 0\right\rangle, \tag{1.136}$$

with $V_{ij}^\mu = \bar{\psi}_i \ \gamma^\mu \ \psi_j$. The longitudinal part is related to $\psi(q^2)$ from the Ward identity of the type of (1.121) which reads to two loops (see (1.133)) :

$$(q^2)^2 \ \Pi_V^{(o)} = \psi(q^2) - \frac{3}{4\pi^2} (m_i - m_j) \left(m_i^3 \ Z_i - m_j^3 \ Z_j\right) \ , \tag{1.137}$$

where, as we have already said, $\psi(q^2)$ can be deduced from $\psi_5(q^2)$ by interchanging m_j into $-m_j$ in (1.134) and (1.135). The lowest order contribution to $\Pi_V^{(1)}(q^2)$ reads[34]:

$$\Pi_V^{(1)} = \left(\frac{3}{4\pi^2}\right) \nu^{-\epsilon} \ \Gamma(\epsilon/2) \int_0^1 dx \ . \left[2x(1-x) + \frac{m_i^2 x + m_j^2 (1-x) - m_i m_j}{- q^2}\right] .$$

$$. \left(\frac{R^2}{\nu^2}\right)^{-\epsilon/2} . \tag{1.138}$$

The complete two-loop expression in the $\overline{MS}$ scheme is[47]:

$$\Pi^{(1)}(q^2) = \nu^{-\epsilon}\,\frac{1}{4\pi^2}\left\{\frac{2}{\epsilon}\left(1+\frac{\alpha_s}{\pi}\right)-\frac{1}{3}\left(1+\frac{\alpha_s}{\pi}\,\frac{15}{4}\right)\right.$$

$$+\,(1-\alpha-\beta-2(\alpha-\beta)^2)K+\alpha\ell_i+\beta\ell_j+2(\alpha-\beta)\,(\alpha Z_i-\beta Z_j)$$

$$+\left(\frac{2\alpha_s}{3\pi}\right)\left\{\frac{1}{2}\,(1-\alpha-\beta-2(\alpha-\beta)^2)\,L+\alpha\ell_i\,(1+2\ell_i)\right.$$

$$+\,\beta\ell_j(1+2\ell_j)-\frac{1}{4}\,(1+2N_A+2N_B)\left(1+6\,\frac{(m_i-m_j)^2}{q^2}\right)$$

$$+\,x_i\,f_i^2+x_j\,f_j^2+\frac{1}{2}\,(N_A-N_B)^2$$

$$-\,(\alpha-\beta)\,(G(x_i)-G(x_j))-(3-2\,(\alpha+\beta))\,K^2$$

$$\left.-\,\frac{1}{2}+\alpha\,(2+\ell_i)+\beta\,(2+\ell_j)\,K\right\}\;, \tag{1.139a}$$

with Z_i defined (1.133b) and :

$$\alpha \equiv -\,m_i^2/q^2 \quad ; \quad \beta \equiv -\,m_j^2/q^2 \quad ; \quad \ell_i \equiv -\log\frac{m_i^2}{\nu^2} \quad ;$$

$$K \equiv 1+\frac{\ell_i}{2}+\frac{1}{2}\,(1+x_i)\,f_i \quad ; \quad f_i \equiv \log x_i/(1-x_i) \quad ;$$

$$x_i \equiv m_i^2/E\,\left[1+\sqrt{1-(m_i m_j/E)^2}\right] \;;$$

$$E \equiv \frac{1}{2}\left(m_i^2 + m_j^2 - q^2\right) \quad ;$$

$$N_A = \alpha(1 + f_i)(1 + x_j f_j) \quad ; \quad N_B = \beta(1+f_j)(1 + x_i f_i) \; ;$$

$$L \equiv 3K^2 + 2K + 6 - 6(1+\alpha+\beta)\, I - 10\, x_i f_i^2 + m_i\left[(3K-2)\frac{\partial K}{\partial m_i} - (1+\alpha+3\beta)\frac{\partial I}{\partial m_i}\right] + (i \leftrightarrow j) \; ;$$

$$G(x) = \int_0^x dy \left(\frac{\log y}{1-y}\right)^2 = \sum_{n=1}^{\infty} \frac{x^n}{n^2}\left(1 + (1-n \log x)^2\right) \; ;$$

$$F(x) = \int_0^x dy \left(\frac{\log y}{1-y}\right)^2 \log\left(\frac{x}{y}\right) = \sum_{x=1}^{\infty} \frac{x^n}{n^3}\left(2 + (2 - n \log x)^2\right) \quad ;$$

$$I \equiv F(1) + F(x_i x_j) - F(x_i) - F(x_j) \quad . \qquad (1.139b)$$

We have not checked these horrible expressions ! It appears that in the limit $m_j = 0$ the result becomes less horrible. In this case, one can deduce from (1.139) the spectral function :

$$\mathrm{Im}\, \Pi_V^{(1)}(m_j = 0) = (2+x)\frac{\mathrm{Im}\, \psi_5(t)}{3\, m^2 t} - \left(\frac{\alpha_s}{6\pi^2}\right)\left\{(3+x)(1-x)^3 \log\frac{x}{1-x} + 2\, x \log x + (3-x^2)(1-x)\right\} , \qquad (1.140)$$

with $x \equiv m^2/t$.

For the equal mass case, the QCD expression of $Q^2 \dfrac{d}{dQ^2}\Pi(Q^2)$ which is related to the $e^+e^- \to$ Hadrons total cross-section is also known to four loops for $m_i = 0$ and to three loops up to $\frac{m^2}{Q^2}$ terms. After renormalization group improvement in the $\overline{MS}$ scheme[38] it reads:

$$\frac{Q^2}{dQ^2}\,\Pi_V^{(1)}(Q^2) = -\frac{3}{12\pi^2}\left\{1 + \frac{\bar{\alpha}_s}{\pi} + \left((1.986 - 0.115n) + \sum_{j=1}^{n} \frac{\bar{m}_j^2}{Q^2}\,1.05\right)\left(\frac{\bar{\alpha}_s}{\pi}\right)^2\right.$$

$$- \frac{\bar{m}^2}{Q^2}\left[6 + 28\,\frac{\bar{\alpha}_s}{\pi} + (269.15 - 12.25n)\left(\frac{\bar{\alpha}_s}{\pi}\right)^2\right] + \left[70.985 - 1.200n - 0.005\,n^2\right.$$

$$\left.\left. - 1.679\left(\sum_n Q_i\right)^2\right]\left(\frac{\alpha_s}{\pi}\right)^3 , \qquad (1.141)$$

where here $\bar{\alpha}_s$ and $\bar{m}$ are the three-loop running parameters defined in (1.93). The m^4 terms of $\Pi_V^{(1)}$ have also been obtained to two loops. In the $\overline{MS}$ scheme[47] it reads :

$$\Pi_V^{(1)}\bigg|_{m^4} = \left(\frac{3}{2\pi^2}\right)\frac{\bar{m}^4}{Q^4}\left\{1 + \log\frac{Q^2}{\bar{m}^2} + 2\left(\frac{\bar{\alpha}_s}{3\pi}\right)\left(5 + 5\log\frac{Q^2}{\bar{m}^2} + 3\log^2\frac{Q^2}{\bar{m}^2}\right)\right\}.$$

$$\cdot\left\{\left(1 + \frac{\bar{\alpha}_s}{3\pi} + \mathcal{O}\left(\frac{\bar{\alpha}_s}{\pi}\right)^2\right) - \frac{3}{4\pi^2}\,\frac{\bar{m}^4}{Q^4}\left(1 + \mathcal{O}\left(\bar{\alpha}_s\right)\right)\right\} , \qquad (1.142a)$$

where one should note that the terms appearing in the first { } of (1.142a) are involved in the current algebra Ward identity obeyed by the quark $\langle\bar\psi\psi\rangle$ condensate in (1.133b). After a resummation of the mass-singular logarithms in this { } , one can rewrite the improved expression of the $\bar m^4$ terms for $SU(n)_F$ as :

$$\Pi_V^{(1)}\Big|_{\bar m^4} \simeq \frac{3}{12\pi^2}\left\{\frac{\bar m^4}{Q^4}\cdot\frac{1}{(15+2n)}\left[-\frac{72\pi}{\bar\alpha_s}+\frac{357-23n}{4}\right]+\frac{8\pi^2}{Q^4}\, m\langle\bar\psi\psi\rangle\left(1+\frac{\bar\alpha_s}{3\pi}\right)\right\} \quad (1.142b)$$

h) **Weinberg Sum Rules to Two Loops**

We are now in a good position to check the validity of the Weinberg superconvergent assumptions done in (1.25) and (1.26). Let us, for instance, analyze the spectral functions to the lowest order. They can be deduced from (1.135) and (1.138) using the usual change m_i into $-m_i$ in order to deduce the expressions of the scalar and axial spectral functions. It is easy to see that for the combination involved in the WSR (1.25) and (1.26), the first one is still zero while the second one reads :

$$\mathrm{Im}\,\Pi_{LR}^{(1)} = -\,\mathrm{Im}\,\Pi_{LR}^{(o)} = \frac{3}{2\pi}\cdot\frac{m_i m_j}{t}\,\lambda^{1/2}\left(1,\frac{m_i^2}{t},\frac{m_j^2}{t}\right) \quad . \quad (1.143)$$

One can compute the next corrections for the first sum rule, which reads[22b]:

$$\Pi_{LR}^{(1)} + \Pi_{LR}^{(o)}\ (-q^2 \gg m^2) \simeq \left(\frac{\alpha_s}{\pi}\right)\frac{1}{\pi^2}\left\{\frac{m_i m_j}{q^2}+\dots\right\} \quad . \quad (1.144)$$

For the second to be convergent, one has to look for some other combinations like[22b] :

$$R_{ijk} = \Pi^{(1)}_{LR}\Big|_{ij} - \frac{m_j}{m_k}\,\Pi^{(1)}_{LR}\Big|_{ik}\,, \qquad (1.145)$$

or to improve the sum rule with, for example, the Laplace transform, as we shall see later on.

i) **Tensor Current Correlator to Two Loops**

We shall be concerned with the current :

$$J_{\mu\nu}(x) = i\,\bar{\psi}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi \qquad (1.146a)$$

$$\hat{J}_{\mu\nu}(x) = :-G_{\mu\alpha}G^{\alpha}_{\nu} + \frac{1}{4}\,g_{\mu\nu}G^{\alpha\beta}G_{\alpha\beta}: \qquad (1.146b)$$

where $\overleftrightarrow{D}_\mu \equiv \overrightarrow{D}_\mu - \overleftarrow{D}_\mu$ is the covariant derivative.
We now study the renormalization of the two currents according to Ref. 48) :

$$\begin{aligned} J^{\mu\nu} &= Z_{11}\,J^{B}_{\mu\nu} + Z_{12}\,\hat{J}^{B}_{\mu\nu} \\ \hat{J}^{\mu\nu} &= Z_{21}\,J^{B}_{\mu\nu} + Z_{22}\,\hat{J}^{B}_{\mu\nu} \end{aligned} \qquad (1.147a)$$

where in the $\overline{MS}$ scheme :

$$Z_{11} = 1 + \left(\frac{\alpha_s}{\pi}\right)\frac{4}{3}\,C_2(R)\,\frac{1}{\epsilon}\,,$$

$$Z_{12} = -\frac{8}{3}\,T(R)\left(\frac{\alpha_s}{\pi}\right)\frac{1}{\epsilon}\,,$$

$$Z_{21} = -\frac{1}{3} C_2(R) \left(\frac{\alpha_s}{\pi}\right) \frac{1}{\epsilon} \quad ,$$

$$Z_{22} = 1 + \frac{n}{3} \left(\frac{\alpha_s}{\pi}\right) \frac{1}{\epsilon} \quad . \qquad (1.147b)$$

The associated two-point correlator can be written as :

$$\Pi_{\mu\nu,\rho\sigma} = \frac{1}{2} \left\{\eta_{\mu\rho}\ \eta_{\nu\sigma} + \eta_{\mu\sigma}\ \eta_{\nu\rho} + \frac{2}{1-d}\ \eta_{\mu\nu}\ \eta_{\rho\sigma}\right\} \Pi(q^2) \quad , \qquad (1.148a)$$

where $d \equiv 4-\epsilon$ is the space-time dimension and $\eta_{\mu\nu} \equiv \left(g_{\mu\nu}-q_\mu q_\nu/q^2\right)$.
$\Pi(q^2)$ can be extracted using the projector.

$$R_{\mu\nu\rho\sigma} = \frac{2}{(d+1)(d-1)(d-2)} \left\{(n-2) q_\mu q_\nu q_\rho q_\sigma/q^4 \right.$$
$$+ \left(q_\mu q_\nu g_{\rho\sigma} + g_{\mu\nu}\ q_\rho q_\sigma\right)\Big/q^2 - g_{\mu\nu}\ g_{\rho\sigma}$$
$$\left. - (d-1) \left(2\ q_\mu q_\rho g_{\nu\sigma}\Big/q^2 - g_{\mu\rho}\ g_{\nu\sigma}\right)\right\} \quad , \qquad (1.148b)$$

which gives

$$\Pi(q^2) = R_{\mu\nu\rho\sigma}\ \Pi^{\mu\nu\rho\sigma} \ . \qquad (1.148c)$$

To the lowest order, the spectral function reads[48b] :

$$\mathrm{Im}\ \Pi(t) = \frac{t^2}{10\pi} (5-2v^2)\ v^3\ \theta(t-4m^2) \qquad (1.148d)$$

where $v \equiv \left(1 - 4\ \dfrac{m^2}{t}\right)^{1/2}$.

The evaluation of the two-loop quark diagram gives for $m_1 = 0$ [48b]:

$$\Pi_B(q^2) = -\frac{3q^4}{10\pi^2}\left\{-\frac{2}{\epsilon} + \gamma - \frac{12}{5} + \log\frac{-q^2}{\nu^2} + \left(\frac{\alpha_s}{9\pi}\right)\left[\frac{32}{\epsilon^2} + \frac{1}{\epsilon}\left(\frac{1049}{15} - 32\gamma - 32\log - \frac{q^2}{4\pi\nu^2}\right) + \left(32\gamma - \frac{1049}{15}\right)\log - \frac{q^2}{4\pi\nu^2} + 16\log^2 - \frac{q^2}{4\pi\nu^2}\right]\right\}, \quad (1.149a)$$

where one should notice the non-local pole $\frac{1}{\epsilon}\log - \frac{q^2}{\nu^2}$. The renormalized correlator is

$$\Pi^R = Z_{11}^2\ \Pi^B \quad (1.149b)$$

Then one gets in the $\overline{MS}$ scheme[48b]:

$$\Pi^R = -\frac{3}{10\pi^2}\left(q^4\log - \frac{q^2}{4\pi\nu^2}\right)\left\{1 + \left(\frac{\alpha_s}{\pi}\right)\left[-\frac{473}{135} + \frac{8}{9}\left(2\gamma + \log - \frac{q^2}{4\pi\nu^2}\right)\right]\right\}. \quad (1.149c)$$

This result is free of the $\frac{1}{\epsilon}\log - \frac{q^2}{\nu^2}$ non-local pole. This serves as a check of the partial validity of the expression. The α_s coefficient differs from that given in Ref. 29) both in the $\log^2$ and in the log terms. In fact, renormalization of the current has not been done carefully there. The above examples have really shown how much care should be taken in dealing with two-loop corrections and renormalizations. Let us now come to one example of renormalization and regularization scheme dependences of the QCD parameters.

j) $\overline{MS}$ and on-shell schemes : Scheme Invariance of the Quark Pole Mass

Let us call the on-shell scheme the QED-like scheme where Green's functions are Pauli-Villars regularized and the renormalization is done on shell :

$$\Pi^B_{OS} = \Pi^B_{PV}(q^2) - \Pi^B_{PV}(q^2 = 0) \quad . \qquad (1.150)$$

The relevance of this question for the sum rules is that one uses (mainly in the heavy-quark case[18,19,29]) the expression obtained in QED for the spectral function but there is often confusion regarding definition of the quark mass to be used in the analysis. For definiteness, let us concentrate on the vector spectral function.

In a QED-like on-shell scheme, the vector spectral function is known from the calculation of Källen and Sabry[49] which is accurately approximated by the Schwinger[50] expression :

$$\mathrm{Im}\,\Pi^{(1)}_V(t)\;(m_i = m_j) = \frac{3}{12\pi}\,\theta(t - 4m^2)\;.\;v\left(\frac{3-v^2}{2}\right)\left\{1 + \frac{4}{3}\,\alpha_s(t)\;f(v)\right\},$$

$$\text{where :}\quad v = \left(1 - 4\,\frac{m^2}{t}\right)^{1/2} \quad\text{and}\quad f(v) = \frac{\pi}{2v} - \frac{3+v}{4}\left(\frac{\pi}{2} - \frac{3}{4\pi}\right) \quad . \qquad (1.151)$$

In QED, the mass appearing in (1.151) is well defined as the electron is observed, i.e it is the mass at $p^2 = M^2 = m^2$, the pole of the electron propagator $S_F(p)$. Radiative corrections do not affect this mass and this is achieved via the mass renormalization constant[51]:

$$Z_m^{QED} = 1 - \left(\frac{\alpha_s}{\pi}\right)\left[\log\frac{\Lambda^2_{UV}}{M^2} + \frac{1}{2}\right] . \qquad (1.152)$$

This pole mass can be related to the Euclidian mass $m(p^2 = -M^2)$. Using the expression of Σ_1 in (1.67), one gets :

$$m(p^2) = M(p^2{=}M^2)\left(1 - \frac{\alpha_s}{\pi}\left(1 - \frac{M^2}{p^2}\right)\left\{\log\left(1 - \frac{p^2}{M^2}\right) - \frac{1}{3}\alpha_G\right.\right. .$$

$$\left.\left. \cdot \left[1 + \frac{M^2}{p^2}\log\left(1 - \frac{p^2}{M^2}\right)\right]\right\}\right) . \qquad (1.153)$$

Now the question is to know how to connect these QED-like masses to the masses used in the $\overline{MS}$ off-shell schemes. This question has been studied in Ref. 47) by calculating the vector correlator in two ways. Firstly, they calculate Re $\Pi(q^2)$ from the dispersive integral using the spectral function in (1.151) where the QED-like pole mass should be used. Secondly, they calculate directly Re $\Pi(q^2)$ in the Euclidian region by using the "light"-quark expansion m^2/q^2. In this case, one should use the quark mass in (1.91) after a leading-log resummation. Using the RGI of $\Pi(q^2)$ which is related to the physical observable e^+e^- into hadrons total cross-section, one obtains after requiring the same radiative corrections :

$$M_{QED}(p^2 = M^2) = \overline{m}_{\overline{MS}}(M)\left\{1 + \frac{4}{3}\frac{\alpha_s}{\pi} + \mathcal{O}\left(\alpha_s^2\right)\right\} . \qquad (1.154)$$

The same procedure has also been applied to other channels[46] and leads to (1.154). Let us now compare Eq. (1.154) with the relation between the pole and running mass in the $\overline{MS}$ scheme. This relation is[34,52]:

$$M_{\overline{MS}}(p^2=M^2) = \bar{m}_{\overline{MS}}(M)\left\{1 + \Sigma_1^{FP}(M^2 = \nu^2)\right\} , \qquad (1.155a)$$

where Σ_1^{FP} is the finite part of the quark self-energy decomposed as in (1.66). In the $\overline{MS}$ scheme, one can easily deduce from (1.67) :

$$\Sigma_1^{FP}(M^2 = \nu^2) = \frac{4}{3}\left(\frac{\alpha_s}{\pi}\right) , \qquad (1.155b)$$

which states that the pole mass in QED and $\overline{MS}$ schemes are equal.
Then, we have the important conclusion[52]:
"*The pole mass is regularization and renormalization schemes invariant*"

These relations in (1.153) to (1.155) are sufficient for a consistent use of the mass definitions appearing in the sum rules analysis as we shall see later. Let us now consider the non-perturbative QCD effects on the correlators which we have discussed previously.