1. THERMODYNAMIC STATES AND THE FIRST LAW (1001-1030)

1001

Describe briefly the basic principle of the following instruments for making temperature measurements and state in one sentence the special usefulness of each instrument: constant-volume gas thermometer, thermocouple, thermistor.

(*Wisconsin*)

Solution:

Constant-volume gas thermometer: It is made according to the principle that the pressure of a gas changes with its temperature while its volume is kept constant. It can approximately be used as an ideal gas thermometer.

Thermocouple thermometer: It is made according to the principle that thermoelectric motive force changes with temperature. The relation between the thermoelectric motive force and the temperature is

$$\varepsilon = a + bt + ct^2 + dt^3 \ ,$$

where ε is the electric motive force, t is the difference of temperatures of the two junctions, a, b, c and d are constants. The range of measurement of the thermocouple is very wide, from −200°C to 1600°C. It is used as a practical standard thermometer in the range from 630.74°C to 1064.43°C.

Thermister thermometer: We measure temperature by measuring the resistance of a metal. The precision of a thermister made of pure platinum is very good, and its range of measurement is very wide, so it is usually used as a standard thermometer in the range from 13.81K to 903.89K.

1002

Describe briefly three different instruments that can be used for the accurate measurement of temperature and state roughly the temperature range in which they are useful and one important advantage of each instrument. Include at least one instrument that is capable of measuring temperatures down to 1K.

(*Wisconsin*)

Solution:

1. *Magnetic thermometer:* Its principle is Curie's law $\chi = C/T$, where χ is the susceptibility of the paramagnetic substance used, T is its absolute temperature and C is a constant. Its advantage is that it can measure temperatures below 1K.

2. *Optical pyrometer:* It is based on the principle that we can find the temperature of a hot body by measuring the energy radiated from it, using the formula of radiation. While taking measurements, it does not come into direct contact with the measured body. Therefore, it is usually used to measure the temperatures of celestial bodies.

3. *Vapor pressure thermometer:* It is a kind of thermometer used to measure low temperatures. Its principle is as follows. There exists a definite relation between the saturation vapor pressure of a chemically pure material and its boiling point. If this relation is known, we can determine temperature by measuring vapor pressure. It can measure temperatures greater than 14K, and is the thermometer usually used to measure low temperatures.

1003

A bimetallic strip of total thickness x is straight at temperature T. What is the radius of curvature of the strip, R, when it is heated to temperature $T + \Delta T$? The coefficients of linear expansion of the two metals are α_1 and α_2, respectively, with $\alpha_2 > \alpha_1$. You may assume that each metal has thickness $x/2$, and you may assume that $x \ll R$.

(*Wisconsin*)

Solution:

We assume that the initial length is l_0. After heating, the lengths of the mid-lines of the two metallic strips are respectively

$$l_1 = l_0(1 + \alpha_1 \Delta T) , \tag{1}$$

$$l_2 = l_0(1 + \alpha_2 \Delta T) . \tag{2}$$

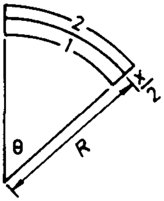

Fig. 1.1.

Assuming that the radius of curvature is R, the subtending angle of the strip is θ, and the change of thickness is negligible, we have

$$l_2 = \left(R + \frac{x}{4}\right)\theta\ , \quad l_1 = \left(R - \frac{x}{4}\right)\theta\ ,$$

$$l_2 - l_1 = \frac{x}{2}\theta = \frac{x}{2}\,\frac{l_1 + l_2}{2R} = \frac{xl_0}{4R}[2 + (\alpha_1 + \alpha_2)\Delta T]\ . \tag{3}$$

From (1) and (2) we obtain

$$l_2 - l_1 = l_0 \Delta T(\alpha_2 - \alpha_1)\ , \tag{4}$$

(3) and (4) then give

$$R = \frac{x}{4}\,\frac{[2 + (\alpha_1 + \alpha_2)\Delta T]}{(\alpha_2 - \alpha_1)\Delta T}\ .$$

1004

An ideal gas is originally confined to a volume V_1 in an insulated container of volume $V_1 + V_2$. The remainder of the container is evacuated. The partition is then removed and the gas expands to fill the entire container. If the initial temperature of the gas was T, what is the final temperature? Justify your answer.

(*Wisconsin*)

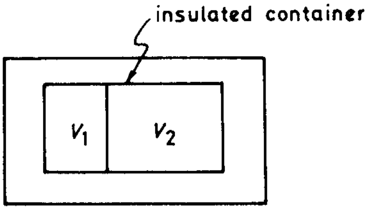

Fig. 1.2.

Solution:

This is a process of adiabatic free expansion of an ideal gas. The internal energy does not change; thus the temperature does not change, that is, the final temperature is still T.

1005

An insulated chamber is divided into two halves of volumes. The left half contains an ideal gas at temperature T_0 and the right half is evacuated. A small hole is opened between the two halves, allowing the gas to flow through, and the system comes to equilibrium. No heat is exchanged with the walls. Find the final temperature of the system.

(*Columbia*)

Solution:

After a hole has been opened, the gas flows continuously to the right side and reaches equilibrium finally. During the process, internal energy of the system E is unchanged. Since E depends on the temperature T only for an ideal gas, the equilibrium temperature is still T_0.

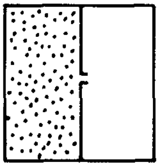

Fig. 1.3.

1006

Define heat capacity C_v and calculate from the first principle the numerical value (in calories/°C) for a copper penny in your pocket, using your best physical knowledge or estimate of the needed parameters.

(*UC, Berkeley*)

Solution:

$C_v = (dQ/dT)_v$. The atomic number of copper is 64 and a copper penny is about 32 g, i.e., 0.5 mol. Thus $C_v = 0.5 \times 3R = 13$ J/K.

1007

Specific heat of granite may be: 0.02, 0.2, 20, 2000 cal/g·K.

(*Columbia*)

Solution:

The main component of granite is $CaCO_3$; its molecular weight is 100. The specific heat is $C = 3R/100 = 0.25$ cal/g· K. Thus the best answer is 0.2 cal/g·K.

1008

The figure below shows an apparatus for the determination of C_p/C_v for a gas, according to the method of Clement and Desormes. A bottle G, of reasonable capacity (say a few litres), is fitted with a tap H, and a manometer M. The difference in pressure between the inside and the outside can thus be determined by observation of the difference h in heights of the two columns in the manometer. The bottle is filled with the gas to be investigated, at a very slight excess pressure over the outside atmospheric pressure. The bottle is left in peace (with the tap closed) until the temperature of the gas in the bottle is the same as the outside temperature in the room. Let the reading of the manometer be h_i. The tap H is then opened for a very short time, just sufficient for the internal pressure to become equal to the atmospheric pressure (in which case the manometer reads $h = 0$). With the tap closed the bottle is left in peace for a while, until the inside temperature has become equal to the outside temperature. Let the final reading of the manometer be h. From the values of h_i and h_f it is possible to find C_p/C_v. (a) Derive an expression for C_p/C_v in terms of h_i and h_f in the above experiment. (b) Suppose that the gas in question is oxygen. What is your theoretical prediction for C_p/C_v at 20°C, within the framework of statistical mechanics?

(*UC, Berkeley*)

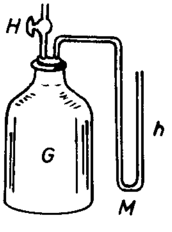

Fig. 1.4.

Solution:

(a) The equation of state of ideal gas is $pV = nkT$. Since the initial and final T, V of the gas in the bottle are the same, we have $p_f/p_i = n_f/n_i$.

Meanwhile, $n_f/n_i = V/V'$, where V' is the volume when the initial gas in the bottle expands adiabatically to pressure p_0. Therefore

$$\frac{V}{V'} = \left(\frac{p_0}{p_i}\right)^{\frac{1}{\gamma}} , \quad \frac{p_f}{p_i} = \left(\frac{p_0}{p_i}\right)^{\frac{1}{\gamma}} ,$$

$$\gamma = \frac{\ln \dfrac{p_i}{p_0}}{\ln \dfrac{p_i}{p_f}} = \frac{\ln\left(1 + \dfrac{h_i}{h_0}\right)}{\ln\left(1 + \dfrac{h_i}{h_0}\right) - \ln\left(1 + \dfrac{h_f}{h_0}\right)} .$$

Since $h_i/h_0 \ll 1$ and $h_f/h_0 \ll 1$, we have $\gamma = h_i/(h_i - h_f)$.

(b) Oxygen consists of diatomic molecules. When $t = 20°\text{C}$, only the translational and rotational motions of the molecules contribute to the specific heat. Therefore

$$C_v = \frac{5R}{2} , \quad C_p = \frac{7R}{2} , \quad \gamma = \frac{7}{5} .$$

1009

(a) Starting with the first law of thermodynamics and the definitions of c_p and c_v, show that

$$c_p - c_v = \left[p + \left(\frac{\partial U}{\partial V}\right)_T\right] \left(\frac{\partial V}{\partial T}\right)_p$$

where c_p and c_v are the specific heat capacities per mole at constant pressure and volume, respectively, and U and V are energy and volume of one mole.

(b) Use the above results plus the expression

$$p + \left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V$$

to find $c_p - c_v$ for a Van der Waals gas

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT .$$

Use that result to show that as $V \to \infty$ at constant p, you obtain the ideal gas result for $c_p - c_v$.

(*SUNY, Buffalo*)

Solution:

(a) From $H = U + pV$, we obtain

$$\left(\frac{\partial H}{\partial T}\right)_p = \left(\frac{\partial U}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p .$$

Let $U = U[T, V(T,p)]$. The above expression becomes

$$\left(\frac{\partial H}{\partial T}\right)_p = \left(\frac{\partial U}{\partial T}\right)_V + \left[p + \left(\frac{\partial U}{\partial V}\right)_T\right]\left(\frac{\partial V}{\partial T}\right)_p .$$

Hence

$$c_p - c_v = \left[p + \left(\frac{\partial U}{\partial V}\right)_T\right]\left(\frac{\partial V}{\partial T}\right)_p .$$

(b) For the Van der Waals gas, we have

$$\left(\frac{\partial p}{\partial T}\right)_V = \frac{R}{(V-b)} ,$$

$$\left(\frac{\partial V}{\partial T}\right)_p = R\Big/\left[\frac{RT}{V-b} - \frac{2a(V-b)}{V^3}\right] .$$

Hence,

$$c_p - c_v = \frac{R}{1 - 2a(1-b/V)^2/VRT} ,$$

When $V \to \infty, c_p - c_v \to R$, which is just the result for an ideal gas.

1010

One mole of gas obeys Van der Waals equation of state. If its molar internal energy is given by $u = cT - a/V$ (in which V is the molar volume, a is one of the constants in the equation of state, and c is a constant), calculate the molar heat capacities C_v and C_p.

(*Wisconsin*)

Solution:

$$C_v = \left(\frac{\partial u}{\partial T}\right)_V = c ,$$

$$C_p = \left(\frac{\partial u}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = \left(\frac{\partial u}{\partial T}\right)_V + \left[\left(\frac{\partial u}{\partial V}\right)_T + p\right] \times \left(\frac{\partial V}{\partial T}\right)_p = c + \left(\frac{a}{V^2} + p\right)\left(\frac{\partial V}{\partial T}\right)_p .$$

From the Van der Waals equation

$$(p + a/V^2)(V - b) = RT ,$$

we obtain

$$\left(\frac{\partial V}{\partial T}\right)_p = R \Big/ \left(p - \frac{a}{V^2} + \frac{2ab}{V^3}\right) .$$

Therefore

$$C_p = c + \frac{R\left(p + \frac{a}{V^2}\right)}{p - \frac{a}{V^2} + \frac{2ab}{V^3}} = c + \frac{R}{1 - \frac{2a(V-b)^2}{RTV^3}} .$$

1011

A solid object has a density ρ, mass M, and coefficient of linear expansion α. Show that at pressure p the heat capacities C_p and C_v are related by

$$C_p - C_v = 3\alpha M p/\rho .$$

(*Wisconsin*)

Solution:

From the first law of thermodynamics $dQ = dU + pdV$ and $\left(\frac{dU}{dT}\right)_p \approx \left(\frac{dU}{dT}\right)_v$ (for solid), we obtain

$$C_p - C_v = \left(\frac{dQ}{dT}\right)_p - \left(\frac{dU}{dT}\right)_v = p\frac{dV}{dT} . \qquad (*)$$

From the definition of coefficient of linear expansion $\alpha = \alpha_{\text{solid}}/3 = \frac{1}{3V}\frac{dV}{dT}$, we obtain

$$\frac{dV}{dT} = 3\alpha V = 3\alpha\frac{M}{\rho} .$$

Substituting this in (*), we find

$$C_p - C_v = 3\alpha\frac{M}{\rho}p .$$

1012

One mole of a monatomic perfect gas initially at temperature T_0 expands from volume V_0 to $2V_0$, (a) at constant temperature, (b) at constant pressure.

Calculate the work of expansion and the heat absorbed by the gas in each case.

(*Wisconsin*)

Solution:

(a) At constant temperature T_0, the work is

$$W = \int_A^B p dV = RT_0 \int_{V_0}^{2V_0} dV/V = RT_0 \ln 2 \ .$$

As the change of the internal energy is zero, the heat absorbed by the gas is

$$Q = W = RT_0 \ln 2 \ .$$

(b) At constant pressure p, the work is

$$W = \int_{V_0}^{2V_0} p dV = pV_0 = RT_0 \ .$$

The increase of the internal energy is

$$\Delta U = C_v \Delta T = \frac{3}{2} R \Delta T = \frac{3}{2} p \Delta V = \frac{3}{2} pV_0 = \frac{3}{2} RT_0 \ .$$

Thus the heat absorbed by the gas is

$$Q = \Delta U + W = \frac{5}{2} RT_0 \ .$$

1013

For a diatomic ideal gas near room temperature, what fraction of the heat supplied is available for external work if the gas is expanded at constant pressure? At constant temperature?

(*Wisconsin*)

Solution:

In the process of expansion at constant pressure p, assuming that the volume increases from V_1 to V_2 and the temperature changes from T_1 to T_2, we have

$$\begin{cases} pV_1 = nRT_1 \\ pV_2 = nRT_2 \ . \end{cases}$$

In this process, the work done by the system on the outside world is $W = p(V_2 - V_1) = nR\Delta T$ and the increase of the internal energy of the system is

$$\Delta U = C_v \Delta T \ .$$

Therefore

$$\frac{W}{Q} = \frac{W}{\Delta U + W} = \frac{nR}{C_v + nR} = \frac{2}{7} \ .$$

In the process of expansion at constant temperature, the internal energy does not change. Hence

$$W/Q = 1 \ .$$

1014

A compressor designed to compress air is used instead to compress helium. It is found that the compressor overheats. Explain this effect, assuming the compression is approximately adiabatic and the starting pressure is the same for both gases.

(*Wisconsin*)

Solution:

The state equation of ideal gas is

$$pV = nRT \, .$$

The equation of adiabatic process is

$$p\left(\frac{V}{V_0}\right)^\gamma = p_0 \, ,$$

where $\gamma = c_p/c_v, p_0$ and p are starting and final pressures, respectively, and V_0 and V are volumes. Because $V_0 > V$ and $\gamma_{\text{He}} > \gamma_{\text{Air}}$ $(\gamma_{\text{He}} = 7/5; \gamma_{\text{Air}} = 5/3)$, we get

$$p_{\text{He}} > p_{\text{Air}} \quad \text{and} \quad T_{\text{He}} > T_{\text{Air}} \, .$$

1015

Calculate the temperature after adiabatic compression of a gas to 10.0 atmospheres pressure from initial conditions of 1 atmosphere and 300K (a) for air, (b) for helium (assume the gases are ideal).

(*Wisconsin*)

Solution:

The adiabatic process of an ideal gas follows the law

$$T_B = (p_B/p_A)^{(\gamma-1)/\gamma}\, T_A = 10^{(\gamma-1)/\gamma} \times 300 \text{ K} .$$

(a) For air, $\gamma = C_p/C_v = 1.4$, thus $T_B = 5.8 \times 10^2\text{K}$.
(b) For helium, $\gamma = C_p/C_v = 5/3$, thus $T_B = 7.5 \times 10^2\text{K}$.

1016

(a) For a mole of ideal gas at $t = 0°\text{C}$, calculate the work W done (in Joules) in an isothermal expansion from V_0 to $10V_0$ in volume.

(b) For an ideal gas initially at $t_i = 0°\text{C}$, find the final temperature t_f (in °C) when the volume is expanded to $10V_0$ reversibly and adiabatically.

(*UC, Berkeley*)

Solution:

(a) $W = \int_{V_0}^{10V_0} pdV = \int_{V_0}^{10V_0} \frac{RT}{V} dV = RT \ln 10 = 5.2 \times 10^3\text{J}$.

(b) Combining the equation of adiabatic process $pV^\gamma = \text{const}$ and the equation of state $pV = RT$, we get $TV^{\gamma-1} = \text{const}$. Thus

$$T_f = T_i \left(\frac{V_i}{V_f}\right)^{\gamma-1} .$$

If the ideal gas molecule is monatomic, $\gamma = 5/3$, we get $t_f = 59\text{K}$ or $-214°\text{C}$.

1017

(a) How much heat is required to raise the temperature of 1000 grams of nitrogen from $-20°\text{C}$ to $100°\text{C}$ at constant pressure?

(b) How much has the internal energy of the nitrogen increased?

(c) How much external work was done?

(d) How much heat is required if the volume is kept constant?

Take the specific heat at constant volume $c_v = 5$ cal/mole °C and $R = 2$ cal/mole·°C.

(*Wisconsin*)

Solution:

(a) We consider nitrogen to be an ideal gas. The heat required is

$$Q = n(c_v + R)\Delta T = \frac{1000}{28}(5+2) \times 120 = 30 \times 10^3 \text{cal} .$$

(b) The increase of the internal energy is

$$\begin{aligned}\Delta U = nc_v\Delta T &= \frac{100}{28} \times 5 \times 120 \\ &= 21 \times 10^3 \text{cal} .\end{aligned}$$

(c) The external work done is

$$W = Q - \Delta U = 8.6 \times 10^3 \text{ cal} .$$

(d) If it is a process of constant volume, the required heat is

$$Q = nc_v\Delta T = 21 \times 10^3 \text{cal} .$$

1018

10 litres of gas at atmospheric pressure is compressed isothermally to a volume of 1 litre and then allowed to expand adiabatically to 10 litres.

(a) Sketch the processes on a pV diagram for a monatomic gas.

(b) Make a similar sketch for a diatomic gas.

(c) Is a net work done on or by the system?

(d) Is it greater or less for the diatomic gas?

(*Wisconsin*)

Solution:

We are given that $V_A = 10l, V_B = 1l, V_C = 10l$ and $p_A = 1$ atm.

$A \to B$ is an isothermal process, thus

$$pV = \text{const. or } p_AV_A = p_BV_B \ ,$$

hence

$$p_B = \frac{V_A}{V_B} p_A = 10 \text{ atm} .$$

(The curve AB of the two kinds of gas are the same).

$B \to C$ is an adiabatic process, thus

$$pV^\gamma = \text{const, or } p_B V_B^\gamma = p_C V_C^\gamma , \quad \text{hence}$$
$$p_C = \left(\frac{V_B}{V_C}\right)^\gamma p_B = 10^{1-\gamma} \text{atm}.$$

(a) For the monatomic gas, we have

$$\gamma = 5/3, p_C = 10^{-2/3} = 0.215 \text{ atm} .$$

(b) For the diatomic gas, we have

$$\gamma = 7/5, p_C = 10^{-2/5} = 0.398 \text{ atm}.$$

The two processes are shown in the figures 1.5. (The curve BC of the monatomic gas (a) is lower than that of the diatomic gas (b)).

(c) In each case, as the curve AB for compression is higher than the curve BC for expansion, net work is done on the system. As p_C (monatomic gas) $<$ p_C (diatomic gas) the work on the monatomic gas is greater than that on the diatomic gas.

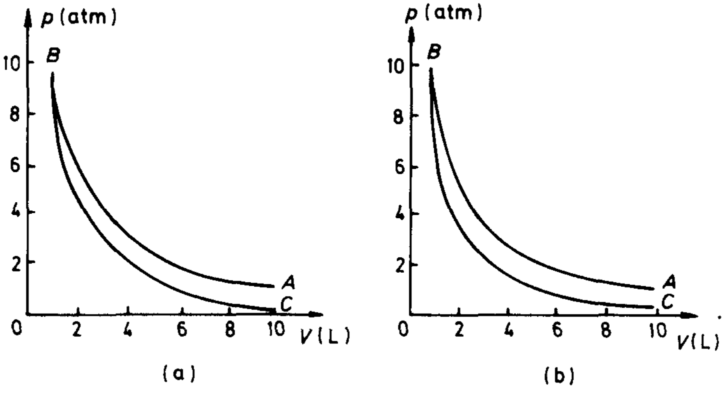

Fig. 1.5.

1019

An ideal gas is contained in a large jar of volume V_0. Fitted to the jar is a glass tube of cross-sectional area A in which a metal ball of mass M fits snugly. The equilibrium pressure in the jar is slightly higher than atmospheric pressure p_0 because of the weight of the ball. If the ball is displaced slightly from equilibrium it will execute simple harmonic motion (neglecting friction). If the states of the gas represent a quasistatic adiabatic process and γ is the ratio of specific heats, find a relation between the oscillation frequency f and the variables of the problem.

(*UC, Berkeley*)

Fig. 1.6.

Solution:

Assume the pressure in the jar is p. As the process is adiabatic, we have

$$pV^\gamma = \text{const} ,$$

giving

$$\frac{dp}{p} + \gamma\frac{\partial V}{V} = 0 .$$

This can be written as $F = Adp = -kx$, where F is the force on the ball, $x = dV/A$ and $k = \gamma A^2 p/V$. Noting that $p = p_0 + mg/A$, we obtain

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{\gamma A^2\left(p_0 + \frac{mg}{A}\right)}{Vm}} .$$

1020

The speed of longitudinal waves of small amplitude in an ideal.gas is

$$C = \sqrt{\frac{dp}{d\rho}}$$

where p is the ambient gas pressure and ρ is the corresponding gas density. Obtain expressions for

(a) The speed of sound in a gas for which the compressions and rarefactions are isothermal.

(b) The speed of sound in a gas for which the compressions and rarefactions are adiabatic.

(*Wisconsin*)

Solution:

The isothermal process of an ideal gas follows $pV = \text{const}$; the adiabatic process of an ideal gas follows $pV^\gamma = \text{const}$. We shall use $pV^t = \text{const}$ for a general process, its differential equation being

$$\frac{dp}{p} + t\frac{dV}{V} = 0 .$$

Thus

$$\left(\frac{dp}{dV}\right) = -t\frac{p}{V} .$$

With $\rho = M/V$, we have

$$\frac{dp}{d\rho} = \frac{dp}{dV}\left(\frac{dV}{d\rho}\right) = \left(-t\frac{p}{V}\right)\left(-\frac{M}{\rho^2}\right) = t\frac{RT}{M} ,$$

Therefore

$$c = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{tRT}{M}} .$$

(a) The isothermal process: $t = 1$, thus $c = \sqrt{RT/M}$.

(b) The adiabatic process: $t = \gamma$, thus $c = \sqrt{\gamma RT/M}$.

1021

Two systems with heat capacities C_1 and C_2, respectively, interact thermally and come to a common temperature T_f. If the initial temperature of system 1 was T_1, what was the initial temperature of system 2? You may assume that the total energy of the combined systems remains constant.

(*Wisconsin*)

Solution:

We assume that the initial temperature of system 2 is T_2. According to the conservation of energy, we know the heat released from system 1 is equal to that absorbed by the other system, i.e.,

$$C_1(T_f - T_1) = C_2(T_2 - T_f) \ .$$

The solution is

$$T_2 = \frac{C_1}{C_2}(T_f - T_1) + T_f \ .$$

1022

A large solenoid coil for a physics experiment is made of a single layer of conductor of cross section 4cm × 2cm with a cooling water hole 2 cm × 1cm in the conductor. The coil, which consists of 100 turns, has a diameter of 3 meters, and a length of 4 meters (the insulation thickness is negligible). At the two ends of the coil are circular steel plates to make the field uniform and to return the magnetic flux through a steel cylindrical structure external to the coil, as shown in the diagram. A magnetic field of 0.25 Tesla is desired. The conductor is made of aluminium.

(a) What power (in kilowatts) must be supplied to provide the desired field, and what must be the voltage of the power supply?

(b) What rate of water flow (litres/second) must be supplied to keep the temperature rise of the water at 40°C? Neglect all heat losses from the coil except through the water.

(c) What is the outward pressure exerted on the coil by the magnetic forces?

(d) If the coil is energized by connecting it to the design voltage calculated in (a), how much time is required to go from zero current to 99% of the design current? Neglect power supply inductance and resistance. The resistivity of aluminium is 3×10^{-8} ohm-meters. Assume that the steel is far below saturation.

(*CUSPEA*)

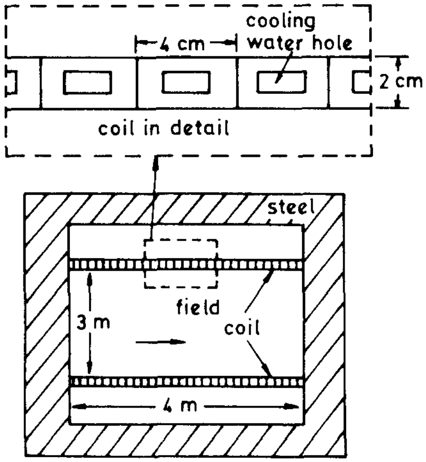

Fig. 1.7.

Solution:

(a) The magnetic field is $B = \mu_0 NI/L$, where N is the number of turns, L is the length of the solenoid coil. The current is therefore

$$I = \frac{BL}{\mu_0 N} = \frac{0.25 \times 4}{4\pi \times 10^{-7} \times 100} = 7960\text{A} .$$

The total resistance of the coil is $R = \rho L/A$. Therefore, the resistance, the voltage and the power are respectively

$$R = \frac{(3 \times 10^{-8})(100 \times 2\pi \times 1.5)}{(4 \times 2 - 2 \times 1) \times 10^{-4}} = 0.0471\Omega$$

$$V = RI = 375\text{V}$$

$$P = VI = 2.99 \times 10^3 \text{ kW} .$$

(b) The rate of flow of the cooling water is W. Then $\rho WC\Delta T = P$, where ρ is the density, C is the specific heat and ΔT is the temperature rise of the water. Hence

$$W = \frac{p}{\rho C \Delta T} = \frac{2.99 \times 10^3 \times 10^3}{1 \times 4190 \times 40} = 17.8 \ l/\text{s} .$$

(c) The magnetic pressure is

$$\rho = \frac{B^2}{2\mu_0} = \frac{(0.25)^2}{2(4\pi \times 10^{-7})} = 2.49 \times 10^4 \text{ N/m}^2 .$$

(d) The time constant of the circuit is

$$\tau = L/R, \quad \text{with} \quad L = N\Phi/I \, ,$$

where L is the inductance, R is the resistance, N is the number of turns, I is the current and Φ is the magnetic flux. Thus we have

$$L = 100 \times 0.25\pi \times (1.5)^2/7960 = 0.0222 \text{ H}$$

and

$$\tau = 0.0222/0.0471 = 0.471 \text{ s} \, .$$

The variation of the current before steady state is reached is given by

$$I(t) = I_{\max}[1 - \exp(-t/\tau)] \, .$$

When $I(t)/I_{\max} = 99\%$,

$$t = \tau \ln 100 = 4.6\tau \approx 2.17 \text{ s} \, .$$

1023

Consider a black sphere of radius R at temperature T which radiates to distant black surroundings at $T = 0$K.

(a) Surround the sphere with a nearby heat shield in the form of a black shell whose temperature is determined by radiative equilibrium. What is the temperature of the shell and what is the effect of the shell on the total power radiated to the surroundings?

(b) How is the total power radiated affected by additional heat shields? (Note that this is a crude model of a star surrounded by a dust cloud.)

(*UC, Berkeley*)

Solution:

(a) At radiative equilibrium, $J - J_1 = J_1$ or $J_1 = J/2$. Therefore $T_1^4 = T^4/2$, or $T_1 = \sqrt[4]{\frac{T^4}{2}} = \frac{T}{\sqrt[4]{2}}$.

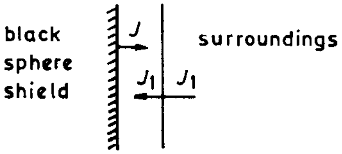

Fig. 1.8.

(b) The heat shield reduces the total power radiated to half of the initial value. This is because the shield radiates a part of the energy it absorbs back to the black sphere.

1024

In vacuum insulated cryogenic vessels (Dewars), the major source of heat transferred to the inner container is by radiation through the vacuum jacket. A technique for reducing this is to place "heat shields" in the vacuum space between the inner and outer containers. Idealize this situation by considering two infinite sheets with emissivity = 1 separated by a vacuum space. The temperatures of the sheets are T_1 and T_2 $(T_2 > T_1)$. Calculate the energy flux (at equilibrium) between them. Consider a third sheet (the heat shield) placed between the two which has a reflectivity of R. Find the equilibrium temperature of this sheet. Calculate the energy flux from sheet 2 to sheet 1 when this heat shield is in place.

For T_2 = room temperature, T_1 = liquid He temperature (4.2 K) find the temperature of a heat shield that has a reflectivity of 95%. Compare the energy flux with and without this heat shield.
$(\sigma = 0.55 \times 10^{-7}$ watts/m^{2}K)

(*UC, Berkeley*)

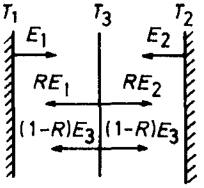

Fig. 1.9.

Solution:

When there is no "heat shield", the energy flux is

$$J = E_2 - E_1 = \sigma(T_2^4 - T_1^4) .$$

When "heat shield" is added, we have

$$J^* = E_2 - RE_2 - (1 - R)E_3 ,$$
$$J^* = (1 - R)E_3 + RE_1 - E_1 .$$

These equations imply $E_3 = (E_1 + E_2)/2$, or $T_3 = [(T_2^4 + T_1^4)/2]^{1/4}$. Hence

$$J^* = (1 - R)(E_2 - E_1)/2 = (1 - R)J/2 \ .$$

With $T_1 = 4.2$ K, $T_2 = 300$K and $R = 0.95$, we have

$$T_3 = 252 \text{ K and } J^*/J = 0.025 \ .$$

1025

Two parallel plates in vacuum, separated by a distance which is small compared with their linear dimensions, are at temperatures T_1 and T_2 respectively ($T_1 > T_2$).

(a) If the plates are non-transparent to radiation and have emission powers e_1 and e_2 respectively, show that the net energy W transferred per unit area per second is

$$W = \frac{E_1 - E_2}{\dfrac{E_1}{e_1} + \dfrac{E_2}{e_2} - 1} \ .$$

where E_1 and E_2 are the emission powers of black bodies at temperatures T_1 and T_2 respectively.

(b) Hence, what is W if T_1 is 300 K and T_2 is 4.2 K, and the plates are black bodies?

(c) What will W be if n identical black body plates are interspersed between the two plates in (b)?
($\sigma = 5.67 \times 10^{-8}\text{W/m}^2\text{K}^4$).

(*SUNY, Buffalo*)

Solution:

(a) Let f_1 and f_2 be the total emission powers (thermal radiation plus reflection) of the two plates respectively. We have

$$f_1 = e_1 + \left(1 - \frac{e_1}{E_1}\right) f_2 \ , \quad f_2 = e_2 + \left(1 - \frac{e_2}{E_2}\right) f_1 \ .$$

The solution is

$$f_1 = \frac{\frac{E_1 E_2}{e_1 e_2}(e_1 + e_2) - E_2}{\frac{E_1}{e_1} + \frac{E_2}{e_2} - 1} ,$$

$$f_2 = \frac{\frac{E_1 E_2}{e_1 e_2}(e_1 + e_2) - E_1}{\frac{E_1}{e_1} + \frac{E_2}{e_2} - 1} .$$

Hence

$$W = f_1 - f_2 = \frac{E_1 - E_2}{\frac{E_1}{e_1} + \frac{E_2}{e_2} - 1} .$$

(b) For black bodies, $W = E_1 - E_2 = \sigma(T_1^4 - T_2^4) = 460\ \mathrm{W/m^2}$.

(c) Assume that the n interspersed plates are black bodies at temperatures $t_1, t_2, \ldots, t_n$. When equilibrium is reached, we have

$$T_1^4 - t_1^4 = t_1^4 - T_2^4 , \quad \text{for } n = 1 ,$$

with solution

$$t_1^4 = \frac{T_1^4 + T_2^4}{2} , \quad W = \sigma(T_1^4 - t_1^4) = \frac{\sigma}{2}(T_1^4 - T_2^4) .$$

$$T_1^4 - t_1^4 = t_1^4 - t_2^4 = t_2^4 - T_2^4 , \quad \text{for } n = 2 ,$$

with solution

$$t_1^4 = \frac{4}{3}\left(\frac{T_1^4}{2} + \frac{T_2^4}{4}\right) , \quad W = \frac{\sigma}{3}(T_1^4 - T_2^4) .$$

Then in the general we have

$$T_1^4 - t_1^4 = t_1^4 - t_2^4 = \ldots = t_n^4 - T_2^4 ,$$

with solution

$$t_1^4 = \frac{n}{n+1}T_1^4 - \frac{1}{n+1}T_2^4 ,$$

$$W = \sigma(T_1^4 - T_2^4) = \frac{\sigma}{n+1}(T_1^4 - T_2^4) .$$

1026

A spherical black body of radius r at absolute temperature T is surrounded by a thin spherical and concentric shell of radius R, black on both sides. Show that the factor by which this radiation shield reduces the rate of cooling of the body (consider space between spheres evacuated, with no thermal conduction losses) is given by the following expression: $aR^2/(R^2+br^2)$, and find the numerical coefficients a and b.

(SUNY, Buffalo)

Solution:

Let the surrounding temperature be T_0. The rate of energy loss of the black body before being surrounded by the spherical shell is

$$Q = 4\pi r^2\sigma(T^4 - T_0^4) .$$

The energy loss per unit time by the black body after being surrounded by the shell is

$$Q' = 4\pi r^2\sigma(T^4 - T_1^4), \text{ where } T_1 \text{ is temperature of the shell} .$$

The energy loss per unit time by the shell is

$$Q'' = 4\pi R^2\sigma(T_1^4 - T_0^4) .$$

Since $Q'' = Q'$, we obtain

$$T_1^4 = (r^2T^4 + R^2T_0^4)/(R^2 + r^2) .$$

Hence $Q'/Q = R^2/(R^2 + r^2)$, i.e., $a = 1$ and $b = 1$.

1027

The solar constant (radiant flux at the surface of the earth) is about 0.1 W/cm^2. Find the temperature of the sun assuming that it is a black body.

(MIT)

Solution:

The radiant flux density of the sun is

$J = \sigma T^4$, where $\sigma = 5.7 \times 10^{-8}$ W/m^{2}K^4. Hence $\sigma T^4(r_S/r_{SE})^2 = 0.1$, where the radius of the sun $r_S = 7.0 \times 10^5$km, the distance between the earth and the sun $r_{SE} = 1.5 \times 10^8$km. Thus

$$T = \left[\frac{0.1}{\sigma}\left(\frac{r_{SE}}{r_S}\right)^2\right]^{\frac{1}{4}} \approx 5 \times 10^3 \text{ K} .$$

1028

(a) Estimate the temperature of the sun's surface given that the sun subtends an angle θ as seen from the earth and the earth's surface temperature is T_0. (Assume the earth's surface temperature is uniform, and that the earth reflects a fraction, ε, of the solar radiation incident upon it). Use your result to obtain a rough estimate of the sun's surface temperature by putting in "reasonable" values for all parameters.

(b) Within an unheated glass house on the earth's surface the temperature is generally greater than T_0. Why? What can you say about the maximum possible interior temperature in principle?

(*Columbia*)

Solution:

(a) The earth radiates heat while it is absorbing heat from the solar radiation. Assume that the sun can be taken as a black body. Because of reflection, the earth is a grey body of emissivity $1 - \varepsilon$. The equilibrium condition is

$$(1 - \varepsilon) J_S 4\pi R_S^2 \cdot \pi R_E^2 / 4\pi r_{S-E}^2 = J_E \cdot 4\pi R_E^2 \ ,$$

where J_S and J_E are the radiated energy flux densities on the surfaces of the sun and the earth respectively, R_S, R_E and r_{S-E} are the radius of the sun, the radius of the earth and the distance between the earth and the sun respectively. Obviously $R_S/r_{S-E} = \tan(\theta/2)$. From the Stefan-Boltzman law, we have

for the sun, $J_S = \sigma T_S^4$;
for the earth $J_E = (1 - \varepsilon)\sigma T_E^4$.

Therefore

$$T_S = T_E\sqrt{\frac{2r_{S-E}}{R_S}} \approx 300 \text{ K} \times \left(2 \times \frac{1.5 \times 10^8 \text{ km}}{7 \times 10^6 \text{ km}}\right)^{1/2}$$
$$\approx 6000 \text{ K} .$$

(b) Let T be temperature of the glass house and t be the transmission coefficient of glass. Then

$$(1-t)T^4 + tT_0^4 = tT^4 ,$$

giving

$$T = \left[\frac{t}{(2t-1)}\right]^{1/4} T_0 .$$

Since $t < 1$, we have $t > 2t - 1$, so that

$$T > T_0 .$$

1029

Consider an idealized sun and earth, both black bodies, in otherwise empty flat space. The sun is at a temperature of $T_S = 6000$ K and heat transfer by oceans and atmosphere on the earth is so effective as to keep the earth's surface temperature uniform. The radius of the earth is $R_E = 6 \times 10^8$ cm, the radius of the sun is $R_S = 7 \times 10^{10}$ cm, and the earth-sun distance is $d = 1.5 \times 10^{13}$ cm.

(a) Find the temperature of the earth.

(b) Find the radiation force on the earth.

(c) Compare these results with those for an interplanetary "chondrule" in the form of a spherical, perfectly conducting black-body with a radius of $R = 0.1$cm, moving in a circular orbit around the sun with a radius equal to the earth-sun distance d.

(Princeton)

Solution:

(a) The radiation received per second by the earth from the sun is approximately

$$q_{\mathrm{SE}} = 4\pi R_{\mathrm{S}}^2(\sigma T_{\mathrm{S}}^4)\frac{\pi R_{\mathrm{E}}^2}{4\pi d^2} .$$

The radiation per second from the earth itself is

$$q_{\mathrm{E}} = 4\pi R_{\mathrm{E}}^2 \cdot (\sigma T_{\mathrm{E}}^4) .$$

Neglecting the earth's own heat sources, energy conservation leads to the relation $q_{\mathrm{E}} = q_{\mathrm{SE}}$, so that

$$T_{\mathrm{E}}^4 = \frac{R_{\mathrm{S}}^2}{4d^2} T_{\mathrm{S}}^4 \ ,$$

i.e.,

$$T_{\mathrm{E}} = \sqrt{R_{\mathrm{S}}/2d} \cdot T_{\mathrm{S}} = 290 \ \mathrm{K} = 17^\circ\mathrm{C} \ .$$

(b) The angles subtended by the earth in respect of the sun and by the sun in respect of the earth are very small, so the radiation force is

$$F_{\mathrm{E}} = \frac{q_{\mathrm{E}}}{c} = \frac{1}{c} \frac{R_{\mathrm{S}}^2}{d^2} \cdot \pi R_{\mathrm{E}}^2 \cdot (\sigma T_{\mathrm{S}}^4) = 6 \times 10^8 \ \mathrm{N} \ .$$

(c) As $R_{\mathrm{E}} \to R, T = T_{\mathrm{E}} = 17^\circ\mathrm{C}$

$$F = (R/R_{\mathrm{E}})^2 F_{\mathrm{E}} = 1.7 \times 10^{-11} \ \mathrm{N} \ .$$

1030

Making reasonable assumptions, estimate the surface temperature of Neptune. Neglect any possible internal sources of heat. What assumptions have you made about the planet's surface and/or atmosphere?

Astronomical data which may be helpful: radius of sun$=7 \times 10^5$ km; radius of Neptune $= 2.2 \times 10^4$ km; mean sun-earth distance $= 1.5 \times 10^8$ km; mean sun-Neptune distance $= 4.5 \times 10^9$ km; $T_{\mathrm{S}} = 6000$ K; rate at which sun's radiation reaches earth $= 1.4$ kW/m^2; Stefan-Boltzman constant $= 5.7 \times 10^{-8}$ W/m^{2}K^4.

(*Wisconsin*)

Solution:

We assume that the surface of Neptune and the thermodynamics of its atmosphere are similar to those of the earth. The radiation flux on the earth's surface is

$$J_{\mathrm{E}} = 4\pi R_{\mathrm{S}}^2 \sigma T_{\mathrm{S}}^4 / 4\pi R_{\mathrm{SE}}^2$$

The equilibrium condition on Neptune's surface gives

$$4\pi R_{\mathrm{S}}^2 \sigma T_{\mathrm{S}}^4 \cdot \pi R_{\mathrm{N}}^2 / 4\pi R_{\mathrm{SN}}^2 = \sigma T_{\mathrm{N}}^4 \cdot 4\pi R_{\mathrm{N}}^2 \ .$$

Hence

$$R_{\mathrm{SE}}^2 J_{\mathrm{E}}/R_{\mathrm{SN}}^2 = 4\sigma T_{\mathrm{N}}^4 ,$$

and we have

$$\begin{aligned} T_{\mathrm{N}} &= (R_{\mathrm{SE}}^2 J_{\mathrm{E}}/4\sigma R_{\mathrm{SN}}^2)^{1/4} \\ &= \left[\frac{(1.5\times 10^8)^2}{(5.7\times 10^9)^2}\cdot\frac{1.4\times 10^3}{4\times 5.7\times 10^{-8}}\right]^{\frac{1}{4}} \\ &= 52\ \mathrm{K} . \end{aligned}$$

2. THE SECOND LAW AND ENTROPY (1031-1072)

1031

A steam turbine is operated with an intake temperature of 400°C, and an exhaust temperature of 150°C. What is the maximum amount of work the turbine can do for a given heat input Q? Under what conditions is the maximum achieved?

(*Wisconsin*)

Solution:

From the Clausius formula

$$\frac{Q_1}{T_1} - \frac{Q_2}{T_2} \leq 0 ,$$

we find the external work to be

$$W = Q_1 - Q_2 \leq \left(1 - \frac{T_2}{T_1}\right) Q_1 .$$

Substituting $Q_1 = Q, T_1 = 673$ K and $T_2 = 423$ K in the above we have

$$W_{\max} = \left(1 - \frac{T_2}{T_1}\right) Q = 0.37Q .$$

As the equal sign in the Clausius formula is valid if and only if the cycle is reversible, when and only when the steam turbine is a reversible engine can it achieve maximum work.