2. Kinematic Variables

In relativistic heavy-ion collisions and in many other high-energy reaction processes, it is convenient to use kinematic variables which have simple properties under a change of the frame of reference. The light-cone variables x_+ and x_-, the rapidity variable y, and the pseudorapidity variable η are kinematic variables which have simple properties under a Lorentz transformation. They are commonly used. The Feynman scaling variable, $x_{_F}$, which is related to the light-cone variables x_+ and x_-, is also often used. It is worthwhile to discuss these variables in detail to establish the proper language for relativistic reactions.

§2.1 Notation and Conventions

In this book, we shall follow the notation of Bjorken and Drell, *Relativistic Quantum Mechanics*, (McGraw-Hill Book Company, N.Y. 1964). We use the *natural units* $c = \hbar = 1$. The space-time coordinates of a point x are denoted by a *contravariant vector* with components x^μ:

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, \boldsymbol{x}) = (t, x, y, z)\,. \qquad (2.1)$$

The momentum vector p is similarly defined by a contravariant vector with components p^μ:

$$p^\mu = (p^0, p^1, p^2, p^3) = (E, \boldsymbol{p}) = (E, \boldsymbol{p}_{_T}, p_z) = (E, p_x, p_y, p_z). \qquad (2.2)$$

We shall adopt the space-time *metric tensor* $g_{\mu\nu}$ in the form

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \qquad (2.3)$$

The *covariant vector* x_μ is related to the contravariant vector x^μ through the metric tensor $g_{\mu\nu}$ by

$$x_\mu \equiv (x_0, x_1, x_2, x_3) \equiv g_{\mu\nu} x^\nu = (t, -x, -y, -z)\,, \qquad (2.4)$$

where we use the notation that a repeated index implies a summation with respect to that index, unless indicated otherwise. Conversely, the contravariant vector x^μ is related to the corresponding covariant vector x_ν by

$$x^\mu \equiv g^{\mu\nu} x_\nu \,, \tag{2.5}$$

where the metric tensor $g^{\mu\nu}$ is

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{2.6}$$

The *scalar product of two vectors* a and b is defined as

$$a \cdot b \equiv a^\mu b_\mu = g_{\mu\nu} a^\mu b^\nu = a^0 b^0 - \boldsymbol{a} \cdot \boldsymbol{b} \,.$$

The *four-momentum operator* p^μ in coordinate representation is

$$p^\mu = i\partial^\mu = (i\partial^0, i\partial^1, i\partial^2, i\partial^3) \tag{2.7a}$$

$$= (i\partial_0, -i\partial_1, -i\partial_2, -i\partial_3) = (i\frac{\partial}{\partial x^0}, -i\frac{\partial}{\partial x^1}, -i\frac{\partial}{\partial x^2}, -i\frac{\partial}{\partial x^3}). \tag{2.7b}$$

The covariant operator ∂_i is the usual *gradient operator* ∇ defined by

$$\nabla = (\nabla_x, \nabla_y, \nabla_z) = (\partial_1, \partial_2, \partial_3) = (\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3) \,. \tag{2.8}$$

⊕⟦ Supplement 2.1

The placement of the indices in the tensor notation of contravariant and covariant vectors is confusing for beginning students when they are dealing with relativistic kinematics. In the "common" notation for vectors, only subscript indices are used and contravariant and covariant vectors are not distinguished. This is permissible in the Euclidean space, where $g_{\mu\nu} = \delta_{\mu\nu}$ and there is no significant distinction between a contravariant vector and a covariant vector. However, in Minkowski space with the metric tensor (2.3), they are different types of vectors as they have different properties under a coordinate transformation. In the tensor notation, the contravariant vectors have superscript indices but the covariant vectors have subscript indices. The tensor notation makes a clear distinction between these two types of vectors. The easiest way to understand the distinction between them is to remember that the suffixes 'contra-' and 'co-' refer to a comparison with the gradient vector. 'Contravariant' corresponds to the nomenclature 'contragradient', and 'covariant' corresponds to the term 'cogradient' [1]. A quantity which transforms like a gradient vector is 'cogradient' and is therefore a covariant vector with a subscript index. A quantity which transforms like the coordinates is 'contragradient' and is a contravariant vector with a superscript index.

Accordingly, the coordinate x^μ and the momentum p^μ are contravariant four-vectors and they have superscript indices. The covariant vectors x_μ and p_μ, with subscript indices derived from these vectors, are different vectors because the signs of their space components are changed as in Eq. (2.4).

On the other hand, because superscript indices are clumsy to use, the common notation for a vector uses subscript indices to refer to the components of any vector. It does not distinguish a contravariant vector from a covariant vector. Unfortunately, the most commonly used vectors, such as the coordinate vector and the momentum vector, are contravariant vectors. They have superscript indices in the tensor notation but subscript indices in the common notation. It is unavoidable that one uses both notations for these quantities. Therefore, when these quantities are written out, it is necessary to make a mental note as to which notation is adopted.

It is worth recommending that one adheres to the tensor notation as much as possible, to insure that one gets the correct signs in an algebraic manipulation with vector quantities. As an example, we can follow the line of reasoning which gives the coordinate representation of the momentum operator (2.7). There, the signs in front of the ∂ operators in Eqs. (2.7a) and (2.7b) are often a source of confusion for beginning students. We begin by recalling that the coordinate vector x^μ and the momentum vector p^μ are contravariant vectors. The gradient operator ∂_μ is a covariant vector, and it is the derivative with respect to the μ component of x. As x is a contravariant vector, ∂_μ is the derivative with respect to x^μ:

$$\partial_\mu = \frac{\partial}{\partial x^\mu}\,.$$

All other vectors such as p_μ, x_μ, and ∂^μ are derived in terms of the basic vectors x^μ, p^μ, and ∂_μ by using the metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$ as in Eqs. (2.4) and (2.5).

Being a contravariant vector, the momentum vector has a coordinate representation which is naturally another contravariant vector $i\partial^\mu$. If we want to write out the operator $i\partial^\mu$ explicitly in terms of the derivatives of the coordinates, it is necessary to express ∂^μ in terms of the covariant operator ∂_μ by using the metric tensor $g^{\mu\nu}$. Thus, we have

$$\begin{aligned} p^\mu &= i\partial^\mu \\ &= i g^{\mu\nu}\partial_\nu\,. \end{aligned}$$

Writing the covariant operator ∂_ν explicitly as the derivative with respect to the contravariant vector x^μ, we have

$$\begin{aligned} p^\mu &= i g^{\mu\nu}\frac{\partial}{\partial x^\nu} \\ &= (i\frac{\partial}{\partial x^0}, -i\frac{\partial}{\partial x^1}, -i\frac{\partial}{\partial x^2}, -i\frac{\partial}{\partial x^3}) \end{aligned}$$

which is the expression in Eq. (2.7b). ⟧⊕

§2.2 Light-Cone Variables

In many high-energy reaction processes, a detected particle can be identified as originating from one of the colliding particles. For example, in the reaction

$$b + a \rightarrow c + X\,,$$