10

Nonlinear Properties Derivable from Small-Amplitude Waves

Dedication

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Hermann Ludwig Ferdinand von Helmholtz (1821–1894) was born in Potsdam, southwest of Berlin. The dedication of this chapter to Helmholtz is in recognition of his extensive contributions to fluid dynamics and physics in general. While he did work in the area of waves, his major contribution to this text is the Helmholtz equation, which governs the motion of waves in harbors.

Helmholtz entered the Pepiniere Berlin University in 1838 to study medicine. During his formal education, Gustav Magnus and others influenced him to expand his interest to natural sciences. In 1842 he graduated, successfully defending his work on ganglia. From 1842 to 1845, simultaneous to Kelvin's activities, he investigated the mechanical equivalent of heat. In 1849 he took a professorship in physiology at Königsburg, where he developed an interest in the importance of electricity in the working of the human body and studied ophthalmology and color vision. In 1855 he moved to Bonn and in 1858 to another chair at Heidelberg. There he developed his theories on vortex motion, free streamline flows, and the viscosity of water. In 1871 he succeeded Magnus at the University of Berlin, where he built a physical sciences institute which educated many well-known scientists, such as Heinrich Hertz and Max Planck. Planck has been quoted as observing: "Wir hatten das Gefühl, dass er sich selber mindestens ebenso langweilte wie wir" ("We had the feeling that he himself was at least as bored as we were"). Clearly, he engendered a testimonial distinct from the one Lamb received from his students.

In 1883 Helmholtz became a Prussian noble in recognition of his scientific contributions. In 1888 he assumed the leadership of the Physical Technical Government Institute (Reichsanstalt) in Charlottenburg, West Berlin.

Other areas of interest for Helmholtz included the physiology of optics, binocular vision, acoustics, and the physiology of the ear, sound (harmony), and electrodynamics.

10.1 INTRODUCTION

Wave energy and power, which were derived in Chapter 4, are nonlinear quantities obtained from the linear wave theory—nonlinear in the sense that they involve the wave height to the second power. In this chapter other nonlinear quantities will be sought which have a bearing on coastal and ocean design. These quantities, which are time averaged, are correct to second order in ak, yet have their origin strictly in *linear* theory. In Chapter 11 a further and more complete study of nonlinear waves is undertaken.

10.2 MASS TRANSPORT AND MOMENTUM FLUX

If a small neutrally buoyant float is placed in a wave tank and its trajectory traced as waves pass by, a small mean motion in the direction of the waves can be observed. The closer to the water surface, the greater the tendency for this net motion. This motion of the float, which is indicative of the mean fluid motion, is a nonlinear effect, as the trajectory of the water particles from linear theory are predicted to be closed ellipses (see Chapter 4).

There are two approaches for examining this mass transport: the Eulerian frame, using a fixed point to measure the mean flux of mass, or the Lagrangian frame, which involves moving with the water particles.

10.2.1 Eulerian Mass Transport

Examining the horizontal velocity at any point below the water surface and averaging over a wave period shows that

$$\overline{u}(x, z) = \frac{1}{T}\int_0^T u(x, z)\, dt = 0 \tag{10.1}$$

However, in the region between the trough and the wave crest, the horizontal velocity must be obtained by the Taylor series. For example, for the surface velocities we have, approximately,[1]

[1]Neglecting some contributions from second-order theory.

$$u(x, \eta) = u(x, 0) + \eta \left.\frac{\partial u}{\partial z}\right|_{z=0} \tag{10.2}$$

$$= \frac{gak}{\sigma} \left.\frac{\cosh k(h+z)}{\cosh kh}\right|_{z=0} \cos(kx - \sigma t) + \frac{ga^2k^2}{\sigma} \tanh kh \cos^2(kx - \sigma t)$$

$$= \frac{gak}{\sigma} \cos(kx - \sigma t) + a^2k\sigma \cos^2(kx - \sigma t)$$

The surface velocity is periodic, yet faster at the wave crest than at the wave trough, as the second term is always positive at these two phase positions. This asymmetry of velocity indicates that more fluid moves in the wave direction under the wave crest than in the trough region. This is, in fact, true. If we average $u(x, \eta)$ over a wave period (an operation denoted by an overbar), there is a mean transport of water[2]

$$\bar{u}(x, \eta) \equiv \frac{1}{T}\int_0^T u(x, \eta)\, dt = \frac{a^2k\sigma}{2} = \frac{(ka)^2C}{2} \tag{10.3}$$

To obtain the total mean flux, or flow of mass, we perform the following integration, where M is defined as the mass transport

$$M \equiv \overline{\int_{-h}^{\eta} \rho u\, dz} = \overline{\int_{-h}^{0} \rho u\, dz} + \overline{\eta \rho u} = \frac{E}{C} \tag{10.4}$$

a result first presented by Starr (1947). Note that the first term in Eq. (10.4) is zero; again, there is no mean flow except due to the contribution of the region bounded vertically by η. The depth-averaged time-mean velocity, due to mass transport, is

$$U = \frac{M}{\rho h} \tag{10.5}$$

10.2.2 Lagrangian Mass Transport

The Eulerian velocity discussed above is obtained by examining the velocity at a fixed point. A Lagrangian velocity is one obtained by moving with a particle as it changes location. The velocity of a particular water particle with a mean position of (x_1, z_1) is $u(x_1 + \zeta, z_1 + \xi)$, where ζ and ξ are locations on the trajectory of the particle. An approximation to the instantaneous velocity is

$$u_L(x_1 + \zeta, z_1 + \xi) = u(x_1, z_1) + \frac{\partial u}{\partial x}\zeta + \frac{\partial u}{\partial z}\xi \tag{10.6}$$

[2]Clearly, $\bar{u}(x, \eta)$ is much less than the phase speed of the wave, C.

Using the values of the trajectory obtained in Chapter 4 [Eqs. (4.9) and (4.10) evaluated at (x_1, z_1)], u_L can be written as

$$u_L = \frac{gak}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx - \sigma t) + \frac{a^2\sigma k}{\sinh^2 kh}[\cosh^2 k(h+z)\sin^2(kx-\sigma t) + \sinh^2 k(h+z)\cos^2(kx-\sigma t)] \tag{10.7}$$

The mean value of u_L is

$$\overline{u}_L(x_1+\zeta, z_1+\xi) = \frac{a^2\sigma k\cosh 2k(h+z)}{2\sinh^2 kh} = \frac{ga^2k^2\cosh 2k(h+z)}{\sigma\sinh 2kh} \tag{10.8}$$

This mean Lagrangian velocity indicates that the water particles drift in the direction of the waves and move more rapidly at the surface than at the bottom.

Integrating over the water column to obtain the total transport and multiplying by the density of the fluid yields, as before,

$$M = \int_{-h}^{0}\rho\overline{u}_L\,dz = \frac{\rho g a^2 k}{2\sigma} = \frac{E}{C} \tag{10.9}$$

10.3 MEAN WATER LEVEL

The Bernoulli equation at the free surface, Eq. (3.13), is

$$\frac{(\partial\phi/\partial x)^2 + (\partial\phi/\partial z)^2}{2} - \frac{\partial\phi}{\partial t} + gz = C(t) \qquad \text{on } z = \eta \tag{10.10}$$

Expanding to the free surface by the Taylor series yields to *first* order in η after time averaging (which is denoted by the overbar),

$$\overline{\frac{(\partial\phi/\partial x)^2 + (\partial\phi/\partial z)^2}{2}} + g\overline{\eta} - \overline{\eta\frac{\partial^2\phi}{\partial t\,\partial z}} = \overline{C(t)} \tag{10.11}$$

where $\overline{\eta}$ is a mean displacement in water level from $z = 0$. Substituting for η and ϕ from the linear *progressive* wave theory, we have

$$\overline{\eta} = -\frac{a^2k}{2\sinh 2kh} + \frac{\overline{C(t)}}{g} \equiv -f(x) + \frac{\overline{C(t)}}{g} \tag{10.12}$$

There are several choices for $C(t)$ here, depending on the problem. If the problem is one of waves propagating from deep to shallow water, a customary boundary condition is $\overline{\eta}$ is zero in deep water, which fixes $\overline{C(t)} = 0$ everywhere. Thus $\overline{\eta}$ is always negative, becoming more so as the wave enters shallow water until breaking commences. This is called the setdown. Alternatively, we can force the x axis ($z = 0$) to be the mean water level at some fixed

x_1 by setting $\overline{C(t)} = f(x_1)g$ in Eq. (10.12), where f is now a constant. As another example, in an enclosed tank where the amount of water in the tank must be conserved, a continuity argument must be invoked for $\overline{C(t)}$. If the tank is of length l, then

$$\frac{1}{l}\int_0^l \bar{\eta}(x)\, dx = 0 \tag{10.13a}$$

or, from Eq. (10.12),

$$\frac{1}{l}\int_0^l f(x)\, dx = \frac{\overline{C(t)}}{g} \tag{10.13b}$$

The mean water level associated with standing waves is

$$\bar{\eta} = \frac{\overline{C(t)}}{g} + \frac{a^2 k}{4 \sinh 2kh}(\cosh 2kh \cos 2kx - 1)$$

This is left as an exercise for the reader (Problem 10.3).

10.4 MEAN PRESSURE

The mean pressure under a wave can be most easily obtained by time-averaging the Bernoulli equation:

$$p(z) = \rho\frac{\partial\phi}{\partial t} - \rho\frac{u^2 + w^2}{2} - \rho g z + C(t) \tag{10.14a}$$

or

$$\bar{p}(z) = -\rho\frac{\overline{u^2 + w^2}}{2} - \rho g z + \overline{C}(t) \tag{10.14b}$$

under a progressive wave. If $\overline{C}(t) = 0$, the case for shoaling progressive waves, then it is clear that the mean pressure is decreased from its hydrostatic value. As (u, w) decrease with depth into the water, the mean pressure approaches hydrostatic with depth. Substituting into the equation above yields

$$\bar{p}(z) = -\frac{\rho g a^2 k \cosh 2k(h+z)}{2 \sinh 2kh} - \rho g z \tag{10.15}$$

Alternatively, if the coordinate system is located at the mean water level such that $\overline{C}(t) = f(x_1)g$ and $\bar{\eta} = 0$, it can be shown that

$$\bar{p}(z^*) = \overline{-\rho w^2} - \rho g z^* \tag{10.16}$$

where z^* differs by $\bar{\eta}$ from the z of the other coordinate system (see Figure 10.1).

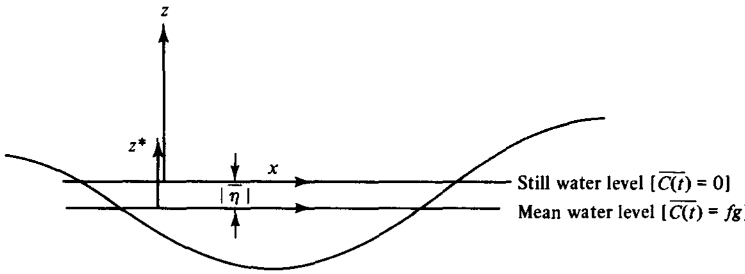

Figure 10.1 The two vertical reference systems and associated Bernoulli constants.

Under a standing wave of amplitude a,

$$\bar{p}(z) = -\frac{\rho g a^2 k}{4 \sinh 2kh}[\cosh 2k(h+z) - \cos 2kx] - \rho g z \tag{10.17}$$

and at the bottom,

$$\bar{p}(-h) = \frac{\rho g a^2 k}{4 \sinh 2kh}(\cos 2kx - 1) + \rho g h \tag{10.18}$$

10.5 MOMENTUM FLUX

At a point above the trough level, there is a mean momentum flux as well as mass flux. The mean vertically averaged momentum flux correct to second order in ka is

$$\overline{\int_{-h}^{\eta} (\rho u) u \, dz} = MC_g \tag{10.19}$$

where C_g is the group velocity, the speed at which the wave energy propagates.

The flux of momentum in the direction of the wave past a section and the pressure force per unit width is defined as

$$I_x = MC_g + \overline{\int_{-h}^{\eta} p(z)\, dz} \tag{10.20}$$

From Newton's second law, this quantity is unchanged between any two sections unless forces are applied. Evaluating the last integral yields the expression

$$I_x = MC_g + \tfrac{1}{2}\rho g h^2 \tag{10.21}$$

I_x can be rewritten as

$$I_x = S_{xx} + \tfrac{1}{2}\rho g(h + \bar{\eta})^2 \tag{10.22}$$

where S_{xx} is the radiation stress in the direction of the waves.

$$S_{xx} \equiv \overline{\int_{-h}^{\eta} p(z)\,dz} - \tfrac{1}{2}\rho g(h+\bar{\eta})^2 + MC_g = E(2n - \tfrac{1}{2}) \tag{10.23}$$

The difference between the two forms for I is that the latter explicitly includes the mean water level $\bar{\eta}$. Each form is important for different applications.

For the flux of momentum transverse to the wave direction, we have

$$\overline{\int_{-h}^{\eta} (\rho v)v\,dz} = 0 \tag{10.24}$$

The sum of momentum flux and pressure force in the transverse direction is

$$I_y = \overline{\int_{-h}^{\eta} p(z)\,dz} = \tfrac{1}{2}\rho g h^2$$

or

$$I_y = S_{yy} + \tfrac{1}{2}\rho g(h+\bar{\eta})^2$$

where

$$\begin{aligned} S_{yy} &\equiv \overline{\int_{-h}^{\eta} p(z)\,dz} - \tfrac{1}{2}\rho g(h+\bar{\eta})^2 \\ &= -\rho g h\bar{\eta} \text{ to } O(ka)^2 \\ &= E(n - \tfrac{1}{2}) \end{aligned}$$

If a progressive wave is propagating at some angle θ to the x axis, then S_{xx} and S_{yy} are modified to the following forms:

$$S_{xx} = E[n\,(\cos^2\theta + 1) - \tfrac{1}{2}] \tag{10.25}$$

$$S_{yy} = E[n\,(\sin^2\theta + 1) - \tfrac{1}{2}] \tag{10.26}$$

in which n is the ratio of group velocity to wave celerity ($n = C_G/C$). In addition, for this case there is an additional term representing the flux in the x direction of the y component of momentum, denoted S_{xy}:

$$S_{xy} = \overline{\int_{-h}^{\eta} \rho uv\,dz} \approx \overline{\int_{-h}^{0} \rho(uv)\,dz} \tag{10.27}$$

and employing linear wave theory, it can be shown that

$$S_{xy} = \frac{E}{2}\,n\sin 2\theta \tag{10.28}$$

It is of interest to note that, if the bathymetry is composed of straight and parallel contours and if no energy dissipation or additions occur, there is no change in S_{xy} from deep to shallow water.

For further information on radiation stresses and their uses, the reader is referred to Longuet-Higgins and Stewart (1964), Longuet-Higgins (1976), and Phillips (1966).

Example 10.1: Wave Setdown and Setup

As waves shoal and break on a beach, the momentum flux in the onshore direction is reduced and results in compensating forces on the water column. Consider a train of waves encountering the coast with normal incidence. For a short distance dx (Figure 10.2), a force balance can be developed

$$I_1 = I_2 - R_x \tag{10.29a}$$

$$I - \frac{dI}{dx}\frac{dx}{2} = I + \frac{dI}{dx}\frac{dx}{2} - R_x \tag{10.29b}$$

or finally,

$$\frac{dI}{dx}dx = R_x \tag{10.29c}$$

using the Taylor series expansion, where I is evaluated at the center and R_x is the reaction force of the bottom in the $(-x)$ direction. Using the radiation stress approach,

$$\begin{aligned}\frac{dI_x}{dx} &= \frac{d}{dx}[S_{xx} + \tfrac{1}{2}\rho g(h + \bar{\eta})^2]\\ &= \frac{dS_{xx}}{dx} + \rho g(h + \eta)\frac{d(h + \bar{\eta})}{dx}\end{aligned} \tag{10.30}$$

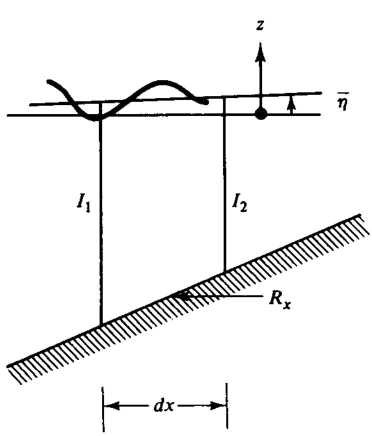

Figure 10.2 Schematic diagram for calculation of wave setup or setdown.

For a mildly sloping bottom, the reaction force R is due to the weight of the column of fluid and thus

$$R_x = \rho g(h + \bar{\eta}) \frac{dh}{dx} dx$$

Substituting yields

$$\boxed{-\frac{1}{\rho g(h + \bar{\eta})} \frac{dS_{xx}}{dx} = \frac{d\bar{\eta}}{dx}} \tag{10.31}$$

There is therefore a change in mean water surface slope whenever there is a change in S_{xx}. The change in $\bar{\eta}$ offshore of the breaker line is described by Eq. (10.12), which describes a gradual reduction of the mean water level as the shoreline is approached. At $x = x_b$, the breaker line, the wave amplitude is $a = \kappa(h + \bar{\eta})/2$, where κ is the breaking index (Chapter 4), and $\bar{\eta}$ (in shallow water) is

$$\bar{\eta} = -\frac{a^2}{4h_b}$$

as given by Longuet-Higgins and Stewart (1964) or

$$\bar{\eta} = -\frac{\kappa^2 h_b}{16} \tag{10.32}$$

The setdown therefore is less than 5% of the breaking depth for $\kappa = 0.8$.

Inside the surf zone, where $a(x) = \kappa(h + \bar{\eta})/2$, based on a spilling breaker model, the setup is found from the force balance, Eq. (10.31):

$$-\frac{1}{\rho g(h + \bar{\eta})} \frac{d}{dx}\left[\frac{1}{2}\rho g \frac{\kappa^2(h + \bar{\eta})^2}{4} \frac{3}{2}\right] = \frac{d\bar{\eta}}{dx}$$

Simplifying yields

$$\frac{d\bar{\eta}}{dx}\left(1 + \frac{3\kappa^2}{8}\right) = -\frac{3\kappa^2}{8} \frac{dh}{dx} \tag{10.33}$$

Finally,

$$\bar{\eta} = -\frac{3\kappa^2/8}{1 + 3\kappa^2/8} h + C \tag{10.34}$$

Evaluating the constant at $x = x_b$, the breaker line, where $\eta = \eta_b$, gives finally

$$\bar{\eta}(x) = \bar{\eta}_b + \frac{3\kappa^2/8}{1 + 3\kappa^2/8} [h_b - h(x)] \tag{10.35}$$

The mean water surface displacement $\bar{\eta}$ thus increases linearly with depth as the shore is approached. This water surface slope provides a hydrostatic pressure gradient directed offshore to counter the change of wave momentum by breaking across the shoreline.

$$\bar{\eta}(0) = \bar{\eta}_b + \frac{3\kappa^2/8}{1 + 3\kappa^2/8} h_b \tag{10.36}$$

or, for $\kappa = 0.8$, $\bar{\eta}(0)$ is about 15% of the breaker depth or about 19% of the breaking wave height.

Example 10.2: Applied Longshore Wave Thrust

For waves propagating obliquely into the surf zone, breaking will result in a reduction in wave energy and an associated decrease in S_{xy} [cf. Eq. (10.28)], which is manifested as an applied longshore wave thrust F_y on the surf zone. For straight and parallel bottom contours, thrust per unit area is given by

$$F_y = -\frac{\partial S_{xy}}{\partial x} \tag{10.37}$$

Thus gradients of the momentum flux terms provide a useful framework for the driving forces in the nearshore zone. In the present case, the longshore wave thrust per unit area is resisted by shear stresses on the bottom and lateral faces of the water column (Longuet-Higgins, 1970).

10.6 SUMMARY

The results of linear wave theory may be used to calculate nonlinear mean quantities, correct to second order in ka. These quantities, such as mass transport and mean momentum flux, play a major role in coastal engineering. In fact, the mean momentum flux of the waves in the longshore direction, relative to a coastline, is related to the currents engendered at the coastline and the amounts of sediments transported along the coast. See, for an overview, the book by Komar (1976). In the open ocean, the mean momentum flux results in the drifting of objects, such as ships, ice flows, and oil slicks.

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PROBLEMS

10.1 Determine the mean water level due to a wave train impinging on a perfectly reflecting vertical wall with an angle θ.

10.2 Calculate the mean water level associated with an edge wave,

$$\phi = \frac{ga}{\sigma} e^{-k(y\cos\beta - z\sin\beta)} \cos kx \sin \sigma t$$

where y is positive offshore, x is alongshore, and β is the bottom slope.

10.3 Show that the setdown under a standing wave system is

$$\bar{\eta}(x) = \frac{a^2 k}{4 \sinh 2kh} (\cosh 2kh \cos 2kx - 1)$$

10.4 Show by two different methods that for the origin of the vertical coordinate taken at the mean water line, the mean pressure for a progressive wave system is

$$p = -\rho g z - \overline{\rho w^2}$$

One method is suggested in the paragraph following Eq. (10.15). A second method involves integration of the vertical equation of motion from an arbitrary depth z up to the free surface, the use of the Leibniz rule, and time averaging over a wave period.

10.5 For the case of straight and parallel bottom contours, combine energy conservation consideration with Snell's law to demonstrate that S_{xy} is the same from deep to shallow water.

10.6 Verify Eqs. (10.25) and (10.26) for the radiation stresses developed by a wave train traveling at an angle θ to the x axis. Use $\phi(x, y, z, t)$ as developed in Chapter 4.