1

Functions of a complex variable

Complex numbers have played a fundamental part in physics since the discovery of quantum mechanics, where the wave function has real arguments but complex values. The theory of complex functions, i.e. functions of complex arguments, on the other hand, has found only subordinate applications of technical character in physics so far.

For convenience this chapter reviews briefly complex numbers and a small portion of complex analysis, essentially Green's real theorem in Cauchy's complex formulation. Its main purpose is the use of the residue theorem to evaluate definite integrals in one real variable which we shall need later on.

1.1 Complex numbers

Although sufficient for any numerical calculation, the field of rational numbers is geometrically incomplete. This incompleteness is cured by the extension to the real numbers $\mathbb{R}$. In the field $\mathbb{R}$ every Cauchy sequence converges, "the real axis has no holes". However, the reals are still algebraically incomplete, there are real polynomials that do not have real roots, e.g. $p(x) = x^2+1$. Therefore it is convenient to extend the real numbers to the field of complex numbers $\mathbb{C}$. A complex number z is a pair of real numbers x and y

$$z = (x, y). \tag{1.1}$$

Addition of complex numbers is defined componentwise

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2) \tag{1.2}$$

and is denoted by the same symbol $+$ as the addition of real numbers. The neutral element of addition, denoted by 0 as for real numbers, is

$$0 = (0, 0) \tag{1.3}$$

and the negative of the complex number (x, y) is

$$-(x, y) = (-x, -y). \tag{1.4}$$

So far $\mathbb{C}$ is just the two-dimensional vector space $\mathbb{R}^2$. Let us denote by

$$1 := (1,0), \quad \mathrm{i} := (0,1) \tag{1.5}$$

its canonical basis. The product of two complex numbers, denoted by juxtaposition, is defined to be

$$(x_1, y_1)(x_2, y_2) := (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) \tag{1.6}$$

and as immediate consequences of this defintion we have

a) $1 = (1,0)$ is the neutral element of multiplication:

$$1z = z1 = z, \tag{1.7}$$

b) the inverse of the complex number (x, y) is

$$(x,y)^{-1} =: \frac{1}{(x,y)} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right), \tag{1.8}$$

if $x^2 + y^2 \neq 0$.

c) commutativity of multiplication:

$$z_1z_2 = z_2z_1 \tag{1.9}$$

d) associativity of multiplication:

$$(z_1z_2)z_3 = z_1(z_2z_3), \tag{1.10}$$

e) and distributivity

$$z(z_1 + z_2) = zz_1 + zz_2. \tag{1.11}$$

Furthermore we have the important equation

$$\mathrm{i}^2 = -1. \tag{1.12}$$

This means that i is a root of the complex polynomial $p(z) = z^2 + 1$, which, over the reals, has no root. In fact a fundamental theorem of complex numbers states that $\mathbb{C}$ is algebraically complete, i.e. every complex polynomial has complex roots.

During the extension from the real to the complex numbers we gained a fundamental property, algebraic completeness, but we also lost one: complex numbers are not ordered. Any set admits an ordering, indeed many and therefore arbitrary ones, e.g. the complex numbers may be ordered "alphabetically". However, in the context of fields we are interested in an ordering that respects the field structure, addition and multiplication. An ordering of a field is defined by a subset of "positive" numbers such that sums and products of positive numbers are positive. However, there cannot be a subset of positive complex numbers. $\mathrm{i}^3 = -\mathrm{i}$ would imply that i

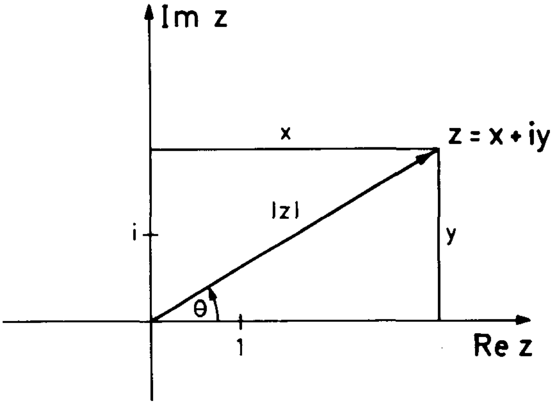

Fig. 1.1. The complex plane

is both positive and negative. Inequalities make sense only for real, not for complex numbers.

A complex number $z = (x, y)$ can be decomposed with respect to the canonical basis 1 and i, $z = x1 + y\mathrm{i}$. Persisting in our abuse of notation, denoting addition, multiplication and their neutral elements for complex numbers by the same symbol as for real numbers, we also drop the 1 in the above equation:

$$z = x1 + y\mathrm{i} = x + \mathrm{i}y, \tag{1.13}$$

thereby identifying the real number x with the complex number $(x, 0)$. The complicated multiplication law (1.6) can now be reconstructed from $\mathrm{i}^2 = -1$ and the distributivity.

The complex number $z = (x, y) = x + \mathrm{i}y$ can be represented by a point in a two-dimensional real plane, "the complex plane". The Cartesian coordinates of z are called real and imaginary part of z

$$\mathrm{Re} z := x, \quad \mathrm{Im} z := y. \tag{1.14}$$

A complex number is called real if its imaginary part vanishes and purely imaginary if its real part vanishes. For any complex number $z = x + \mathrm{i}y$ we define its complex conjugate by

$$\bar{z} := x - \mathrm{i}y. \tag{1.15}$$

In the complex plane complex conjugation is the reflection with respect to the real axis. It is also a field automorphism:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \tag{1.16}$$

$$\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}. \tag{1.17}$$

It is often convenient to use polar coordinates in the complex plane: The absolute value (or norm or magnitude) of $z = x + \mathrm{i}y$ is by definition the real number

$$|z| := \sqrt{x^2 + y^2}. \tag{1.18}$$

The polar angle θ (also phase or argument) of z is given by

$$x =: |z| \cos\theta, \quad y =: |z| \sin\theta, \quad 0 \le \theta < 2\pi. \tag{1.19}$$

and is well defined except at the origin $z = 0$. For any complex number we have the polar decomposition

$$z = |z|(\cos\theta + \mathrm{i}\sin\theta), \quad 0 \le \theta < 2\pi. \tag{1.20}$$

The following equations are immediate consequences of these definitions:

$$\mathrm{Re}z = \frac{1}{2}(z + \bar{z}), \tag{1.21}$$

$$\mathrm{Im}z = \frac{1}{2\mathrm{i}}(z - \bar{z}), \tag{1.22}$$

$$|z|^2 = z\bar{z}. \tag{1.23}$$

Exercises

1. Show that the multiplication of complex numbers is associative and distributive.
2. Verify
$$\frac{a + ib}{c + id} = \frac{ac + bd}{c^2 + d^2} + \mathrm{i}\frac{bc - ad}{c^2 + d^2}.$$
3. Show that the complex conjugation preserves addition and multiplication, equations (1.16) and (1.17).
4. Verify $|z|^2 = z\bar{z}$.

1.2 Roots

While the addition of complex numbers has a simple form in Cartesian coordinates, the multiplication takes a convenient form in polar coordinates:

$$\begin{aligned} z_1 z_2 =& |z_1|(\cos\theta_1 + \mathrm{i}\sin\theta_1)|z_2|(\cos\theta_2 + \mathrm{i}\sin\theta_2) \\ =& |z_1||z_2|[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \mathrm{i}(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ =& |z_1||z_2|[\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2)]. \end{aligned} \tag{1.24}$$

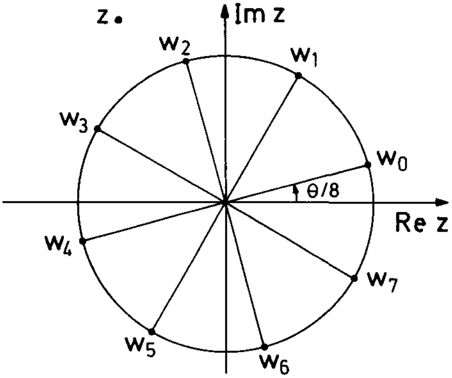

Fig. 1.2 The 8th roots of z

"Under multiplication absolute values multiply and polar angles are added modulo 2π." In particular

$$|z_1 z_2| = |z_1||z_2|. \tag{1.25}$$

On the other hand note the triangle inequality

$$|z_1 + z_2| \le |z_1| + |z_2|. \tag{1.26}$$

Definition: An n-th root w of $z \in \mathbb{C}, n = 1, 2, 3..$ is a complex number w such that

$$w^n = z. \tag{1.27}$$

Iterating (1.25) we get

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta, \quad n \in \mathbb{N} \tag{1.28}$$

and if z is written in polar coordinates (1.20) an obvious n-th root of z, the so-called principal root is given by

$$w_0 = \sqrt[n]{|z|}(\cos\frac{\theta}{n} + \mathrm{i}\ \sin\frac{\theta}{n}). \tag{1.29}$$

More generally, if z is non-zero, there are n distinct n-th roots of z:

$$w_k = \sqrt[n]{|z|}\left(\cos\frac{\theta + 2\pi k}{n} + \mathrm{i}\ \sin\frac{\theta + 2\pi k}{n}\right), k = 0, 1, 2, \cdots n - 1. \tag{1.30}$$

For example $z = -1$, its absolute value is 1, its polar angle $\theta = \pi$ and its second roots are $w_0 = \mathrm{i},\ w_1 = -\mathrm{i}$

Exercises

1. Prove the triangle inequality.
2. Calculate all third roots of -1.

1.3 Complex functions

A complex function $f(z)$ is a map from the complex numbers into the complex numbers

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\mapsto f(z). \end{aligned}$$

Alternatively, f may be viewed as a function from $\mathbb{R}^2$ into $\mathbb{R}^2$ by decomposing argument and value into real and imaginary part, $z = x + \mathrm{i}y$ and

$$f(z) = u(x, y) + \mathrm{i}v(x, y). \tag{1.31}$$

Examples are the complex conjugation

$$f(z) = \bar{z}, \quad u(x, y) = x, \quad v(x, y) = -y, \tag{1.32}$$

polynomials, e.g.

$$f(z) = z^2 + 1, \quad u(x, y) = x^2 - y^2 + 1, \quad v(x, y) = 2xy \tag{1.33}$$

or the n-th principal root. Note that the first two examples are continuous functions while the principal root is discontinuous along the positive real axis. A most important complex function is the exponential. We extend the definition of the real exponential function to the complex plane by

$$e^{x+\mathrm{i}y} := e^x(\cos y + \mathrm{i}\sin y), \qquad \text{“Euler's formula''}. \tag{1.34}$$

A remarkable property of this definition is that the complex exponential also satisfies the functional equation

$$e^0 = 1, \quad e^{z_1+z_2} = e^{z_1}e^{z_2} \tag{1.35}$$

i.e. it defines a group homomorphism from the additive to the multiplicative group of complex numbers. The last equation follows from Euler's formula using the trigonometric angle sum relations as in equation (1.24). Further consequences are:

$$|e^z| = e^{\mathrm{Re}\, z}, \tag{1.36}$$

in particular

$$|e^{\mathrm{i}x}| = 1, \quad x \in \mathbb{R}, \tag{1.37}$$

$$\cos x = \frac{1}{2}(e^{\mathrm{i}x} + e^{-\mathrm{i}x}), \tag{1.38}$$

$$\sin x = \frac{1}{2\mathrm{i}}(e^{\mathrm{i}x} - e^{-\mathrm{i}x}). \tag{1.39}$$

One major advantage in using exponential functions with purely imaginary argument rather than trigonometric functions in physical problems concerned with periodic arrangements is that the simple functional equation (1.35) replaces many complicated trigonometric identities. For example:

$$\begin{aligned} \cos 2x = \mathrm{Re} e^{2\mathrm{i}x} &= \mathrm{Re}(e^{\mathrm{i}x} e^{\mathrm{i}x}) = \mathrm{Re} e^{\mathrm{i}x}\ \mathrm{Re} e^{\mathrm{i}x} - \mathrm{Im} e^{\mathrm{i}x}\ \mathrm{Im} e^{\mathrm{i}x} \\ &= \cos^2 x - \sin^2 x. \end{aligned} \tag{1.40}$$

Exercise

1. Solve the damped harmonic oscillator

$$\frac{d^2}{dt^2}x(t) + 2\lambda\frac{d}{dt}x(t) + \omega_0^2 x(t) = 0,$$

λ and real ω_0, with initial conditions

$$x(0) = x_0, \quad \frac{dx}{dt}(0) = 0$$

using the complex exponential.

1.4 Differentiation

We generalize the derivative from real to complex functions.

Definition: Let f be a complex function. If the limit

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

exists for all sequences Δz approaching zero, it is called the (complex) derivative of f at z, denoted by $f'(z)$ or $\frac{df}{dz}(z)$.

The definition of the complex derivative is much more restrictive than the one of partial derivatives of the component functions $u(x,y)$ and $v(x,y)$, where horizontal limits, $\Delta z = \Delta x$ real, and vertical limits, $\Delta z = \mathrm{i}\Delta y$ purely imaginary, may well differ. Let us illustrate this point by a complex function, which is not differentiable in the complex sense. Consider the complex conjugation $f(z) = \bar{z}$ at the origin $z = 0$. The limit to examine is

$$\lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}. \tag{1.41}$$

This limit is 1 for Δz approaching 0 on the real axis, but -1 for Δz approaching 0 on the imaginary axis. Although its component functions $u(x, y) = x$ and $v(x, y) = -y$ have well-defined partial derivatives, the complex conjugation is not differentiable in the complex sense, because if we spiral down to the origin

$$\Delta z = \frac{1}{n} e^{in\pi/2}, \quad n = 1, 2, 3 \ldots \tag{1.42}$$

the limit (1.44) oscillates between -1 and +1 and fails to exist.

For any complex function $f = u + \mathrm{i}v$ complex differentiability at $z = x + \mathrm{i}y$ means that all simultaneous limits

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[u(x + \Delta x, y + \Delta y) - u(x, y)] + \mathrm{i}[v(x + \Delta x, y + \Delta y) - v(x, y)]}{\Delta x + \mathrm{i}\Delta y} \tag{1.43}$$

exist and are all equal. In particular for $\Delta y \equiv 0$

$$f' = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} \tag{1.44}$$

and for $\Delta x = 0$

$$f' = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y}. \tag{1.45}$$

Equating real and imaginary part we arrive at the so-called Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{1.46}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.47}$$

They are necessary and also sufficient equations for a differentiable function f to admit a complex derivative.

In the above example $f(z) = \bar{z}$ the first Cauchy-Riemann equation is not satisfied in any point, the complex conjugation admits nowhere a complex derivative.

Definition: A complex function f is said to be holomorphic (or regular or analytic) at a point $z \in \mathbb{C}$ if it possesses a complex derivative in every point of some neighbourhood of z. The function is called holomorphic if it is holomorphic in every point of the complex plane.

A fundamental theorem of complex functions states that a holomorphic function admits complex derivatives of any order and that its Taylor series converges to this function everywhere, whence the name analytic. Both statements fail of course in the case of real functions.

On the other hand the following theorems of real analysis hold true in the complex setting:

a) If f and g are holomorphic functions then so are their sum, complex multiples, product and composition and

$$(f+g)' = f' + g', \tag{1.48}$$

$$(cf)' = cf', \ c \in \mathbb{C}, \tag{1.49}$$

$$(fg)' = f'g + fg', \tag{1.50}$$

$$\frac{d}{dz}\left(g(f(z))\right) = g'(f(z))f'(z), \quad \text{“chain rule”}.$$

b) For any positive integer n $f(z) = z^n$ is holomorphic and

$$\frac{d}{dz}(z^n) = nz^{n-1}. \tag{1.51}$$

c) The exponential is holomorphic and

$$\frac{d}{dz}e^z = e^z. \tag{1.52}$$

The proofs of a) and b) are literally the same as for functions of one real variable. To prove c) we first verify the Cauchy-Riemann conditions and then use (1.44).

A final remark: by partial differentiating the Cauchy-Riemann equations we find that both component functions of a holomorphic function are harmonic:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v = 0. \tag{1.53}$$

Exercises

1. Verify the Cauchy-Riemann condition for $f(z) = z^3$.
2. Calculate the derivative of the exponential by equations (1.44) and (1.45).
3. Verify equation (1.53).
4. Is the function

$$f(z) := \begin{cases} e^{-1/z^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

holomorphic at the origin? Are the Cauchy-Riemann equations satisfied at the origin?

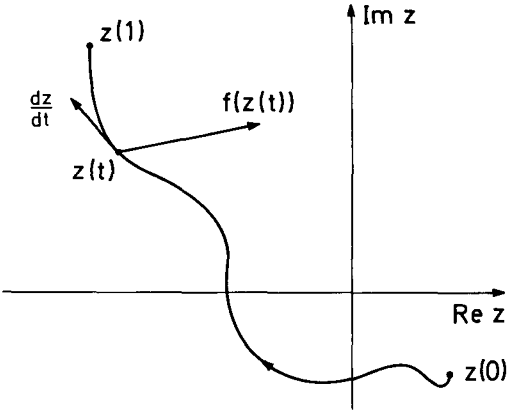

Fig. 1.3. A curve in the complex plane

1.5 Integration

We would like now to define an integral of a complex function. A complex function can be considered as vector field in the complex plane. This point of view suggests to use some line integral. However, we replace the scalar product appearing in the ordinary line integral of a vector field by the product of complex numbers.

Definition: Let C be an (oriented, piecewise smooth) curve in the complex plane and $f = u + \mathrm{i}v$ a complex function. The (complex) integral of f along C is

$$\int_C f := \int_C f(z)dz := \int_C (udx - vdy) + \mathrm{i} \int_C (vdx + udy) \tag{1.54}$$

if the real line integrals on the rhs exist.

This definition is easy to remember by putting $dz = dx + \mathrm{i}dy$ and by decomposing $f\ dz$ into real and imaginary part. Let us recall the definition of the line integral of a real function. Choose a parameter representation $z(t)$ of the curve C

$$\begin{aligned} z : [0,1] &\longrightarrow \mathbb{C} \\ t &\mapsto z(t) = x(t) + \mathrm{i}y(t). \end{aligned} \tag{1.55}$$

Then one of the integrals in (1.54) is

$$\int_C udx := \int_0^1 u(x(t), y(t))\frac{dx}{dt}dt. \tag{1.56}$$

Example: Let C be the upper half circle of radius R centered around the origin, oriented in a counter clockwise sense: We choose the parameter representation

$$z(t) = R\, e^{\mathrm{i}t}, \quad t \in [0, \pi] \tag{1.57}$$

or

$$x(t) = R\cos t, \quad y(t) = R\sin t. \tag{1.58}$$

Let $f(z) = z^2$ with $u = x^2 - y^2$ and $v = 2xy$. Then

$$\begin{aligned}\int_C f = &\int_C (x^2 - y^2)dx - 2\int_C xydy + \mathrm{i}\int_C 2xydx + \mathrm{i}\int_C (x^2 - y^2)dy \\ = &\int_0^\pi (R^2\cos^2 t - R^2\sin^2 t)(-R\sin t)dt - \int_0^\pi 2R^2\cos t\sin tR\cos t\ dt \\ &+ \mathrm{i}\int_0^\pi 2R^2\cos t\sin t(-R\sin t)dt + \mathrm{i}\int_0^\pi (R^2\cos^2 t - R^2\sin^2 t)R\cos tdt \\ = &- 3R^3\int_0^\pi \cos^2 t\sin tdt + R^3\int_0^\pi \sin^3 tdt \\ = &- 4R^3\int_0^\pi \cos^2 t\sin tdt + R^3\int_0^\pi \sin tdt \\ = &\ 4R^3\int_1^{-1} w^2dw + R^3\int_0^\pi \sin tdt = -\frac{8}{3}R^3 + 2R^3 = -\frac{2}{3}R^3.\end{aligned} \tag{1.59}$$

The same calculation becomes more transparent in the compact complex notation:

$$z(t) = Re^{\mathrm{i}t},\ dz = \mathrm{i}Re^{\mathrm{i}t}dt \quad \text{and} \quad f(z(t)) = R^2e^{2\mathrm{i}t}.$$

In this form the integral is not plagued by trigonometric functions. (Compare the remark following equation (1.39).)

$$\begin{aligned}\int_C z^2dz &= \int_0^\pi R^2e^{2\mathrm{i}t}\mathrm{i}R\, e^{\mathrm{i}t}dt = R^3\int_0^\pi e^{3\mathrm{i}t}\mathrm{i}dt \\ &= \frac{R^3}{3}e^{3\mathrm{i}t}\Big|_0^\pi = -\frac{2}{3}R^3.\end{aligned} \tag{1.60}$$

From of the real line integral the complex integral inherits the following properties:

a) Let f and g be two complex functions, C a curve and k a complex number. Then

$$\int_C (f+g) = \int_C f + \int_C g, \tag{1.61}$$

$$\int_C kf = k\int_C f \tag{1.62}$$

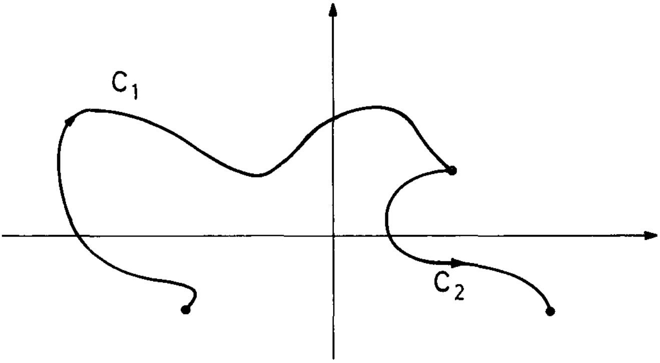

Fig. 1.4 The addition of curves

if all integrals exist.

b) If C_1 and C_2 are curves such that the end point of C_1 coincides with the initial point of C_2, then

$$\int_{C_1+C_2} f = \int_{C_1} f + \int_{C_2} f \tag{1.63}$$

where $C_1 + C_2$ is explained in fig. 1.4.

c) If $-C$ denotes the curve having the same trace as C but opposite orientation, then

$$\int_{-C} f = -\int_C f. \tag{1.64}$$

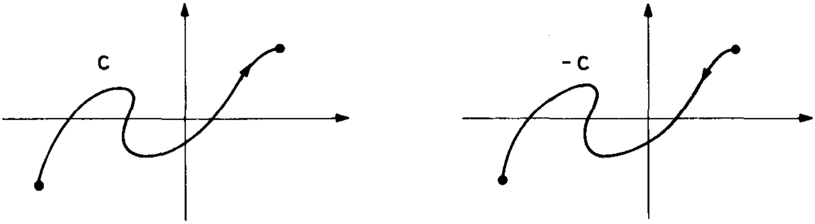

Fig. 1.5. The negative of a curve

d) If L is the length of the curve C

$$L := \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{1.65}$$

then

$$\left|\int_C f\right| \leq L \max_C |f(z)|. \tag{1.66}$$

Recall from mechanics that a force field f in $\mathbb{R}^3$ which is conservative

$$\text{curl } f = 0 \tag{1.67}$$

does not do work around closed trajectories, the line integral of f over any closed curve c vanishes due to Stokes' theorem

$$\oint_C f \cdot dx = \int\int_S \text{curl } f \cdot dS = 0 \tag{1.68}$$

where S is any surface bounded by C. In two dimensions the Cauchy-Riemann equations are the analogue of equation (1.67) with respect to the complex integral and Stokes' theorem in two dimensions is better known as

Green's theorem: Let $P(x,y)$ and $Q(x,y)$ be two real functions defined in a simply connected domain D of $\mathbb{R}^2$ (i.e. without holes). Suppose both functions have continuous first partial derivatives in D. Let C be a closed curve in D, oriented counter-clockwise and let S be the interior of C. Then

$$\oint_C (Pdx + Qdy) = \int\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy. \tag{1.69}$$

In terms of complex functions Green's theorem yields

Cauchy's theorem: If $f(z)$ is a holomorphic function defined in a simply connected domain D of the complex plane and C a closed curve in D, then

$$\oint_C f(z)dz = 0. \tag{1.70}$$

Example: Let C be the circle of radius R centered at the origin and $f(z) = z^n, \quad n = 1,2,3...$

$$\int_C z^n dz = R^{n+1} \int_0^{2\pi} e^{(n+1)rit} \mathrm{i} dt = R^{n+1} \frac{1}{n+1} e^{(n+1)\mathrm{i}t}\Big|_0^{2\pi} = 0. \tag{1.71}$$

As important consequence of Cauchy's theorem we see that the integral of a holomorphic function does not depend on the curve C but only on the initial and final point of C.

Definition: Let f be a function holomorphic in a simply connected domain D and let a and b be complex numbers in D. The integral of f from a to b

$$\int_a^b f := \int_a^b f(z)dz \tag{1.72}$$

is the integral of f over any curve in D connecting a to b.

Now the complex integral resembles the real integral and we also have a

Fundamental theorem of differentiation and integration: Let $f(z)$ be holomorphic in a simply connected domain D and let z_0 be a point in D. Then the function

$$F(z) := \int_{z_0}^{z} f(\zeta)d\zeta \tag{1.73}$$

is also holomorphic in D and

$$F' = f. \tag{1.74}$$

Proof: Let $z = x + \mathrm{i}y,\ z_0 = x_0 + \mathrm{i}y_0,\ f = u + \mathrm{i}v,\ F = U + \mathrm{i}V$. Then

$$U = \int_{(x_0,y_0)}^{(x,y)} (udx - vdy) \tag{1.75}$$

and

$$V = \int_{(x_0,y_0)}^{(x,y)} (vdx + udy). \tag{1.76}$$

Using the fundamental theorem for real line integrals we verify the Cauchy-Riemann equations

$$\frac{\partial U}{\partial x} = u = \frac{\partial V}{\partial y} \quad \text{and} \quad \frac{\partial U}{\partial y} = -v = -\frac{\partial V}{\partial x}. \tag{1.77}$$

Therefore F is holomorphic and $F' = \partial U/\partial x + \mathrm{i}\partial V/\partial x = u + \mathrm{i}v = f$.

In our example $\int_C z^2 dz$ over the upper half circle, the calculation can now be simplified by computing the integral from R to $-R$ along the real line and even better we can bypass the choice of a curve altogether and get the integral from a primitive just as in the real case.

$$\int_R^{-R} z^2 dz = \frac{z^3}{3}\bigg|_R^{-R} = -\frac{2}{3}R^3. \tag{1.78}$$

Exercises

1. Calculate $\int_C z^3 dz$ where C is the straight line from 1 to i without using the fundamental theorem.
2. Prove the inequality (1.66).
3. Prove Cauchy's theorem.
4. Compute $\int_1^{\mathrm{i}} z^3 dz$.

1.6 Residues

We now consider examples where the domain of holomorphy has holes.

Definition: Let f be a complex function. A complex number z_0 is called an isolated singularity of f if there is a neighbourhood D of z_0 in the complex plane such that f is holomorphic at every point of D except at z_0 itself.

Example: The origin is an isolated singularity of $1/z^n$, $n = 1, 2, 3....$

Since $D - \{z_0\}$ is not simply connected we expect Cauchy's theorem to fail. Indeed if C is the circle of radius R around z_0 parametrized by

$$z(t) = z_0 + Re^{\mathrm{i}t}, \quad t \in [0, 2\pi] \tag{1.79}$$

then

$$\oint_C \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{1}{Re^{\mathrm{i}t}} \mathrm{i}Re^{\mathrm{i}t} dt = \mathrm{i} \int_0^{2\pi} dt = 2\pi r \mathrm{i} \tag{1.80}$$

The integral does not vanish, however, its value does not depend on the radius R. More generally, for any function f with isolated singularity z_0 any closed counterclockwise curve K surrounding z_0 once and lying in the domain D where f is holomorphic the integral of f over K does not depend on the choice of such a curve.

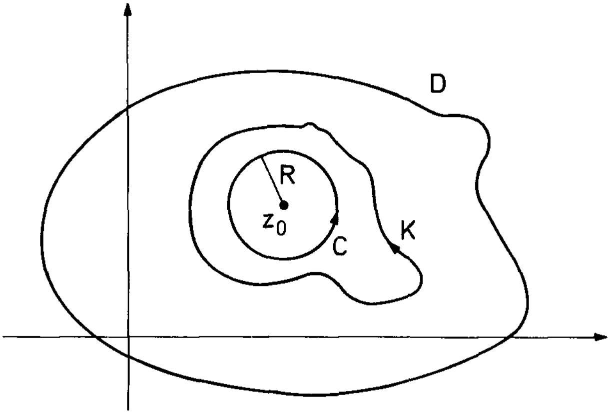

Fig. 1.6. Curves around an isolated singularity

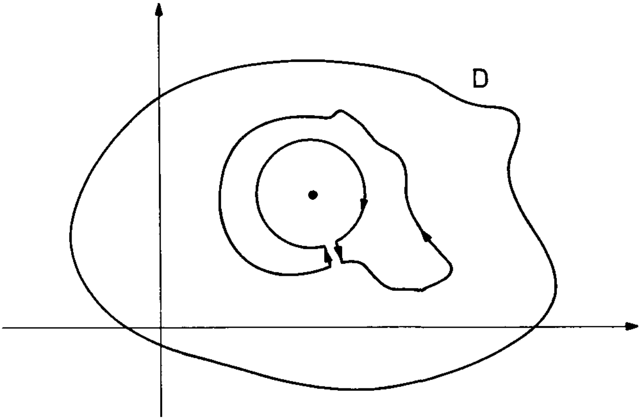

Fig. 1.7. To apply Cauchy's theorem

The proof relies on a simple universal trick. Choose a circular curve C with small enough radius so that C lies inside K. Construct the curve $\tilde{K}$ by cutting a small channel between K and C as in fig. 1.7.

By construction $\tilde{K}$ is contained in a simply connected subset of D. By Cauchy's theorem $\oint_{\tilde{K}} f = 0$. In the limit where the channel width ϵ tends to zero, the two integrals along the channel cancel because they have opposite orientations and we remain with

$$0 = \lim_{\epsilon \to 0} \oint_{\tilde{K}} f = \oint_{K} f - \oint_{C} f \tag{1.81}$$

where the minus sign comes from the clockwise orientation of the circle C.

Definition: Let $f(z)$ be a complex function holomorphic in a neighbourhood D of z_0 except possibly at z_0. We define the residue of f at z_0 by

$$\text{Res } f(z)\big|_{z_0} := \frac{1}{2\pi\mathrm{i}} \oint f(z)dz \tag{1.82}$$

where the integral runs over any counter clockwise curve surrounding z_0 once.

The normalization has been chosen such that

$$\text{Res } \frac{1}{z - z_0}\bigg|_{z_0} = 1. \tag{1.83}$$

If f is holomorphic at z_0 then by Cauchy's theorem

$$\text{Res } f(z)\bigg|_{z_0} = 0. \tag{1.84}$$

On the other hand, an isolated singularity can have vanishing residue, e.g.

$$\operatorname{Res} \left.\frac{1}{(z-z_0)^n}\right|_{z_0} = 0, \quad n = 2,3,4... \tag{1.85}$$

The main result of this section is the residue theorem. It is a two-dimensional analogue of Gauss' law in electrodynamics expressing the additivity of electric charge.

Residue theorem: If $f(z)$ is holomorphic on and inside a closed counter-clockwise curve C except for a finite number of isolated singularities $z_1, z_2, ...z_n$, all located inside C, then

$$\oint_C f = 2\pi i \sum_{k=1}^{n} \operatorname{Res} f\big|_{z_k}. \tag{1.86}$$

The proof uses the channel trick as illustrated in fig. 1.8.

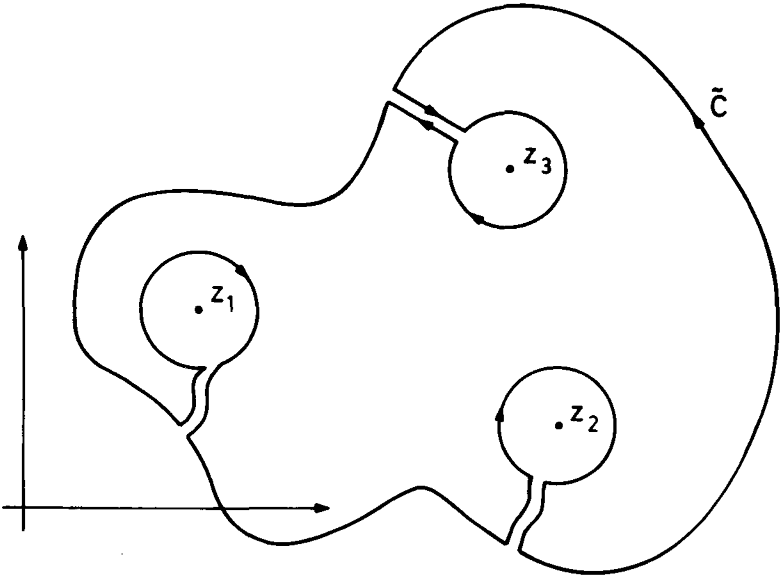

Fig. 1.8. The residue theorem

In later applications the lhs of equation (1.86) will be expressed as a real integral we want to compute. There is a variety of methods to evaluate the residues appearing on the rhs without doing integrals. For our purpose the following theorem, which we cite without proof, will be sufficient.

Quotient theorem: Let f and g be two functions holomorphic in a neighbourhood of z_0 and satisfying

$$f(z_0) \neq 0, \quad g(z_0) = 0, \quad g'(z_0) \neq 0.$$

Then

$$\operatorname{Res} \frac{f}{g}\bigg|_{z_0} = \frac{f(z_0)}{g'(z_0)}. \tag{1.87}$$

An astonishing corollary is obtained by putting $g(z) = z - z_0$:

$$f(z_0) = \operatorname{Res} \frac{f}{z - z_0}\bigg|_{z_0} = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{z - z_0} dz. \tag{1.88}$$

It tells us that the values of a holomorphic function inside a given closed curve are already determined by its values on this curve.

This situation is analogous to the static soap film or drum skin suspended freely from a closed frame. Its shape $f(x, y)$ is subject to the differential equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = 0 \tag{1.89}$$

as in equation (1.53). The closed curve C is the projection of the frame onto the $x-y$ plane and the shape of the frame $f\big|_C$ determines the shape of the whole drum skin.

Exercises

1. Calculate the residue of z^n at the origin for any integer n.

2. Calculate the residues at $z_0 = 1$ and $z_0 = \mathrm{i}$ of

$$a)\quad \frac{1}{z^2 - (1+\mathrm{i})z + \mathrm{i}}$$

$$b)\quad \frac{1}{z^2 + (1-\mathrm{i})z - \mathrm{i}}$$

$$c)\quad \frac{1}{z^2 - 2\mathrm{i}z - 1}.$$

1.7 Applications to the real world

In view of some definite integrals of functions with real argument which we shall encounter later, let us discuss a few applications of the residue theorem.

$$\text{a)}\quad \int_0^{2\pi} \frac{dt}{a + \sin t}, \quad a > 1$$

We rewrite this integral as a complex integral along the unit circle C of an appropriate function and put

$$z(t) = e^{\mathrm{i}t}, \tag{1.90}$$

$$dt = \frac{dz}{\mathrm{i}z} \tag{1.91}$$

and from equation (1.39)

$$\sin t = \frac{1}{2\mathrm{i}}(z - \frac{1}{z}). \tag{1.92}$$

Then our integral becomes

$$\int_0^{2\pi} \frac{dt}{a + \sin\ t} = \oint_C \frac{dz}{\mathrm{i}z[a + \frac{1}{2\mathrm{i}}(z - 1/z)]} = \frac{1}{\mathrm{i}} \oint_C \frac{2\mathrm{i}dz}{z^2 + 2\mathrm{i}az - 1}. \tag{1.93}$$

The integrand has only one isolated singularity inside the unit circle

$$z_0 = -\mathrm{i}a + \mathrm{i}\sqrt{a^2 - 1} \tag{1.94}$$

and by the residue theorem

$$\int_0^{2\pi} \frac{dt}{a + \sin t} = 2\pi \text{ Res } \frac{2\mathrm{i}}{z^2 + 2\mathrm{i}az - 1}\bigg|_{z_0}. \tag{1.95}$$

By the quotient theorem this residue is

$$\frac{\mathrm{i}}{z_0 + \mathrm{i}a} = \frac{1}{\sqrt{a^2 - 1}} \tag{1.96}$$

and our integral

$$\int_0^{2\pi} \frac{dt}{a + \sin t} = \frac{2\pi}{\sqrt{a^2 - 1}}. \tag{1.97}$$

This example generalizes to integrals of the form

$$\int_0^{2\pi} R(\sin\ t, \cos\ t)dt$$

where $R(x, y)$ is a rational function with no singularities on the unit circle.

b) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{x^2 + a^2}, \quad a > 0$

Although the integrand has an elementary primitive let us evaluate the integral by extending it to the complex plane. Let C be the closed curve along the real axis

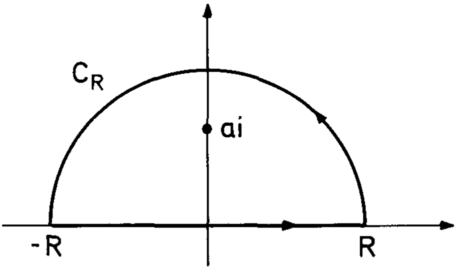

Fig. 1.9 Integrating along a curve

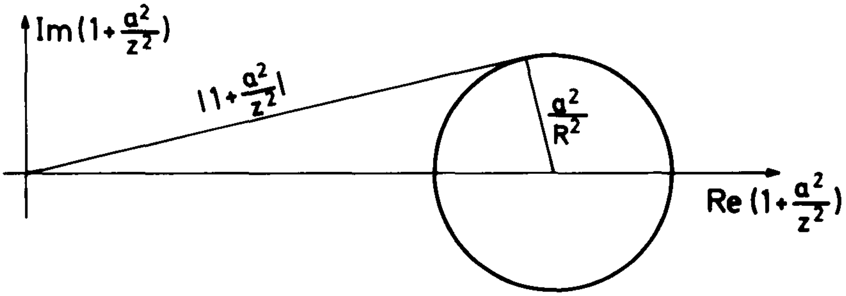

Fig. 1.10 An estimate

between $-R$ and R and around the upper half circle C_R of radius R as illustrated in fig. 1.9.

First we prove that

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2+a^2} = \lim_{R\to\infty} \int_C \frac{dz}{z^2+a^2} = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+a^2} + \lim_{R\to\infty} \int_{C_R} \frac{dz}{z^2+a^2}. \quad (1.98)$$

We have to show that the second limit vanishes: If z is on the half circle C_R with radius $R > \sqrt{2}a$, then $1 + a^2/z^2$ lies on the circle centered at 1 with radius $a^2/z^2 < \frac{1}{2}$ and by fig. 1.10:

$$|1 + \frac{a^2}{z^2}| > \frac{1}{2}. \quad (1.99)$$

Then

$$\left|\frac{1}{z^2+a^2}\right| = \frac{1}{|z|^2} \frac{1}{|1+a^2/z^2|} < \frac{2}{R^2} \quad (1.100)$$

and using inequality (1.66)

$$\left|\int_{C_R} \frac{dz}{z^2+a^2}\right| \le \pi R \max_{C_R} \left|\frac{1}{z^2+a^2}\right| < \pi R \frac{2}{R^2} = \frac{2\pi}{R} \xrightarrow[R\to\infty]{} 0. \quad (1.101)$$

For $R > \sqrt{2}a$ the function $1/(z^2+a^2)$ has one isolated singularity inside the curve C, at $z_0 = \mathrm{i}a$ and

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2+a^2} = 2\pi\mathrm{i}\,\mathrm{Res}\,\frac{1}{z^2+a^2}\bigg|_{\mathrm{i}a} = 2\pi\mathrm{i}\frac{1}{2z}\bigg|_{\mathrm{i}a} = \frac{\pi}{a}. \tag{1.102}$$

This technique works for convergent integrals of the form

$$\int_{-\infty}^{+\infty} R(x)dx,$$

$R(x)$ being a rational function without singularities on the real axis.

$$\text{c)} \quad \int_{-\infty}^{+\infty} \frac{e^{\mathrm{i}x}}{x}dx$$

Let us continue the integrand to the complex plane and integrate along the closed curve as in fig. 1.11:

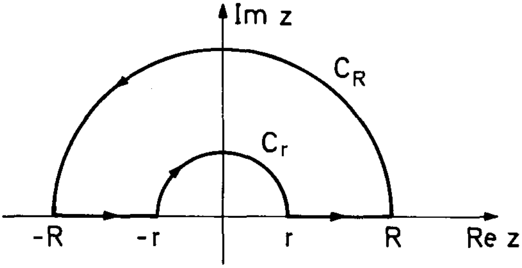

Fig. 1.11 Avoiding the pole

$$\int_{-R}^{-r} + \int_{r}^{R} + \int_{C_r} + \int_{C_R} \frac{e^{\mathrm{i}z}}{z}dz = 0 \tag{1.103}$$

by Cauchy's theorem.

First we show that

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{\mathrm{i}z}}{z}dz = 0. \tag{1.104}$$

This time the length of C_R increases at the same rate as the integrand decreases with R and we have to work a little harder than in the preceding example. Integration by parts applies to holomorphic functions just as to real functions because it only relies on the product rule for differentiation and the fundamental theorem of differentiation and integration. In the case at hand integration by parts yields:

$$\begin{aligned}\int_{C_R} \frac{e^{\mathrm{i}\,z}}{z} dz =& \frac{e^{\mathrm{i}\,z}}{\mathrm{i}z}\bigg|_{-R}^{R} + \int_{C_R} \frac{e^{\mathrm{i}\,z}}{\mathrm{i}z^2} dz \\ =& \frac{2}{\mathrm{i}R} \cos R + \int_{C_R} \frac{e^{\mathrm{i}\,z}}{\mathrm{i}z^2} dz.\end{aligned} \tag{1.105}$$

Now as R tends to infinity also the second term tends to zero as in b) since $|e^{\mathrm{i}\,z}| < 1$ in the upper half plane.

Next we show that

$$\lim_{r\to 0} \int_{C_r} \frac{e^{\mathrm{i}\,z}}{z} dz = -\pi \mathrm{i}. \tag{1.106}$$

The exponential $e^{\mathrm{i}\,z}$ is holomorphic at the origin and there it takes the value 1. Thus

$$|e^{\mathrm{i}\,z} - 1| < \epsilon |z| \tag{1.107}$$

for all z with sufficiently small absolute values and

$$\left| \int_{C_r} \frac{e^{\mathrm{i}\,z} - 1}{z} dz \right| \leq \pi r \max_{C_r} \frac{|e^{\mathrm{i}\,z} - 1|}{|z|} \leq \pi r \epsilon. \tag{1.108}$$

Consequently

$$\lim_{r\to 0} \int_{C_r} \frac{e^{\mathrm{i}\,z} - 1}{z} dz = 0 \tag{1.109}$$

and

$$\begin{aligned}\lim_{r\to 0} \int_{C_r} \frac{e^{\mathrm{i}\,z}}{z} dz =& \lim_{r\to 0} \int_{C_r} \frac{e^{\mathrm{i}\,z} - 1}{z} dz + \lim_{r\to 0} \int_{C_r} \frac{dz}{z} \\ =& \lim_{r\to 0} -\mathrm{i} \int_0^{\pi} dt = -\pi \mathrm{i}.\end{aligned} \tag{1.110}$$

In the limit $R \to \infty$ and $r \to 0$ the first two terms in equation (1.103) tend to the integral we are after

$$\int_{-\infty}^{+\infty} \frac{e^{\mathrm{i}\,x}}{x} dx = \pi \mathrm{i}. \tag{1.111}$$

This technique can be generalized to other integrals of the form

$$\int_{-\infty}^{+\infty} e^{\mathrm{i}\,x} f(x) dx$$

which will appear in Fourier transforms.

Exercises

1. Compute

$$\int_0^{2\pi} \frac{dt}{1 - 2c\cos t + c^2}, \quad c \in \mathbb{C}, \quad |c| \neq 1.$$

2. Compute

$$\int_{-\infty}^{+\infty} \frac{x^2}{x^6 + 1} dx.$$

3. Compute

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{x^2 + a^2} dx, \quad a, k \in \mathbb{R}, \quad a \neq 0.$$

4. Calculate the Fresnel integrals

$$\int_0^{\infty} \sin(x^2) dx \quad \text{and} \quad \int_0^{\infty} \cos(x^2) dx.$$

Hint: Integrate e^{-z^2} over an eighth of a circle.