Preface

This book is based on lectures I held in Berkeley and Heidelberg during 1988-1990. It is intended for undergraduate students in mathematics, physics, and engineering. The only prerequisites on the mathematical side are linear algebra and real analysis while introductory courses on electrodynamics and quantum mechanics are needed to appreciate the physical motivations.

Distributions are often cited to examplify the interplay between mathematics and physics. Born from physical intuition, Huygens' principle in wave mechanics, point-like charges in electrodynamics and the wave function of a particle with well-defined position in quantum theory, distributions are today inherent to our way of thinking about physics. The manipulations involved when dealing with distributions were made rigorous by mathematicians and then distributions evolved quite independently to a tool used in such diverse branches of mathematics as the theory of linear differential equations, probability theory, group theory, and manifolds. All these branches offer results relevant to modern physics.

In the same way Fourier transforms constitute a solid link between physics and mathematics. Their applications to physics are almost universal: spectral analysis in acoustics, optics and electronics, "rhythm is the secret". The superposition principle, the particle-wave duality and the uncertainty principle are all encompassed by the Fourier transformation. Mathematicians use the Fourier transformation to solve linear differential equations with constant coefficients and have found fruitful generalizations in functional analysis.

The purpose of this text is to define distributions and Fourier transformations in $\mathbb{R}^n$ and to illustrate some of their applications to physics. Chapter one collects a few tools from complex analysis needed later to evaluate some real integrals. It is otherwise disconnected from the main stream.

In chapter two we motivate distributions with several physical examples, give their definition and discuss some of their properties. We prefer Temple's approach as introduced in Lighthill's book (Lighthill, 1962) to the original functional approach due to Laurent Schwartz. For our purpose Temple's definition of a distribution in terms of sequences of functions has two advantages: It uses Riemann's integral and by-passes most of the functional analysis as Lebesque's integral and continuous linear forms. It emphasizes the close relation between distributions and functions, facilitating the justification of some heuristic manipulations, e.g.

$$\delta(x^2 - m^2) = \frac{1}{2m}[\delta(x+m) - \delta(x-m)], \quad m > 0.$$

The Green function, the outstanding application of distributions, is defined and calculated for our leitmotiv example, the harmonic oscillator.

Chapter three is devoted to Fourier series of periodic functions and periodic distributions. In chapter four, motivated by a special limit of the Fourier series, we define the Fourier transform of functions. Then we generalize the Fourier transform to distributions. In this framework the Fourier series in turn becomes a special case of the Fourier transformation. Finally, the use of Fourier transforms to calculate Green functions is illustrated. Chapter five contains a discussion of some differential equations playing an important role in physics as well as their Green functions.

In chapter six we review linear algebra with special emphasis on infinite dimensional spaces and Hilbert spaces and explain how differential operators and Fourier transforms fit naturally into this frame. This point of view becomes fundamental in quantum mechanics. It also prepares the presentation of chapter seven, dealing with some systems of special functions which appear in almost every branch of physics.

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