1. BASIC LAWS OF ELECTROSTATICS (1001–1023)

1001

A static charge distribution produces a radial electric field

$$\mathbf{E} = A\frac{e^{-br}}{r}\mathbf{e}_r\,,$$

where A and b are constants.

(a) What is the charge density? Sketch it.

(b) What is the total charge Q?

(*MIT*)

Solution:

(a) The charge density is given by Maxwell's equation

$$\rho = \nabla\cdot\mathbf{D} = \varepsilon_0\nabla\cdot\mathbf{E}\,.$$

As $\nabla\cdot u\mathbf{v} = \nabla u\cdot\mathbf{v} + u\nabla\cdot\mathbf{v}$,

$$\nabla\cdot\mathbf{E} = A\left[\nabla(e^{-br})\cdot\frac{\mathbf{e}_r}{r^2} + e^{-br}\nabla\cdot\left(\frac{\mathbf{e}_r}{r^2}\right)\right].$$

Making use of Dirac's delta function $\delta(\mathbf{r})$ with properties

$$\begin{aligned}\delta(\mathbf{r}) &= 0 \quad \text{for} \quad \mathbf{r}\neq 0\,,\\ &= \infty \quad \text{for} \quad \mathbf{r} = 0\,,\end{aligned}$$

$$\begin{aligned}\int_V \delta(\mathbf{r})dV &= 1 \quad \text{if } V \text{ encloses} \quad \mathbf{r} = 0\,,\\ &= 0 \quad \text{if otherwise}\,,\end{aligned}$$

we have

$$\nabla^2\left(\frac{1}{r}\right) = \nabla\cdot\nabla\left(\frac{1}{r}\right) = \nabla\cdot\left(-\frac{\mathbf{e}_r}{r^2}\right) = -4\pi\delta(\mathbf{r})\,.$$

Thus

$$\begin{aligned}\rho &= \varepsilon_0 A\left[-\frac{be^{-br}}{r^2}\mathbf{e}_r\cdot\mathbf{e}_r + 4\pi e^{-br}\delta(\mathbf{r})\right]\\ &= -\frac{\varepsilon_0 Ab}{r^2}e^{-br} + 4\pi\varepsilon_0 A\,\delta(\mathbf{r})\,.\end{aligned}$$

Hence the charge distribution consists of a positive charge $4\pi\varepsilon_0 A$ at the origin and a spherically symmetric negative charge distribution in the surrounding space, as shown in Fig. 1.1.

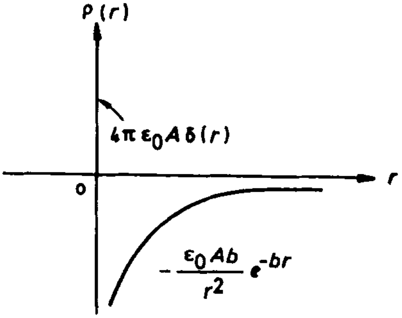

Fig. 1.1

(b) The total charge is

$$Q=\int_{\text{all space}}\rho dV$$

$$=-\int_0^\infty \frac{\varepsilon_0 Abe^{-br}}{r^2}\cdot 4\pi r^2 dr+\int_{\text{all space}} 4\pi\varepsilon_0 A\,\delta(\mathbf{r})dV$$

$$=4\pi\varepsilon_0 A[e^{-br}]_0^\infty+4\pi\varepsilon_0 A$$

$$=-4\pi\varepsilon_0 A+4\pi\varepsilon_0 A=0\,.$$

It can also be obtained from Gauss' flux theorem:

$$Q=\lim_{r\to\infty}\oint_S \varepsilon_0\mathbf{E}\cdot d\mathbf{S}$$

$$=\lim_{r\to\infty}\frac{\varepsilon_0 Ae^{-br}}{r^2}\cdot 4\pi r^2$$

$$=\lim_{r\to\infty}4\pi\varepsilon_0 Ae^{-br}=0\,,$$

in agreement with the above.

1002

Suppose that, instead of the Coulomb force law, one found experimentally that the force between any two charges q_1 and q_2 was

$$\mathbf{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0} \cdot \frac{(1 - \sqrt{\alpha r_{12}})}{r_{12}^2} \mathbf{e}_r ,$$

where α is a constant.

(a) Write down the appropriate electric field $\mathbf{E}$ surrounding a point charge q.

(b) Choose a path around this point charge and calculate the line integral $\oint \mathbf{E} \cdot d\mathbf{l}$. Compare with the Coulomb result.

(c) Find $\oint \mathbf{E} \cdot d\mathbf{S}$ over a spherical surface of radius r_1 with the point charge at this center. Compare with the Coulomb result.

(d) Repeat (c) at radius $r_1 + \Delta$ and find $\nabla \cdot \mathbf{E}$ at a distance r_1 from the point charge. Compare with the Coulomb result. Note that Δ is a small quantity.

(*Wisconsin*)

Solution:

(a) The electric field surrounding the point charge q is

$$\mathbf{E}(r) = \frac{q}{4\pi\varepsilon_0} \frac{1}{r^2} (1 - \sqrt{\alpha r}) \mathbf{e}_r ,$$

where r is the distance between a space point and the point charge q, and $\mathbf{e}_r$ is a unit vector directed from q to the space point.

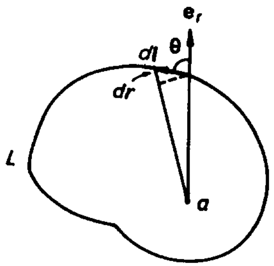

Fig. 1.2

(b) As in Fig. 1.2, for the closed path L we find

$$d\mathbf{l} \cdot \mathbf{e}_r = dl \cos\theta = dr$$

and

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = \oint \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}(1-\sqrt{\alpha r})dr$$
$$= \frac{q}{4\pi\varepsilon_0}\left[-\oint_L d\left(\frac{1}{r}\right) + 2\sqrt{\alpha}\oint_L d\left(\frac{1}{\sqrt{r}}\right)\right] = 0\,.$$

From Coulomb's law $\mathbf{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}^2}\mathbf{e}_{r_{12}}$, we can obtain the electric field of the point charge

$$\mathbf{E}(r) = \frac{q}{4\pi\varepsilon_0 r^2}\mathbf{e}_r\,.$$

Clearly, one has

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = 0\,.$$

So the Coulomb result is the same as that of this problem.

(c) Let S be a spherical surface of radius r_1 with the charge q at its center. Defining the surface element $d\mathbf{S} = dS\mathbf{e}_r$, we have

$$\oint_s \mathbf{E}\cdot d\mathbf{S} = \oint_s \frac{q}{4\pi\varepsilon_0}\frac{1}{r_1^2}(1-\sqrt{\alpha r_1})dS$$
$$= \frac{q}{\varepsilon_0}(1-\sqrt{\alpha r_1})\,.$$

From Coulomb's law and Gauss' law, we get

$$\oint_s \mathbf{E}\cdot d\mathbf{S} = \frac{q}{\varepsilon_0}\,.$$

The two results differ by $\frac{q}{\varepsilon_0}\sqrt{\alpha r_1}$.

(d) Using the result of (c), the surface integral at $r_1 + \Delta$ is

$$\oint_s \mathbf{E}\cdot d\mathbf{S} = \frac{q}{\varepsilon_0}(1-\sqrt{\alpha(r_1+\Delta)})\,.$$

Consider a volume V' bounded by two spherical shells S_1 and S_2 with radii $r = r_1$ and $r = r_1 + \Delta$ respectively. Gauss' divergence theorem gives

$$\oint_{S_1+S_2} \mathbf{E}\cdot d\mathbf{S} = \int_{V'} \nabla\cdot\mathbf{E}\,dV\,.$$

As the directions of $d\mathbf{S}$ on S_1 and S_2 are outwards from V', we have for small Δ

$$\frac{q}{\varepsilon_0}\left[-\sqrt{\alpha(r_1+\Delta)}+\sqrt{\alpha r_1}\right]=\frac{4\pi}{3}[(r_1+\Delta)^3-r_1^3](\Delta\cdot\mathbf{E})|_{r=r_1}\,.$$

As $\frac{\Delta}{r_1}\ll 1$, we can approximately set

$$\left(1+\frac{\Delta}{r_1}\right)^n\approx 1+n\frac{\Delta}{r_1}\,.$$

Thus one gets

$$\nabla\cdot\mathbf{E}(r=r_1)=-\frac{\sqrt{\alpha}\,q}{8\pi\varepsilon_0 r_1^{5/2}}\,.$$

On the other hand, Coulomb's law would give the divergence of the electric field produced by a point charge q as

$$\nabla\cdot\mathbf{E}(r)=\frac{q}{\varepsilon_0}\delta(r)\,.$$

1003

Static charges are distributed along the x-axis (one-dimensional) in the interval $-a\le x'\le a$. The charge density is

$$\begin{matrix}\rho(x') & \text{for} & |x'|\le a\\ 0 & \text{for} & |x'|>a\,.\end{matrix}$$

(a) Write down an expression for the electrostatic potential $\Phi(x)$at a point x on the axis in terms of $\rho(x')$.

(b) Derive a multipole expansion for the potential valid for $x>a$.

(c) For each charge configuration given in Fig. 1.3, find

(i) the total charge $Q=\int\rho dx'$,

(ii) the dipole moment $P=\int x'\rho dx'$,

(iii) the quadrupole moment $Q_{xx}=2\int x'^2\rho dx'$,

(iv) the leading term (in powers of $1/x$) in the potential Φ at a point $x>a$.

(*Wisconsin*)

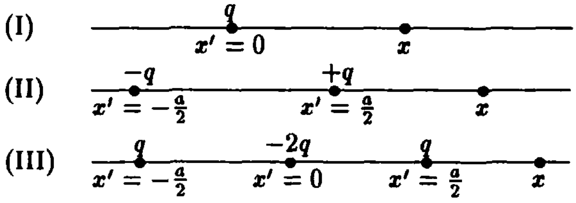

Fig. 1.3

Solution:

(a) The electrostatic potential at a point on x-axis is

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int_{-a}^{a} \frac{\rho(x')}{|x - x'|} dx' .$$

(b) For $x > a, a > x' > -a$, we have

$$\frac{1}{|x - x'|} = \frac{1}{x} + \frac{x'}{x^2} + \frac{x'^2}{x^3} + \dots .$$

Hence the multipole expansion of $\Phi(x)$ is

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\left[\int_{-a}^{a} \frac{\rho(x')}{x} dx' + \int_{-a}^{a} \frac{\rho(x')x'}{x^2} dx' + \int_{-a}^{a} \frac{\rho(x')x'^2}{x^3} dx' + \dots\right].$$

(c) The charge configuration (I) can be represented by

$$\rho(x') = q\delta(x') ,$$

for which

$$\text{(i)}\quad Q = q ; \quad \text{(ii)}\quad P = 0 ; \quad \text{(iii)}\quad Q_{xx} = 0 ; \quad \text{(iv)}\quad \Phi(x) = \frac{q}{4\pi\varepsilon_0 x} .$$

The charge configuration (II) can be represented by

$$\rho(x') = -q\delta\left(x' + \frac{a}{2}\right) + q\delta\left(x' - \frac{a}{2}\right) ,$$

for which

$$\text{(i)}\quad Q = 0 ; \quad \text{(ii)}\quad P = qa ; \quad \text{(iii)}\quad Q_{xx} = 0 ; \quad \text{(iv)}\quad \Phi(x) = -\frac{qa}{4\pi\varepsilon_0 x^2} .$$

The charge configuration (III) can be represented by

$$\rho(x') = q\delta\left(x' + \frac{a}{2}\right) + q\delta\left(x' - \frac{a}{2}\right) - 2q\delta(x'),$$

for which

$$\text{(i)}\quad Q = 0\,;\quad \text{(ii)}\quad P = 0\,;\quad \text{(iii)}\quad Q_{xx} = qa^2\,;\quad \text{(iv)}\quad \Phi(x) = \frac{qa^2}{8\pi\varepsilon_0 x^3}\,.$$

1004

Two uniform infinite sheets of electric charge densities $+\sigma$ and $-\sigma$ intersect at right angles. Find the magnitude and direction of the electric field everywhere and sketch the lines of **E**.

(*Wisconsin*)

Solution:

First let us consider the infinite sheet of charge density $+\sigma$. The magnitude of the electric field caused by it at any space point is

$$E = \frac{\sigma}{2\varepsilon_0}\,.$$

The direction of the electric field is perpendicular to the surface of the sheet. For the two orthogonal sheets of charge densities $\pm\sigma$, superposition of their electric fields yields

$$E = \frac{\sqrt{2}\,\sigma}{2\varepsilon_0}\,.$$

The direction of **E** is as shown in Fig. 1.4.

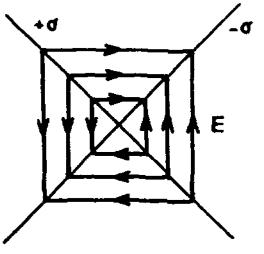

Fig. 1.4

1005

Gauss' law would be invalid if

(a) there were magnetic monopoles,

(b) the inverse-square law were not exactly true,

(c) the velocity of light were not a universal constant.

(CCT)

Solution:

The answer is (b).

1006

An electric charge can be held in a position of stable equilibrium:

(a) by a purely electrostatic field,

(b) by a mechanical force,

(c) neither of the above.

(CCT)

Solution:

The answer is (c).

1007

If $\mathbf{P}$ is the polarization vector and $\mathbf{E}$ is the electric field, then in the equation $\mathbf{P} = \alpha\mathbf{E}$, α in general is:

(a) scalar, (b) vector, (c) tensor.

(CCT)

Solution:

The answer is (c).

1008

(a) A ring of radius R has a total charge $+Q$ uniformly distributed on it. Calculate the electric field and potential at the center of the ring.

(b) Consider a charge $-Q$ constrained to slide along the axis of the ring. Show that the charge will execute simple harmonic motion for small displacements perpendicular to the plane of the ring.

(*Wisconsin*)

Solution:

As in Fig. 1.5, take the z-axis along the axis of the ring. The electric field and the potential at the center of the ring are given by

$$\mathbf{E} = 0\,, \quad \varphi = \frac{Q}{4\pi\varepsilon_0 R}\,.$$

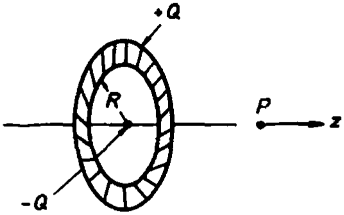

Fig. 1.5

The electric field at a point P on the z-axis is given by

$$\mathbf{E}(z) = \frac{Qz}{4\pi\varepsilon_0(R^2+z^2)^{3/2}}\mathbf{e}_z\,.$$

Thus a negative charge $-Q$ at point p is acted upon by a force

$$\mathbf{F}(z) = -\frac{Q^2 z}{4\pi\varepsilon_0(R^2+z^2)^{3/2}}\mathbf{e}_z\,.$$

As $z \ll R, F(z) \propto z$ and $-Q$ will execute simple harmonic motion.

1009

An amount of charge q is uniformly spread out in a layer on the surface of a disc of radius a.

(a) Use elementary methods based on the azimuthal symmetry of the charge distribution to find the potential at any point on the axis of symmetry.

(b) With the aid of (a) find an expression for the potential at any point $\mathbf{r}$ ($|\mathbf{r}| > a$) as an expansion in angular harmonics.

(*Wisconsin*)

Solution:

(a) Take coordinate axes as in Fig. 1.6 and consider a ring formed by circles with radii ρ and $\rho + d\rho$ on the disc. The electrical potential at a point $(0, 0, z)$ produced by the ring is given by

$$d\varphi = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{\pi a^2} \cdot \frac{2\pi\rho d\rho}{\sqrt{\rho^2 + z^2}} .$$

Integrating, we obtain the potential due to the whole ring:

$$\begin{aligned}\varphi(z) &= \int_0^a \frac{q}{2\pi\varepsilon_0 a^2} \cdot \frac{\rho d\rho}{\sqrt{\rho^2 + z^2}} \\ &= \frac{q}{2\pi\varepsilon_0 a^2}(\sqrt{a^2 + z^2} - |z|) .\end{aligned}$$

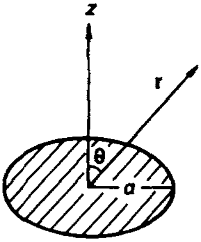

Fig. 1.6

(b) At a point $|\mathbf{r}| > a$, Laplace's equation $\nabla^2\varphi = 0$ applies, with solution

$$\varphi(r, \theta) = \sum_{n=0}^{\infty} \left(a_n r^n + \frac{b_n}{r^{n+1}}\right) P_n(\cos\theta) .$$

As $\varphi \to 0$ for $r \to \infty$, we have $a_n = 0$.

In the upper half-space, $z > 0$, the potential on the axis is $\varphi = \varphi(r, 0)$. As $P_n(1) = 1$, we have

$$\varphi(r, 0) = \sum_{n=0}^{\infty} \frac{b_n}{r^{n+1}} .$$

In the lower half-space, $z < 0$, the potential on the axis is $\varphi = \varphi(r, \pi)$. As $P_n(-1) = (-1)^n$, we have

$$\varphi(r, \pi) = \sum_{n=0}^{\infty}(-1)^n \frac{b_n}{r^{n+1}} .$$

Using the results of (a) and noting that for a point on the axis $|\mathbf{r}| = z$, we have for $z > 0$

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{b_n}{r^{n+1}} &= \frac{2q}{4\pi\varepsilon_0 a^2}(\sqrt{a^2 + r^2} - r) \\ &= \frac{qr}{2\pi\varepsilon_0 a^2}\left(\sqrt{1 + \frac{a^2}{r^2}} - 1\right) .\end{aligned}$$

However, as

$$\begin{aligned}\left(1 + \frac{a^2}{r^2}\right)^{1/2} &= 1 + \frac{1}{2}\left(\frac{a^2}{r^2}\right) + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2!}\left(\frac{a^2}{r^2}\right)^2 + \dots \\ &\quad + \frac{\frac{1}{2}(\frac{1}{2} - 1)\dots\dots(\frac{1}{2} - n + 1)}{n!}\left(\frac{a^2}{r^2}\right)^n + \dots ,\end{aligned}$$

the equation becomes

$$\sum_{n=0}^{\infty} \frac{b_n}{r^{n+1}} = \frac{qr}{2\pi\varepsilon_0 a^2} \sum_{n=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2} - 1)\dots(\frac{1}{2} - n + 1)}{n!}\left(\frac{a^2}{r^2}\right)^n .$$

Comparing the coefficients of powers of r gives

$$b_{2n-1} = 0 , \quad b_{2n-2} = \frac{q}{2\pi\varepsilon_0 a^2} \frac{\frac{1}{2}(\frac{1}{2} - 1)\dots(\frac{1}{2} - n + 1)}{n!} a^{2n} .$$

Hence, the potential at any point $\mathbf{r}$ of the half-plane $z > 0$ is given by

$$\begin{aligned}\varphi(\mathbf{r}) &= \frac{q}{2\pi\varepsilon_0 a} \sum_{n=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2} - 1)\dots(\frac{1}{2} - n + 1)}{n!} \\ &\quad \times \left(\frac{a}{r}\right)^{2n-1} P_{2n-2}(\cos\theta) , \quad (z > 0) .\end{aligned}$$

Similarly for the half-plane $z < 0$, as $(-1)^{2n-2} = 1$ we have

$$\varphi(\mathbf{r}) = \frac{q}{2\pi\varepsilon_0 a}\sum_{n=1}^{\infty}\frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)}{n!} \times \left(\frac{a}{r}\right)^{2n-1} P_{2n-2}(\cos\theta) \qquad (z < 0)\,.$$

Thus the same expression for the potential applies to all points of space, which is a series in Legendre polynomials.

1010

A thin but very massive disc of insulator has surface charge density σ and radius R. A point charge $+Q$ is on the axis of symmetry. Derive an expression for the force on the charge.

(*Wisconsin*)

Solution:

Refer to Problem **1009** and Fig. 1.6. Let Q be at a point $(0, 0, z)$ on the axis of symmetry. The electric field produced by the disc at this point is

$$E = -\frac{\sigma}{2\varepsilon_0}\left(\frac{z}{\sqrt{a^2+z^2}} - 1\right),$$

whence the force on the point charge is

$$F = QE = \frac{\sigma Q}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right).$$

By symmetry the direction of this force is along the axis of the disc.

1011

The cube in Fig. 1.7 has 5 sides grounded. The sixth side, insulated from the others, is held at a potential ϕ_0. What is the potential at the center of the cube and why?

(*MIT*)

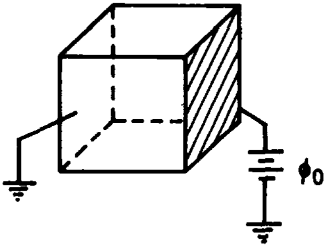

Fig. 1.7

Solution:

The electric potential ϕ_c at the center of the cube can be expressed as a linear function of the potentials of the six sides, i.e.,

$$\phi_c = \sum_i C_i \phi_i ,$$

where the C_i's are constants. As the six sides of the cube are in the same relative geometrical position with respect to the center, the C_i's must have the same value, say C. Thus

$$\phi_c = C \sum_i \phi_i .$$

If each of the six sides has potential ϕ_0, the potential at the center will obviously be ϕ_0 too. Hence $C = \frac{1}{6}$. Now as the potential of one side only is ϕ_0 while all other sides have potential zero, the potential at the center is $\phi_0/6$.

1012

A sphere of radius R carries a charge Q, the charge being uniformly distributed throughout the volume of the sphere. What is the electric field, both outside and inside the sphere?

(*Wisconsin*)

Solution:

The volume charge density of the sphere is

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} .$$

Take as the Gaussian surface a spherical surface of radius r concentric with the charge sphere. By symmetry the magnitude of the electric field at all points of the surface is the same and the direction is radial. From Gauss' law

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int \rho dV$$

we immediately obtain

$$\mathbf{E} = \frac{Q\mathbf{r}}{4\pi\varepsilon_0 r^3} \qquad (r \geq R),$$

$$\mathbf{E} = \frac{Q\mathbf{r}}{4\pi\varepsilon_0 R^3} \qquad (r \leq R).$$

1013

Consider a uniformly charged spherical volume of radius R which contains a total charge Q. Find the electric field and the electrostatic potential at all points in the space.

(*Wisconsin*)

Solution:

Using the results of Problem 1012

$$\mathbf{E}_1 = \frac{Q\mathbf{r}}{4\pi\varepsilon_0 R^3}, \qquad (r \leq R)$$

$$\mathbf{E}_2 = \frac{Q\mathbf{r}}{4\pi\varepsilon_0 r^3}, \qquad (r \geq R)$$

and the relation between electrostatic field intensity and potential

$$\varphi(p) = \int_p^\infty \mathbf{E} \cdot d\mathbf{l},$$

we obtain

$$\begin{aligned} \varphi_1(r) &= \int_r^R \mathbf{E}_1 \cdot d\mathbf{r} + \int_R^\infty \mathbf{E}_2 \cdot d\mathbf{r} \\ &= \int_r^R \frac{Qrdr}{4\pi\varepsilon_0 R^3} + \int_R^\infty \frac{Qdr}{4\pi\varepsilon_0 r^2} \\ &= \frac{Q}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) \qquad (r \leq R), \end{aligned}$$

$$\varphi_2(r) = \int_r^\infty \mathbf{E}_2 \cdot d\mathbf{r} = \frac{Q}{4\pi\varepsilon_0 r} \qquad (r \geq R).$$

1014

For a uniformly charged sphere of radius R and charge density ρ,

(a) find the form of the electric field vector $\mathbf{E}$ both outside and inside the sphere using Gauss' law;

(b) from $\mathbf{E}$ find the electric potential ϕ using the fact that $\phi \to 0$ as $r \to \infty$.

(*Wisconsin*)

Solution:

(a) Same as for Problem **1013**.

(b) Referring to Problem **1013**, we have

$$\text{for} \quad r > R\,, \; \phi = \frac{R^3\rho}{3\varepsilon_0 r}\,,$$

$$\text{for} \quad r < R\,, \; \phi = \frac{\rho R^3}{6\varepsilon_0}\left(3 - \frac{r^2}{R^2}\right).$$

1015

In the equilibrium configuration, a spherical conducting shell of inner radius a and outer radius b has a charge q fixed at the center and a charge density σ uniformly distributed on the outer surface. Find the electric field for all r, and the charge on the inner surface.

(*Wisconsin*)

Solution:

Electrostatic equilibrium requires that the total charge on inner surface of the conducting shell be $-q$. Using Gauss' law we then readily obtain

$$\mathbf{E}(r) = \frac{q}{4\pi\varepsilon_0 r^2}\mathbf{e}_r \qquad \text{for } r < a\,,$$

$$\mathbf{E} = 0 \qquad \text{for } a < r < b\,,$$

$$\mathbf{E}(r) = \frac{1}{4\pi\varepsilon_0}\frac{4\pi b^2\sigma}{r^2}\mathbf{e}_r = \frac{\sigma b^2}{\varepsilon_0 r^2}\mathbf{e}_r \qquad \text{for } r > b\,.$$

1016

A solid conducting sphere of radius r_1 has a charge of $+Q$. It is surrounded by a concentric hollow conducting sphere of inside radius r_2 and outside radius r_3. Use the Gaussian theorem to get expressions for

(a) the field outside the outer sphere,

(b) the field between the spheres.

(c) Set up an expression for the potential of the inner sphere. It is not necessary to perform the integrations.

(*Wisconsin*)

Solution:

Because of electrostatic equilibrium the inner surface of the hollow conducting sphere carries a total charge $-Q$, while the outer surface carries a total charge $+Q$. Using Gauss' law

$$\oint_s \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{tot}}}{\varepsilon_0} ,$$

where Q_{tot} is the algebraic sum of all charges surrounded by a closed surface s, we obtain

(a) $\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{e}_r \qquad (r > r_3)$

(b) $\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{e}_r \qquad (r_2 > r > r_1)$

(c) Using the expression for the potential $\varphi(p) = \int_p^\infty \mathbf{E} \cdot d\mathbf{l}$, we find the potential of the inner sphere:

$$\varphi(r_1) = \int_{r_1}^{r_2} \frac{Q}{4\pi\varepsilon_0 r^2} dr + \int_{r_3}^{\infty} \frac{Q}{4\pi\varepsilon_0 r^2} dr .$$

1017

The inside of a grounded spherical metal shell (inner radius R_1 and outer radius R_2) is filled with space charge of uniform charge density ρ. Find the electrostatic energy of the system. Find the potential at the center.

(*Wisconsin*)

Solution:

Consider a concentric spherical surface of radius $r(r < R_1)$. Using Gauss' law we get

$$\mathbf{E} = \frac{r}{3}\frac{\rho}{\varepsilon_0}\mathbf{e}_r .$$

As the shell is grounded, $\varphi(R_1) = 0, E = 0\,(r > R_2)$. Thus

$$\varphi(r) = \int_r^{R_1} E dr = \frac{\rho}{6\varepsilon_0}(R_1^2 - r^2).$$

The potential at the center is

$$\varphi(0) = \frac{1}{6\varepsilon_0}\rho R_1^2.$$

The electrostatic energy is

$$W = \int \frac{1}{2}\rho\varphi dV = \frac{1}{2}\int_0^{R_1} \frac{\rho}{6\varepsilon_0}(R_1^2 - r^2)\cdot\rho\cdot 4\pi r^2 dr = \frac{2\rho^2 R_1^5}{45\varepsilon_0}.$$

1018

A metal sphere of radius a is surrounded by a concentric metal sphere of inner radius b, where $b > a$. The space between the spheres is filled with a material whose electrical conductivity σ varies with the electric field strength E according to the relation $\sigma = KE$, where K is a constant. A potential difference V is maintained between the two spheres. What is the current between the spheres?

(*Wisconsin*)

Solution:

Since the current is

$$I = j\cdot S = \sigma E\cdot S = KE^2\cdot S = KE^2\cdot 4\pi r^2,$$

the electric field is

$$E = \frac{1}{r}\sqrt{\frac{I}{4\pi K}}$$

and the potential is

$$V = -\int_b^a E\cdot dr = -\int_b^a \sqrt{\frac{I}{4\pi K}}\frac{1}{r}dr = \sqrt{\frac{I}{4\pi K}}\ln\left(\frac{b}{a}\right).$$

Hence the current between the spheres is given by

$$I = 4\pi K V^2/\ln(b/a).$$

1019

An isolated soap bubble of radius 1 cm is at a potential of 100 volts. If it collapses to a drop of radius 1 mm, what is the change of its electrostatic energy?

(*Wisconsin*)

Solution:

If the soap bubble carries a charge Q, its potential is

$$V = \frac{Q}{4\pi\varepsilon_0 r}.$$

For $r = r_1 = 1$ cm, $V = V_1 = 100$ V, we have $Q = 4\pi\varepsilon_0 r_1 V_1$. As the radius changes from r_1 to $r_2 = 1$ mm, the change of electrostatic energy is

$$\begin{aligned}\Delta W &= \frac{Q^2}{8\pi\varepsilon_0 r_2} - \frac{Q^2}{8\pi\varepsilon_0 r_1} = 2\pi\varepsilon_0 (r_1 V_1)^2 \left(\frac{1}{r_2} - \frac{1}{r_1}\right) \\ &= 2\pi \times 8.85 \times 10^{-12} \times (10^{-12} \times 100)^2 \times \left(\frac{1}{10^{-3}} - \frac{1}{10^{-2}}\right) \\ &= 5 \times 10^{-8} \text{ J}.\end{aligned}$$

1020

A static electric charge is distributed in a spherical shell of inner radius R_1 and outer radius R_2. The electric charge density is given by $\rho = a + br$, where r is the distance from the center, and zero everywhere else.

(a) Find an expression for the electric field everywhere in terms of r.

(b) Find expressions for the electric potential and energy density for $r < R_1$. Take the potential to be zero at $r \to \infty$.

(*SUNY, Buffalo*)

Solution:

Noting that ρ is a function of only the radius r, we can take a concentric spherical surface of radius r as the Gaussian surface in accordance with the symmetry requirement. Using Gauss' law

$$\oint_s \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int \rho(r) dr,$$

we can get the following results:

(a) Electric field strength.

For $r < R_1$, $\mathbf{E}_1 = 0$.

For $R_1 < r < R_2$, using the relation $4\pi r^2 E_2 = \frac{4\pi}{\varepsilon_0}\int_{R_1}^{r}(a + br')r'^2 dr'$ we find

$$\mathbf{E}_2 = \frac{1}{\varepsilon_0 r^3}\left[\frac{a}{3}(r^3 - R_1^3) + \frac{b}{4}(r^4 - R_1^4)\right]\mathbf{r}.$$

For $R_2 > r$, from $4\pi r^2 E_3 = \frac{4\pi}{\varepsilon_0}\int_{R_1}^{R_2}(a + br')r'^2 dr'$ we get

$$\mathbf{E}_3 = \frac{1}{\varepsilon_0 r^3}\left[\frac{a}{3}(R_2^3 - R_1^3) + \frac{b}{4}(R_2^4 - R_1^4)\right]\mathbf{r}.$$

(b) Potential and the energy density for $r < R_1$.

Noting that $\varphi(\infty) = 0$, the potential is

$$\begin{aligned}\varphi(r) &= \int_r^\infty \mathbf{E}\cdot d\mathbf{l} = \left(\int_r^{R_1} + \int_{R_1}^{R_2} + \int_{R_2}^{\infty}\right)\mathbf{E}\cdot d\mathbf{r} \\ &= \frac{1}{\varepsilon_0}\left[\frac{a}{3}(R_2^2 - R_1^2) + \frac{b}{4}(R_2^3 - R_1^3)\right].\end{aligned}$$

Also, as $\mathbf{E}_1 = 0\,(r < R_1)$, the energy density for $r < R_1$ is

$$W = \frac{\varepsilon_0}{2}E_1^2 = 0.$$

1021

An electric charge Q is uniformly distributed over the surface of a sphere of radius r. Show that the force on a small charge element dq is radial and outward and is given by

$$dF = \frac{1}{2}E dq,$$

where $E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$ is the electric field at the surface of the sphere.

(*Wisconsin*)

Solution:

The surface charge density is given by

$$\sigma = \frac{Q}{4\pi r^2}.$$

As shown in Fig. 1.8, we consider a point P inside the sphere close to an area element ds. The charge dq on this area element will produce at the point P an electric field which is approximately that due to a uniformly charged infinite plate, namely,

$$\mathbf{E}_{1P} = -\frac{\sigma}{2\varepsilon_0}\mathbf{n},$$

where $\mathbf{n}$ is a unit vector normal to ds in the outward direction.

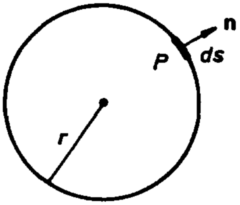

Fig. 1.8

The electric field is zero inside the sphere. Hence, if we take $\mathbf{E}_{2P}$ as the electric field at P due to all the charges on the spherical surface except the element ds, we must have

$$\mathbf{E}_P = \mathbf{E}_{1P} + \mathbf{E}_{2P} = 0.$$

Therefore,

$$\mathbf{E}_{2P} = \frac{\sigma}{2\varepsilon_0}\mathbf{n} = \frac{Q}{8\pi\varepsilon_0 r^2}\mathbf{n}.$$

As P is close to ds, $\mathbf{E}_{2P}$ may be considered as the field strength at ds due to the charges of the spherical surface. Hence, the force acting on ds is

$$d\mathbf{F} = dq\mathbf{E}_{2P} = \frac{1}{2}Edq\mathbf{n},$$

where $E = Q/4\pi\varepsilon_0 r^2$ is just the field strength on the spherical surface.

1022

A sphere of radius R_1 has charge density ρ uniform within its volume, except for a small spherical hollow region of radius R_2 located a distance a from the center.

(a) Find the field $\mathbf{E}$ at the center of the hollow sphere.

(b) Find the potential ϕ at the same point.

(*UC, Berkeley*)

Solution:

(a) Consider an arbitrary point P of the hollow region (see Fig. 1.9) and let

$$\mathrm{OP} = \mathbf{r}\,,\ \mathrm{Q'P} = \mathbf{r}'\,,\ \mathrm{OO'} = \mathbf{a},\quad \mathbf{r}' = \mathbf{r} - \mathbf{a}\,.$$

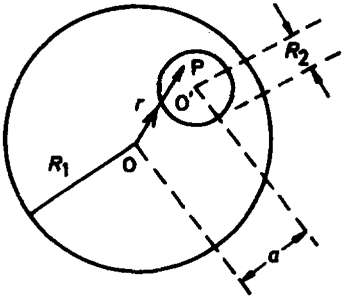

Fig. 1.9

If there were no hollow region inside the sphere, the electric field at the point P would be

$$\mathbf{E}_1 = \frac{\rho}{3\varepsilon_0}\mathbf{r}\,.$$

If only the spherical hollow region has charge density ρ the electric field at P would be

$$\mathbf{E}_2 = \frac{\rho}{3\varepsilon_0}\mathbf{r}'\,.$$

Hence the superposition theorem gives the electric field at P as

$$\mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2 = \frac{\rho}{3\varepsilon_0}\mathbf{a}\,.$$

Thus the field inside the hollow region is uniform. This of course includes the center of the hollow.

(b) Suppose the potential is taken to be zero at an infinite point. Consider an arbitrary sphere of radius R with a uniform charge density ρ. We can find the electric fields inside and outside the sphere as

$$\mathbf{E}(r) = \begin{cases} \frac{\rho \mathbf{r}}{3\varepsilon_0}\,, & r < R\,, \\ \frac{\rho R^3}{3\varepsilon_0 r^3}\mathbf{r}\,, & r > R\,. \end{cases}$$

Then the potential at an arbitrary point inside the sphere is

$$\phi = \left(\int_r^R + \int_R^\infty\right)\mathbf{E}\cdot d\mathbf{r} = \frac{\rho}{6\varepsilon_0}(3R^2 - r^2)\,, \qquad (1)$$

where r is the distance between this point and the spherical center.

Now consider the problem in hand. If the charges are distributed throughout the sphere of radius R_1, let ϕ_1 be the potential at the center O′ of the hollow region. If the charge distribution is replaced by a small sphere of uniform charge density ρ of radius R_2 in the hollow region, let the potential at O′ be ϕ_2. Using (1) and the superposition theorem, we obtain

$$\begin{aligned}\phi_{O'} = \phi_1 - \phi_2 &= \frac{\rho}{6\varepsilon_0}(3R_1^2 - a^2) - \frac{\rho}{6\varepsilon_0}(3R_2^2 - 0)\\ &= \frac{\rho}{6\varepsilon_0}[3(R_1^2 - R_2^2) - a^2]\,.\end{aligned}$$

1023

The electrostatic potential at a point P due to an idealized dipole layer of moment per unit area $\boldsymbol{\tau}$ on surface S is

$$\phi_P = \frac{1}{4\pi\varepsilon_0}\int\frac{\boldsymbol{\tau}\cdot\mathbf{r}}{r^3}dS\,,$$

where $\mathbf{r}$ is the vector from the surface element to the point P.

(a) Consider a dipole layer of infinite extent lying in the x-y plane of uniform moment density $\boldsymbol{\tau} = \tau\mathbf{e}_z$. Determine whether ϕ or some derivative of it is discontinuous across the layer and find the discontinuity.

(b) Consider a positive point charge q located at the center of a spherical surface of radius a. On this surface there is a uniform dipole layer $\boldsymbol{\tau}$ and a uniform surface charge density σ. Find $\boldsymbol{\tau}$ and σ so that the potential inside the surface will be just that of the charge q, while the potential outside will be zero. (You may make use of whatever you know about the potential of a surface charge.)

(*SUNY, Buffalo*)

Solution:

(a) By symmetry the electrostatic potential at point P is only dependent on the z coordinate. We choose cylindrical coordinates (R, θ, z) such that P is on the z-axis. Then the potential at point P is

$$\phi_P = \frac{1}{4\pi\varepsilon_0}\int \frac{\tau\cdot\mathbf{r}}{r^3}dS = \frac{1}{4\pi\varepsilon_0}\int \frac{\tau z}{r^3}dS .$$

As $r^2 = R^2 + z^2, dS = 2\pi R dR$, we get

$$\phi_P = \frac{2\pi\tau z}{4\pi\varepsilon_0}\int_0^\infty \frac{RdR}{\sqrt{(R^2+z^2)^3}} = \begin{cases} \frac{\tau}{2\varepsilon_0}, & z > 0 , \\ -\frac{\tau}{2\varepsilon_0}, & z < 0 . \end{cases}$$

Hence, the electrostaic potential is discontinous across the x-y plane (for which $z = 0$). The discontinuity is given by

$$\Delta\phi = \frac{\tau}{2\varepsilon_0} - \left(-\frac{\tau}{2\varepsilon_0}\right) = \frac{\tau}{\varepsilon_0} .$$

(b) It is given that $\phi = 0$ for $r > a$. Consequently $\mathbf{E} = 0$ for $r > a$. Using Gauss' law

$$\oint \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\varepsilon_0} ,$$

we find that $\sigma \cdot 4\pi a^2 + q = 0$. Thus

$$\sigma = -\frac{q}{4\pi a^2} .$$

If the potential at infinity is zero, then the potential outside the spherical surface will be zero everywhere. But the potential inside the sphere is $\varphi = \frac{q}{4\pi\varepsilon_0 r}$. For $r = a, \varphi = \frac{q}{4\pi\varepsilon_0 a}$, so that the discontinuity at the spherical surface is

$$\Delta\phi = -\frac{q}{4\pi\varepsilon_0 a} .$$

We then have $\frac{\tau}{\varepsilon_0} = -\frac{q}{4\pi\varepsilon_0 a}$, giving

$$\tau = -\frac{q}{4\pi a}\mathbf{e}_r .$$