1.3 A review of relativistic particle motion

Particle accelerator physics is a realm of applied special relativity. In this section we review the most important and useful relations. Following the standard method we define the relativistic velocity β, and the Lorentz factor γ as

$$\beta = \frac{v}{c} \tag{1.7}$$

$$\gamma = (1-\beta^2)^{-\frac{1}{2}} \tag{1.8}$$

with v being the velocity of the particle, and c being the velocity of light in a vacuum. Rearranging this gives

$$\beta\gamma = \sqrt{\gamma^2 - 1}, \quad \text{and} \quad \gamma^2 = (\beta\gamma)^2 + 1. \tag{1.9}$$

The total energy, momentum, and kinetic energy for a particle of rest mass, m, are, respectively:

$$U = \gamma m c^2, \tag{1.10}$$

$$p = \beta\gamma m c = \beta\frac{U}{c}, \quad \text{and} \tag{1.11}$$

$$W = (\gamma - 1) m c^2. \tag{1.12}$$

The relation between energy and momentum is

$$U = \sqrt{(pc)^2 + (mc^2)^2} = \sqrt{p^2 + m^2}, \tag{1.13}$$

where we have used the ever popular set of units, with $c = 1$, in the last expression. The most frequently accelerated particles are the electron with mass $m_e = 0.511\text{MeV}$, and the proton with mass, $m_p = 938\text{MeV}$.

The following divisions are frequently used

$\gamma \simeq 1$	Non-relativistic	N. R.
$\gamma > 1$	Relativistic	—
$\gamma \gg 1$	Ultra-Relativistic	U. R.

The non-relativistic case can be checked by expanding Eq. (1.8) for small β and obtaining $\gamma \simeq 1 + \frac{1}{2}\beta^2$, which when inserted into Eq. (1.12) produces

$$W \simeq \frac{1}{2} m c^2 \beta^2 = \frac{1}{2} m v^2. \tag{1.14}$$

In the ultra-relativistic case the mass becomes negligible and Eqs. (1.10, 1.11, and 1.12) collapse into the simpler relation $U \simeq W \simeq pc$.

Now a set of relations, particularly useful to accelerator physics, will be deduced. Differentiating respectively (1.8) and (1.9), we obtain:

$$d\gamma = \beta(1-\beta^2)^{-\frac{3}{2}}d\beta = \beta\gamma^3 d\beta \tag{1.15}$$

$$d(\beta\gamma) = \gamma d\beta + \beta d\gamma = \gamma(1+\beta^2\gamma^2)d\beta = \gamma^3 d\beta = \frac{d\gamma}{\beta}. \tag{1.16}$$

Squaring and differentiating Eq. (1.13) gives $2UdU = 2pdp$ or

$$\frac{dU}{U} = \frac{p^2}{U^2}\frac{dp}{p} = \beta^2\frac{dp}{p}. \tag{1.17}$$

By dividing Eq. (1.15) by $\beta^2\gamma^3$, the fractional change in velocity can be found as

$$\frac{d\beta}{\beta} = \frac{1}{(\beta\gamma)^2}\frac{d\gamma}{\gamma} = \frac{1}{(\beta\gamma)^2}\frac{dU}{U} = \frac{1}{\gamma^2}\frac{dp}{p}. \tag{1.18}$$

For acceleration in one dimension, Newton's second law becomes

$$F = \frac{dp}{dt} = mc\frac{d}{dt}(\beta\gamma) = \gamma^3 m\frac{dv}{dt} = m^*\frac{dv}{dt}, \tag{1.19}$$

having considered Eq. (1.16), and defining as effective mass,

$$m^* = \frac{dp}{dv} = \frac{d(\gamma m v)}{dv} = m\gamma^3. \tag{1.20}$$

The electromagnetic force is what accelerates charged particles, and is described mathematically by the Lorentz equation,

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}). \tag{1.21}$$

If there is no electric field and only a uniform magnetic field, then the force equation may be written as

$$\vec{F} = q\vec{v} \times \vec{B} = \frac{d}{dt}(\gamma m\vec{v}) = m(\gamma\frac{d\vec{v}}{dt} + \frac{d\gamma}{dt}\vec{v}) = \gamma m\frac{d\vec{v}}{dt}, \tag{1.22}$$

since $\beta = |\vec{\beta}|$ is a constant, which implies that $(d\gamma/dt) = 0$. The velocity $\vec{v} \mp \vec{\omega} \times \vec{\rho}$, with the angular velocity, $\vec{\omega}$ being constant for a central force of constant magnitude.

The *cyclotron radius* ρ is just the radius of the particle's orbit. Eq. (1.22) now becomes

$$q\vec{v} \times \vec{B} = \gamma m\vec{\omega} \times \frac{d\vec{\rho}}{dt} = \gamma m\vec{\omega} \times \vec{v}, \tag{1.23}$$

or for a particle moving in a plane perpendicular to $\vec{B}$,

$$evB = \gamma m\omega v = \gamma m\frac{v^2}{\rho}, \tag{1.24}$$

i. e., the Lorentz force is the centripetal force which keeps the particle of charge q and mass m on a circular orbit. Dividing Eq. (1.24) by v/ρ, we get a relation for the momentum in terms of the orbit radius, magnetic field and charge of the particle:

$$p = \beta\gamma mc = eB\rho. \tag{1.25}$$

For a particle with same charge as the electron, it is useful to remember

$$p(\mathrm{GeV/c}) \simeq 0.3B(\mathrm{T})\rho(\mathrm{m}). \tag{1.26}$$

Another popular formula is the one for the angular velocity or cyclotron frequency:

$$\omega = \frac{eB}{\gamma m} \tag{1.27}$$

It may also be useful to remember the Lorentz transformations of the electromagnetic field:

$$\vec{E}'_\perp = \gamma(\vec{E}_\perp + \vec{v} \times \vec{B}_\perp), \tag{1.28}$$
$$\vec{E}'_\parallel = \vec{E}_\parallel, \tag{1.29}$$
$$\vec{B}'_\perp = \gamma(\vec{B}_\perp - \vec{v} \times \vec{E}_\perp), \quad \text{and} \tag{1.30}$$
$$\vec{B}'_\parallel = \vec{B}_\parallel, \tag{1.31}$$

where the $\|$ designates the component of the field parallel to the boost velocity $\vec{v}$, and $\perp$ indicates the component perpendicular to the boost.

1.4 Linear accelerators with oscillating electric fields

Since it is very difficult to produce dc voltages more than a few million volts, it was necessary to find a new method for acceleration to energies beyond a few MeV. In 1928 Wideröe proposed an accelerating structure using a series of cylindrical