condition,[11]

$$B_g = \frac{1}{2}\bar{B}, \tag{1.35}$$

typical of betatrons, which gives rise to the rather cumbersome structure of these machines.

Working out a few realistic numbers, one obtains $W_{\max} \simeq 50\,MeV$, which for $B_g = 0.5\,\mathrm{T}$, gives $R = 33\,cm$ and $\bar{\tau}_{\text{rev}} = (2\pi R)/\bar{v} \simeq 14\,\text{ns}$, since $\bar{v} \simeq c/2$. For $\Delta t_{\text{cycle}} = (1/4 f_{\text{main}}) = 5\,\text{ms}$, we get

$$N_{\text{rev}} = \frac{\Delta t_{\text{cycle}}}{\bar{\tau}} \simeq 357,000 \text{ turns}. \tag{1.36}$$

Now the problem of focusing the circulating beam, mildly tackled in cyclotrons, becomes of primary importance. In the next chapter, we will derive the trajectory equations for a charged particle traveling through various shaped magnetic fields. Also, we will see that several effects will cause focusing in the dimensions transverse to the beam.

1.6 Momentum compaction and the synchronous particle

If a particle of fixed energy moves along a closed trajectory, the integral of the curvature, $1/\rho$, around the closed path must be

$$\oint \frac{ds}{\rho} = 2\pi. \tag{1.37}$$

Since $1/\rho = qB_\perp/p$, this means that the momentum

$$p = \frac{q}{2\pi} \oint B_\perp\, ds. \tag{1.38}$$

Leaving the magnetic guide field unchanged, consider another particle also with a closed trajectory but a slightly different momentum. This second particle's closed orbit must differ from the path of the first particle. By varying the momentum, the path length, $L = \oint ds$, of the closed trajectory can be varied. The fractional deviation of this path length divided by the fractional deviation of the momentum is frequently called *momentum compaction*

$$\alpha_p = \frac{dL}{L} \Big/ \frac{dp}{p} = \frac{p}{L}\frac{dL}{dp}. \tag{1.39}$$

The name momentum compaction (see Appendix A) is reviled by some authors since an increase in momentum implies a lengthening of the orbit if $\alpha_p > 0$, i. e., a dilation rather than a compaction.

For example, it is easy to show that the momentum compaction for a satellite of mass m in a circular orbit* around the earth at a radius r has a momentum compaction of -2, ignoring elliptical orbits:

$$F = \frac{GMm}{r^2} = \frac{p^2}{mr}, \tag{1.40}$$

$$\frac{dr}{dp} = -\frac{2pr^2}{GMm}, \tag{1.41}$$

and thus

$$\alpha_p = \frac{p}{r}\frac{dr}{dp} = -\frac{2p^2 r}{GMm} = -2. \tag{1.42}$$

In order to understand the acceleration of a particle due to an rf field, let us first consider a proton synchrotron with a fixed magnetic guide field and a single rf cavity driven by an oscillating electric field of constant amplitude and frequency ω_{rf}. The electric field on the axis of the cavity can be described by

$$E_s(x = 0, y = 0, s, t) = E_0(s)\cos(\omega_{\text{rf}} t), \tag{1.43}$$

where $E_0(s)$ is the s-component of the amplitude of the electric field along the axis. For this simple example, let us ignore Maxwell and approximate $E_0(s)$ by a constant inside the cavity and zero everywhere else. This constant would then be the maximum voltage across the gap divided by the length of the gap, V_0/g.

As a particle of velocity v crosses the gap it will experience a varying acceleration. The change in energy can be calculated by

$$\Delta U = \int_{t_0}^{t_0+T} \frac{eV_0}{g}\cos(\omega_{\text{rf}} t)\, v\, dt, \tag{1.44}$$

where the particle enters the gap at time t_0 and position s_0. The transit time T through the cavity must satisfy the condition

$$g = \int_{t_0}^{t_0+T} v\, dt. \tag{1.45}$$

* Here, it is assumed the increment dp of momentum is applied in such a way as to keep the satellite in a circular orbit. If this were not done, then the momentum would not be constant, and our definition of momentum compaction would not be applicable.

Particles arriving at different times will accrue different energy shifts on passing through the cavity, some positive, some negative. There will be a specific phase relative to the rf oscillations, for which $\Delta U = 0$. In this simple case, a *synchronous particle* is defined as a hypothetical particle that moves along the design trajectory and passes through the rf cavity with a phase, such that there is no change in energy. The synchronous particle must have a path length which is an integral number of $\beta\lambda$, where λ is the wavelength of the rf field, and $\beta = v/c$ for the particle.

When the accelerator is ramped up in energy (by ramping up the fields of the guide magnets), the meaning of the synchronous particle changes slightly. The ramping is assumed to be slow, so that the particles gain only a very small increase in energy per revolution. It is then possible to consider the ramping to increase as a step function, so that on each revolution the momentum corresponding to the closed orbit increases in steps. The synchronous particle is now defined as the hypothetical particle whose momentum increases exactly by this amount. Clearly the phase of this synchronous particle with respect to the rf must be shifted slightly from the case of no acceleration, i. e., the particle must see a net electric potential averaged over the time that it spends in the gap of the cavity.

This concept of synchronous particle may also be extended to a linac. A linac is designed to give a specific increase in energy for each cell. These increments typically vary from cell to cell. For the linac, the synchronous particle is defined as the hypothetical particle that obtains exactly the increment of energy of the design at each cell.

The next few chapters will study the transverse component of motion in beam lines and accelerators. The concept of phase stability and acceleration of particles will be discussed starting in Chapter 7.

Problems

1–1 a) If the bunches can be described by Gaussian ellipsoids with

$$\rho \propto \exp\left(-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2} + \frac{z^2}{2\sigma_z^2}\right)\right),$$

show that the luminosity reduces to

$$\mathcal{L} = f_0 N_b \frac{N_+ N_-}{\pi(4\sigma_x\sigma_y)},$$

where it is assumed that the beams move along the z-axis. The number of particles per bunch in the electron and positron beams are N_- and N_+, respectively. The number of bunches in each beam is given by N_b. (Assume the bunches do not change shape either due to the accelerator optics or the interaction of the beams passing through each other.) b) An e^+e^- storage ring (CESR) operates at 5.3GeV with 7 bunches of e^+ and 7 bunches of e^- orbiting in opposite directions. Assume the current per bunch is initially 8mA, and the ring circumference is 768m. For $\sigma_x = 8.4\times10^{-4}$m, $\sigma_y = 3.5\times10^{-5}$m, and $\sigma_z = 2.2$cm, what is the initial luminosity in one of the experiments? What is the integrated luminosity of this experiment for a 3 hr run if the beam lifetimes are both 2 hr? (Assume that the beam currents decay exponentially.)

1–2 Calculate the brightness of a NdYAG laser with the following parameters:

$$\lambda = 1.064\mu\text{m}$$
$$\text{Power} = 20\text{W}$$
$$\text{Bandwidth } \frac{\Delta\omega}{2\pi} = 120\text{GHz}$$
$$\text{Beam divergence} = 10 \text{ mrad.}$$

1–3 Consider two successive Lorentz boosts, one along the x-axis and a second along the y-axis. Find the net velocity, $\vec{v}^*$, of a coordinate system which was at rest before the boosts. Apply a third boost along $-\vec{v}^*$ to bring the initial system back to rest. Show that the result of these three successive rotations is just a simple rotation of coordinates about the z-axis by an angle ϕ given by

$$\tan\phi = \frac{\beta_1\beta_2(\gamma_1\gamma_2 - 1)}{\beta_1^2\gamma_1 + \beta_2^2\gamma_2}.$$

(This is the basis of Thomas precession.)

1–4 a) The proposed SSC will collide 20TeV protons head on with 20TeV protons. What would be the required energy for protons in a fixed target accelerator that would produce the same collision energy in the center of mass? b) The electron-proton accelerator HERA[12] is under construction. The energy of the electron beam will be 30GeV, and the energy of the proton beam will be 800GeV. Assuming a zero crossing angle, what will be the center-of-mass energy, and what will be the average momentum of the collisions in the lab.

1–5 In a fixed Cartesian coordinates system, show that the equations of motion for a charged particle moving in a magnetic field may be written as

$$x'' = \frac{q}{p}(1 + x'^2 + y'^2)^{\frac{1}{2}}[y'B_z - (1 + x'^2)B_y + x'y'B_x], \quad \text{and}$$
$$y'' = -\frac{q}{p}(1 + x'^2 + y'^2)^{\frac{1}{2}}[x'B_z - (1 + y'^2)B_x + x'y'B_y],$$

where the primes denote derivatives with respect to z (i. e., $x' = dx/dz$). Here it has been assumed that the electric field is zero and that $dz/dt \neq 0$.

References for Chapter 1

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