RELAXATION LABELING ALGORITHM BASED ON LEARNING AUTOMATA

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ABSTRACT

Relaxation labeling processes resolve the problem of assigning labels to objects in a manner which is consistent with respect to some specific constraints (or compatibilities). This problem is resolved here by a team of learning automata interacting with an environment that gives responses as to the consistency of labeling selected by the automata. In the environment model proposed here, consistency includes not only compatibilities based on binary relations, but also the tertiary type. An analysis of the proposed relaxation process is given here, by using an explicit definition of consistency.

Afterwards, these processes are applied to the stereoscopic matching of edge points from constraints previously defined. The algorithm is very good in real-time applications, since parallel processing is able, and it is fairly robust to the noise component in the environment. In this application noise may be due to changes in both images, which are acquired from different perspectives.

1. Introduction

Relaxation labeling processes resolve the problem of assigning labels to objects in a manner which is consistent with respect to some domain-specific constraints (world model). Many problems in image processing and artificial intelligence can be formulated in these terms. These processes were initially proposed in vision systems to reduce ambiguity and noise [1]. Here, these processes are applied to solve the matching problem in stereopsis [5]. This problem consists in establishing correspondence between edge points from each image of the stereoscopic pair, in order to obtain a tridimensional structure of a scene. Ambiguity can arise in stereo matching, since each image was initially acquired from a different perspective.

In a labeling problem, one is given [4]:

1) a set of objects, $U = \{1,2, \dots , n\}$,
2) a set of labels for each object, $\Lambda = \{1, \dots ,m\}$,
3) a neighbour relation over the objects that specifies which pairs (or N-tuples) of objects constrain each other, and
4) a set of constraints over labels at pairs (or N-tuples) of neighbouring objects.

Objects and labels sets are understood here like edge points belonging to both images of the stereoscopic pair. A probability vector $\underline{P}_i^k=(P_i^k(1),\dots,P_i^k(m))'$ is associated to each object, where $P_i^k(\alpha)$ expresses the probability of assigning label α to node i at the kth stage of the iterative proccess ($P_i(\alpha)$ value is one if α is always assigned to i, otherwise it is zero). Initial value of the process P_i^0 is obtained through some low-level noisy measurements (for stereo matching, comparing features of two edge points, each one from a different stereo image). As $\underline{P}_i^k$ can take values for a continuous range, then a continuous labeling problem is concerned.

In relaxation labeling, constraints are expressed by real-valued functions $r_{ij}(\alpha,\beta)$ that specify the compatibility degree of assigning α to i, whether β is assigned to j. Here, we consider that these functions can take real values in a continuous range. Most of the compatibility functions are based on relâtions of binary order, since they add the least computation. However, there are some situations which require the use of tertiary order constraints. In a stereo matching when occlusion problems occur, a label can be assigned to objects in an image without correspondence in the other image (*null* label). Here, tertiary order constraints, like $r_{ijk}(m,\alpha,\beta)$, are involved for *null* labels (in this work, label m always represents these labels), due to the impossibility of defining binary order type.

Several continuous relaxation labeling methods were applied to vision systems. The first method was proposed by Rosenfeld et al. [1] which is a non-linear scheme based on heuristic considerations. In [2] another non-linear one based on a bayesian analysis is shown, where compabilities are understood as conditional probabilities. Faugeras and Berthod [3] propose an objective function whose minimum value is reached by a projected gradient algorithm. The final result for [3] depends on the initial values of probabilities. Hummel and Zucher [4] give a definition of consistency to be satisfied by consistent labelings, and an optimization algorithm to maximize a local medium consistency function.

Here, an algorithm based on learning automata is used, that includes not only binary type compatibilities like in [12], but also tertiary type. The same analysis method of the iterative process proposed in [12] is used here to obtain the identical local convergence results. This is a proof that relaxation labeling based on learning automata is extendable to higher order compatibilities also. Afterwards, these processes are applied to the stereoscopic matching of edge points from the constraints previously defined in [14]. Although, compatibility functions can take values in a continuous range, they were intended for the stereo matching algorithms to be of the "all of none" type that express relations of each edge point with respect to its neighbours. It was possible to define constraints of that type, and an increase of speed was noticed. For this case, relaxation labeling consists of a team of automata (as many automata as objects) where each automation selects a tentative labeling according to its label probability vector. The response of the environment to each automata depends upon the consistency of the labelings chosen by the automata team. Afterwards, automata update its probabilities vectors and the cycle is repeated until the convergency is reached.

Learning-automata relaxation labeling actually uses a probabilistic iterative scheme that becomes fairly robust to noise components in the environment. In stereo vision noise is due to changes in both images, since they are acquired from different perspectives. Stochastic relaxation was also used in [8] for vision at which intensity values of pixels belonging to an image neighbourhood are related as couplings of atoms in a chemical system. Annealing processes are simulated which guarantee convergence to the global maxima of the posterior distribution. In [9] an expression of potential energy for stereo matching is proposed from the energy analysis of a spring model, and the problem is to find a disparity map with minimal energy. Although a coarse-to-fine strategy is used in [9] a large computational cost is required to reach convergency. A model based on learning automata better accepts local parallel operations.

2. Consistency and Unambiguity

The definition of consistency given above is the same as in [4]. The variables α, β, Γ and γ will be used in the sequel as notation for labels, and variables i, j and k for objects. An unambiguous labeling assignment is a function mapping from U, the set of objects, to Λ, the set of labels. This can be represented by an m-vector $\underline{P}_i$ = $(P_i(1),\ldots,P_i(m))'$ for each object i. The space of unambiguous labelings is defined by

$$K^{*}= \{\underline{P} \in \Re^{mn} / P= (\underline{P}'_1,\ldots,\underline{P}'_n)'$$

$$\underline{P}_i= (P_i(1),\ldots,P_i(m))' \in \Re^{m}$$

$$P_i(\alpha)= 0 ó 1, \forall\ \alpha,\ i;\ \sum_{\alpha=1}^{m} P_i(\alpha)= 1, \forall\ i\} \tag{1}$$

Given a vector $\underline{P}$ in K^*, the corresponding unambiguous assignment is determined exactly, i.e., the set of vectors in K* is in one-to-one correspondence with the set of mappings from objects to labels. The space K* can be extended to the space of weighted labeling assignments by

$$K = \{P \in \Re^{mn} \,/\, \underline{P} = (\underline{P}'_1,\ldots,\underline{P}'_n)'$$

$$\underline{P}_i = (P_i(1),\ldots,P_i(m))' \in \Re^{m}$$

$$0 \leq P_i(\alpha) \leq 1,\ \sum_{\alpha=1}^{m} P_i(\alpha) = 1,\ \forall\, i\} \tag{2}$$

The space K^{*} can represent ambiguous labelings, and expresses the probability values previous to convergence in the process. The label m is considered when compatibility functions $r_{ijk}(m,\alpha,\beta)$ take place, since they could not be expressed by binary order relations (cases like $r_{ijj}(m,\alpha,\alpha)$ are also considered). Compatibility function values belong to the interval [0,1] (where value one means maximum compatibility). Cases like $r_{ii}(\alpha,\alpha)$ are always zero. Symmetry always exists for the tertiary case $r_{ijk}(m, \alpha, \beta) = r_{ijk}(m,\beta,\alpha)$, but this is not true for the binary case.

Definition 1) The support for labels α and m at object i by the label assignment $\underline{P}$ is given by

$$S_i(\alpha,\underline{P}) = \sum_{j=1}^{n} \sum_{\beta=1}^{m} r_{ij}(\alpha,\beta)\ .\ P_j(\beta) \tag{3}$$

$$S_i(m,\underline{P}) = \sum_{j=1}^{n} \sum_{\beta=1}^{n-1} \sum_{k=1}^{n} \sum_{\Gamma=1}^{m-1} r_{ijk}(m,\beta,\Gamma)\ .\ P_j(\beta)\ .\ P_k(\Gamma) \tag{4}$$

Definition 2) Let us suppose $\underline{P} \in K$ is a weighted labeling assignment, then $\underline{P}$ is consistent if

$$\sum_{\alpha=1}^{m} P_i(\alpha)\ .\ S_i(\alpha,\underline{P}) \geq \sum_{\alpha=1}^{m} v_i(\alpha)\ .\ S_i(\alpha,\underline{P}),\ \ \forall\, i,\ \ \forall\, \underline{v} \in K \tag{5}$$

Consistency is strict in Eq. 5, if inequality intervenes only.

3. Learning Automaton

A learning automaton is a stochastic automaton connected in a feedback loop with a random environment. A stochastic automaton is defined by a sextuple

<I,O,Q,P,h,A>, where I, O and Q are the inputs, outputs and states alphabets of the stochastic automaton. Output function h: Q → S sets up a mapping from each state to one output, and P is the transition probabilities function IxQ → $[0,1]^n$. The function P is usually simplified by an action probability vector $\underline{P}$, where a probability value is assigned to each state. Vector $\underline{P}$ can be expressed as $\underline{P}^k=(P_1^k,\dots,P_n^k)$, where P_i^k is the election probability of the q_i state at the stage k. Values of P_i^k follow

$$0 \leq P_i^k \leq 1,\ i=1,..,n \quad \text{and} \quad \sum_{i=1}^{n} P_i^k=1 \tag{6}$$

Updating scheme A can be expressed at the stage k by $\underline{P}^{k+1} = A(\underline{P}^k,q^k,e^k)$ (e^k is the the automaton input at the stage k). A updates the probability vector for the next stage.

The operation of a learning automaton at the stage k consists of choosing an action i (i= 1,...,n, $q_i \in Q$) according to the probability vector $\underline{P}^k$. For this choice the environment responds with a random reward C_i. Then, the automaton updates the probability vector for the stage k+1, according to the updating scheme A. The cycle is repeated by choosing another action at k+1.

The environment is defined by the probabilities C_i (i =1,..,n), where each C_i is a constant value for a stationary environment. Three types of environment models exist [11], depending on the reactions of the environment. The P-model of an environment is for $I = \{0,1\}$, the Q-model is for $I = \{e_i,\dots,e_p\}$, and the S-model for $I = [0,1]$.

4. Relaxation Labeling with Learning Automata including Tertiary Compatibilities

This relaxation labeling method consists of several learning automata, one for each object, at which each automation interacts with an environment. In Fig. 1 the relaxation labeling model is represented. Automata actions are associated with the possible labels of the considered object. If it is supposed that the automation A_i has chosen label α, according to its action probability vector $\underline{P}_i$, the environment response to the automaton A_i is symbolizied as $\beta_{i\alpha}$. This variable depends on the consistency of the labeling chosen by the automata team. If label ß is chosen by A_j, the contribution of A_j to A_i is one with probability $r_{ij}(\alpha,\beta)$. If α is equal to m (or *null*) label, then compatibilities of the type $r_{ijk}(m,\beta,\Gamma)$ are involved and a second automaton A_k is considered. A_i will use a simple average of all these as the environment reward. Then, environment responses are defined by

$$\beta_{i\alpha} = 1/A(r) . \sum_{j=1}^{n} X_{ij}^{\alpha} , \quad \alpha < m \tag{7}$$

$$\beta_{im} = 1/A'(r) . \sum_{j=1}^{n} \sum_{k=1}^{n} X_{ijk}^{m} , \quad 0 \leq r < n-1 \tag{8}$$

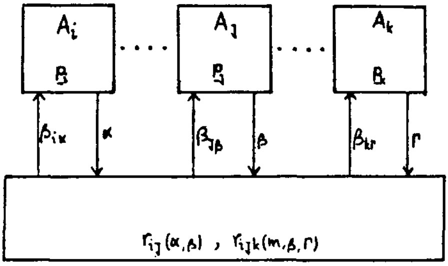


Fig. 1. Relaxation labeling with learning automata.

In Eqs. 7 and 8 X_{ij}^{α} and X_{ijk}^{m} are both random variables that can take 0 or 1 with the following distributions

$$\text{Prob } [X_{ij}^{\alpha}=1] = \sum_{\beta=1}^{m-1} r_{ij}(\alpha,\beta) . P_j(\beta) \tag{9}$$

$$\text{Prob } [X_{ijk}^{m}=1] = \sum_{\beta=1}^{m-1} \sum_{\Gamma=1}^{m-1} r_{ijk}(m,\beta,\Gamma) . P_j(\beta) . P_k(\Gamma) \tag{10}$$

A(r) and A'(r) in Eqs. 7 and 8 mean the number of binary and tertiary relations of each object with respect to the remainder of the objects. Variables A(r) and A'(r) depend on the number r of objects that have chosen label m, since X_{ij}^{α} always takes zero provided that label m has been chosen by A_j. It is the same for X_{ijk}^{m}, whether A_j and/ or A_k have chosen label m.

It is easy to see that $\beta_{i\alpha}$ can take a finite set of values belonging to the interval [0,1]. Also, the distribution of $\beta_{i\alpha}$ is time varying because it depends on $\underline{P}$. Thus, we have a nonstationary Q-model environment for each automaton.

Let us rewrite X_{ij}^{α} and X_{ijk}^{m} by the following mass functions

$$\text{Prob}\ [X_{ij}^{\alpha} = r_{ij}(\alpha,\beta)] = P_j(\beta) \tag{11}$$

$$\text{Prob}\ [X_{ijk}^{m} = r_{ijk}(m,\beta,\Gamma)] = P_j(\beta)\ .\ P_k(\Gamma) \tag{12}$$

Then, the expressions of the environment responses enter into the analysis as

$$E[\beta_{i\alpha}/\underline{P}] = E[\Phi]\ .\sum_{j=1}^{n}\ \sum_{\beta=1}^{m-1} r_{ij}(\alpha,\beta)\ .\ P_j(\beta),\ \alpha < m \tag{13}$$

$$E[\beta_{im}/\underline{P}] = E[\Psi]\ .\sum_{j,k=1}^{n}\ \sum_{\beta,\Gamma=1}^{m-1} r_{ijk}(m,\beta,\Gamma)\ .\ P_j(\beta)\ .\ P_k(\Gamma) \tag{14}$$

In Eqs. 13 and 14 the expressions of $E[\Phi]$ and $E[\Psi]$ are

$$E[\Phi] = \sum_{r=0}^{n-1} 1/A(r)\ .\ P_r[X=r]$$

$$= 1/V_n\ .\sum_{i=0}^{n} [1-P_i(m)] + \sum_{r=1}^{n-1} 1/A(r)\ .\ P_r[X=r] \tag{15}$$

$$E[\Psi] = \sum_{r=0}^{n-1} 1/A(r)\ .\ P_r[X=r]$$

$$= 1/V'_n\ .\sum_{i=0}^{n} [1-P_i(m)] + \sum_{r=1}^{n-1} 1/A(r)\ .\ P_r[X=r] \tag{16}$$

In Eqs. 15 and 16, $P_r[X=r]$ indicates the probability of r objects choosing label m from the total number V_n (V'_n represents in Eq. 16 the total number for binary relations). $P_r[X=r\}$ depends only on $P_i(m)$, $i=1,\ldots,n$. A classical updating scheme in learning automata theory, called Linear Reward Inaction (denoted by L_{R-I}), is used in this case as the updating scheme. If the automaton A_i has chosen the label α at the kth stage, updating of probabilities at the next stage from the environment response $\beta_{i\alpha}(k)$ is

$$P_{i\alpha}(k+1) = P_{i\alpha}(k) + a.\ [1 - P_{i\alpha}(k)]\ .\ \beta_{i\alpha}(k) \tag{17}$$

$$P_{i\beta}(k+1) = P_{i\beta}(k) - a\ .\ P_{i\beta}(k)\ .\ \beta_{i\alpha}(k),\quad \beta \neq \alpha \tag{18}$$

5. Analysis of the Proposed Relaxation Process

Let us express the ordinary differential equation ODE of the proposed process as

$$\Delta \underline{P}(k) = E[\underline{P}(k+1)-\underline{P}(k)/\underline{P}(k)]. \tag{19}$$

Let

$$\Delta \underline{P}(k) = (\Delta \underline{P}_1(k),\ldots,\Delta \underline{P}_n(k)) \tag{20}$$

and

$$\Delta \underline{P}_i(k) = (\Delta P_{i1}(k),\ldots,\Delta P_{im}(k)) \tag{21}$$

Since $\underline{P}_i(k)$ is a Markovian process, the ODE in Eq. 19 can be expressed as $\dot{\underline{P}} = \underline{f}(\underline{P})$. Now, using Eqs. 11-16, we get the expression of each element in Eq. 21

$$\Delta P_{i\alpha}(k) = E[\Phi] \,.\, a P_{i\alpha} \,.\, \{\sum_{\beta=\alpha}^{m-1} P_{i\beta}.[\sum_{j=1}^{n} \sum_{\Gamma=1}^{m-1} \{r_{ij}(\alpha,\Gamma)-r_{ij}(\beta,\Gamma)\}.\, P_{j\Gamma}]$$

$$+ a P_{i\alpha} P_{im} \,.\, [\sum_{j=1}^{n} \sum_{\Gamma=1}^{m-1} \{E[\Phi] \,.\, r_{ij}(\alpha,\Gamma) - \sum_{k=1}^{n} \sum_{\gamma=1}^{m-1} E[\Psi] \,.\, r_{ijk}(m,\Gamma,\gamma)$$

$$.\, P_{k\gamma}\}.\, P_{j\Gamma}]\} = a f_{i\alpha}(\underline{P}),\ \alpha \neq m \tag{22}$$

$$\Delta P_{im}(k) = a P_{im} \,.\, \{\sum_{\beta=1}^{m-1} P_{i\beta} \,.\, [\sum_{j=1}^{n} \sum_{\Gamma=1}^{m-1} \{\sum_{k=1}^{n} \sum_{\gamma=1}^{m-1} E[\Psi] \,.\, r_{ijk}(\alpha,\Gamma,\gamma) \,.\, P_{k\gamma}$$

$$- E[\Phi] \,.\, r_{ij}(\beta,\Gamma)\} \,.\, P_{j\Gamma}]\} = a f_{im}(\underline{P}) \tag{23}$$

Let us suppose that we are considering a labeling with unambiguous and strictly consistent solutions of the type $\underline{P}^0=(e'_{\alpha 1},\ldots,e'_{\alpha l},e'_m,\ldots,e'_m)'$, where each $e'_{\alpha i}$ $(i=1,\ldots,l)$ is an m-dimensional unit vector with the α_i-th unity component. Vectors of $\underline{P}^0$ from l+1 $(1<l\leq n)$ to n have the mth unity component (there is no special significance for considering these elements the last components). Vectors of the type $\underline{P}^0$ belong to the K* space (Eq. 1), since they represent the unambiguous labelings. Vectors like $\underline{P}^0$ will be referred to subsequently as *corners*. The first step for the analysis of the ODE in Eq. 19 is the study of the zeros of f (Eqs. 22-23), and then to investigate if solutions like $\underline{P}^0$ are also zeros of $\underline{f}$. This is demonstrated by the following lemma (its proof appears in Appendix A).

Lemma 1

All the solutions of $\underline{P}^0$ are zeros of $\underline{f}$. Others zeros $\underline{P}$ of $\underline{f}$ satisfy

If $P_{im} = 0$, then

$$\sum_{j=1}^{n} \sum_{\beta=1}^{m-1} \{r_{ij}(1,\beta) - r_{ij}(\alpha,\beta)\} \cdot P_{j\beta}=0, \quad \alpha < m, P_{i\alpha} \neq 0 \tag{24}$$

and

If $P_{im} \neq 0$, besides Eq. 24 they also satisfy

$$\sum_{j=1}^{n} \sum_{\beta=1}^{m-1} \quad [E[\Phi] \cdot r_{ij}(1,\beta) - \{\sum_{j=1}^{n} \sum_{\beta=1}^{m-1} E[\Psi] \cdot r_{ijk}(m,\beta,\Gamma) \cdot P_{k\Gamma}\}] \cdot P_{j\beta} = 0 \tag{25}$$

Consider the ordinary differential equation (ODE) given by

$$\dot{\underline{P}} = \underline{f}(\underline{P}) \tag{26}$$

From Lemma 1 we know all the stationary points of Eq. 26. The following lemma is concerned with its stability properties (its proof appears in Appendix B). The stability analysis of a *corner* vector $\underline{P}^0=(e'_{i1},\ldots,e'_{il},e'_{m},\ldots,e'_{m})'$ is made in a local zone of $\underline{P}^0$, as the function $\underline{f}(\underline{P})$ is non-linear. The function $\underline{f}(\underline{P})$ is approximated in the local zone of $\underline{P}^0$ by a linear function in order to analyse the stability of the linear function. The analysis method used in this case is by Lyapunov.

Lemma 2

1) If the vector $\underline{P}$ is a *corner* which represents a strictly consistent labeling, then it is an asymptotically stable stationary point of Eq. 26.

2) Each *corner* that is a stable stationary point of Eq. 26 represents a consistent labeling.

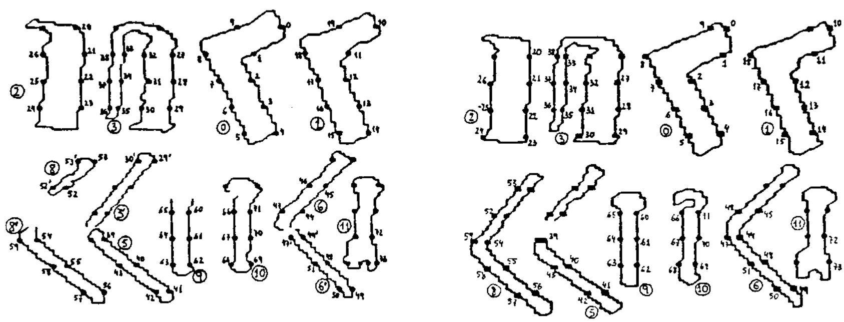
3) Each nonconsistent *corner* and each interior zero of $\underline{f}$ (Eq. 26) is an unstable stationary point of Eq. 26.

Now that the solutions to Eq. 26 are well characterized, the next step in the analysis is to show that the algorithm converges to the solution of Eq. 26. It was proved in [12] that the sequence of interpolated processes $\{\underline{P}^a(.)\}$ converges weakly to the process

$\underline{X}$ (.), which satisfies the ODE $\dot{\underline{X}} = \underline{f}(X)$, as $\underline{P}^a(0) = \underline{X}(0)$ and $a \to 0$. The same proof given in [12] has application for the process proposed here, since the environment responses $\beta_{i\alpha}$ (α < m) and β_{im} have the same properties as in [12] to follow this Theorem.

6. Experimental Results

The so-called block scenes have been used in the experiments. The images were taken by applying a linear shift to the cameras and the digitized images were processed by a spatial resolution of 128x128 pixels. For the first stereoscopic pair, we have included in the scene objects of repeated shapes, with the purpose of testing the way in which the algorithms solve the ambiguity problem. In Fig. 2 it is shown results of stereo matching for edge points and edges. Correspondence for edge points is pointed out by the same number in both images of Fig. 2, whereas for edges it is pointed out by the same number included in a cycle.



(a) (b)

Fig. 2. Matching of edge points and edges of images. (a) Object points and object edges. (b) Label points and label edges.

If there appears wrong matching (like those corresponding to the object edges 3, 3' and 8 in Fig. 2) it is due to the fact that the characteristics obtained from these points differ from those of their equivalent points 3' and 8 is the edge break. Points in edge 11 appear without correspondence. This last problem has rather no significance, since labeling algorithms get the *null* label when the consistency value is small. Tables 1 and 2 show

0 (1.00)	3 (1.00)	6 (0.75)	8'(1.00)	11 (1.00)
1 (1.00)	3'(1.00)	6'(1.00)	9 (1.00)	
2 (1.00)	5 (0.75)	8 (1.00)	10 (0.88)	

Table 1. Correspondences in one per cent between edges shown in Fig. 2.

the average in one per cent of correspondences for edge points and edges, when relaxation processes were run eight times.

0 (1.00)	17 (1.00)	32 (0.88)	48 (1.00)	63 (1.00)
1 (1.00)	18 (1.00)	33 (0.63)	49 (1.00)	64 (1.00)
2 (1.00)	19 (1.00)	34 (0.63)	50 (1.00)	65 (0.88)
3 (1.00)	20 (0.50)	35 (0.63)	51 (1.00)	66 (1.00)
4 (1.00)	21 (1.00)	36 (0.63)	52 (0.50)	67 (1.00)
5 (1.00)	22 (1.00)	37 (0.63)	52' (0.50)	68 (1.00)
6 (1.00)	23 (1.00)	38 (0.50)	53 (0.50)	69 (1.00)
7 (1.00)	24 (1.00)	39 (0.75)	53' (0.50)	70 (1.00)
8 (1.00)	25 (1.00)	40 (0.75)	54 (0.63)	71 (0.88)
9 (0.88)	26 (1.00)	41 (0.75)	55 (0.63)	72 (0.38)
10 (1.00)	27 (0.75)	42 (0.63)	56 (0.50)	73 (0.38)
11 (1.00)	28 (0.88)	43 (0.75)	57 (0.63)	
12 (1.00)	29 (0.88)	44 (0.88)	58 (0.63)	
13 (1.00)	29' (1.00)	45 (0.63)	59 (0.63)	
14 (1.00)	30 (0.88)	46 (0.50)	60 (1.00)	
15 (1.00)	30' (1.00)	47 (0.75)	61 (1.00)	
16 (1.00)	31 (0.88)	47' (1.00)	62 (1.00)	

Table 2. Correspondences in one per cent between edge points shown in Fig. 2.

A second stereoscopic pair is shown with the purpose of testing the way in which the algorithms detect occlusion. Note that total occlusion is detected in all the cases as it is shown in Fig. 3. Tables 3 and 4 show the average in one per cent of correspondences for edge points and edges, when relaxation processes were also run eight times.

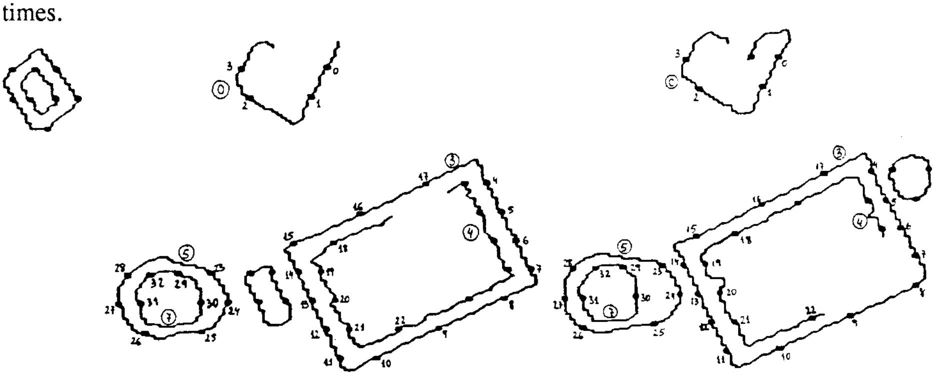


Fig. 3. Matching of edge points and edges of images. (a) Object points and object edges. (b) Label points and label edges.

7. Conclusions

A relaxation algorithm based on learning automata has been proposed. Automata interact with an environment which includes besides binary order compatibilities also tertiary order. After the analysis of the algorithm, we get the same conclusions as the environment that included only compatibilities of binary type. Then, this relaxation algorithm was applied to obtain stereoscopic matching. Deductions made in the process analysis were confirmed by the good experimental results that were obtained. The type of environment used was a P-model (responses all or none), not a Q-model, in order to decrease the interactions number that reached the convergence.

0 (1.00)	3 (1.00)	4 (1.00)	5 (1.00)	7 (1.00)

Table 3. Correspondences in one per cent between edges shown in Fig. 3.

0 (1.00)	8 (1.00)	17 (1.00)	25 (1.00)
1 (1.00)	9 (1.00)	18 (1.00)	26 (1.00)
2 (1.00)	10 (1.00)	19 (1.00)	27 (1.00)
3 (1.00)	12 (1.00)	20 (1.00)	28 (1.00)
4 (1.00)	13 (1.00)	21 (0.88)	29 (0.63)
5 (1.00)	14 (1.00)	22 (1.00)	30 (1.00)
6 (1.00)	15 (1.00)	23 (0.63)	31 (1.00)
7 (1.00)	16 (1.00)	24 (1.00)	32 (0.88)

Table 4. Correspondences in one per cent between edge points shown in Fig. 3.

Appendix A

By inspection from Eqs. 22 and 23 it is obvious that for a *corner* $\underline{P}^0$, $f_{i\alpha}(\underline{P}^0) = 0$, $\alpha \neq m$ and $f_{im}(\underline{P}^0) = 0$. In the case of $P_{im} \neq 0$ we will analyse other zeros of $\underline{f}$. The following expressions are defined, since they allow more simplification after

$$C_{i\alpha} = E[\Phi] \, . \sum_{j=1}^{n} \sum_{\beta=1}^{m-1} r_{ij}(\alpha,\beta) \, . \, P_{j\beta} \tag{27}$$

and

$$g_{i\alpha} = \sum_{\beta \neq \alpha}^{m-1} P_{i\beta} \, . \, [\sum_{j=1}^{n} \sum_{\Gamma=1}^{m-1} \{E[\Psi] \, . \, r_{ij}(\alpha,\Gamma) - E[\Phi] \, . \, r_{ij}(\beta,\Gamma)\} \, . \, P_{j\Gamma}]$$

$$= \sum_{\beta \neq \alpha}^{m-1} P_{i\beta} \,.\, [C_{i\alpha}-C_{i\beta}] \tag{28}$$

Let us suppose that for the first 1 labels ($1 \leq m-1$) probability is distinct of zero (there is no significance considering $P_{i\alpha} \neq 0$, for $\alpha \neq 1$), and the rest of the probabilities are zeroes. Then, we must have $g_{i\alpha} = 0$, $\alpha \leq 1$. Operating with $g_{i1} - g_{i\beta}$ in Eqs. 27 and 28, it follows

$$\sum_{\alpha=1}^{1} P_{i\alpha} = 1, \; i=1, \ldots ,n \tag{29}$$

We notice

$$g_{i1} - g_{i\alpha} = C_{i1} - C_{i\alpha} \tag{30}$$

Let us consider Eqs. 22 and 23. From Eq. 30 we see that the zeros of $\underline{f}$ in this case follow the lemma.

The next step is the analysis of $\underline{f}$ zeros when $P_{im} \neq 0$, $P_{i\alpha} \neq 0$, $\alpha \leq 1$ ($1 \leq m-1$) and the remainder of the probabilities are zero. Besides Eqs. 27 and 28, other functions in this case are defined

$$C_{im} = E[\Psi] \,.\, \sum_{j=1}^{n} \sum_{\beta=1}^{m-1} \sum_{k=1}^{n} \sum_{\Gamma=1}^{m-1} r_{ijk}(m,\beta,\Gamma) \,.\, P_{j\beta} \,.\, P_{k\Gamma} \tag{31}$$

and

$$g_{im} = \sum_{\beta=1}^{m-1} P_{i\beta} \,.\, [\sum_{j=1}^{n} \sum_{\Gamma=1}^{m-1} \sum_{k=1}^{n} \sum_{\gamma=1}^{m-1} \{E[\Psi] \,.\, r_{ijk}(m,\Gamma,\gamma) P_{k\gamma} - E[\Phi]$$

$$.\, r_{ij}(\alpha,\Gamma)\} \,.\, P_{j\Gamma}] = \sum_{\beta=\alpha}^{m-1} P_{i\beta} \,.\, [C_{im}-C_{i\beta}] \tag{32}$$

From Eqs. 31 and 32, we get

$$g_{i\alpha} + P_{im} [C_{i\alpha} - C_{im}] = 0 \tag{33}$$

and

$$g_{im} = 0,\ \alpha \leq 1,\ \alpha \neq m \tag{34}$$

Let us operate the following expressions

$$g_{i1} + P_{im}\,[C_{i1}-C_{im}] - g_{i2} - P_{im}\,[C_{i2}-C_{im}]$$

$$\vdots$$

$$\begin{aligned} &g_{i1} + P_{m}\,[C_{i1}-C_{im}] - g_{il} - P_{im}\,[C_{il}-C_{im}] \\ &g_{i1} + P_{im}\,[C_{i1}-C_{im}] - g_{im} \end{aligned} \tag{35}$$

If each expression in Eq. 35 is equalized to zero, we infer

$$C_{i1} - C_{i\alpha} = 0,\ \ \forall\, \alpha \leq 1 \tag{36}$$

and

$$C_{i1} - C_{im} = 0 \tag{37}$$

From Eq. 37, and considering Eqs. 22 and 23, we see that the zeros of f satisfy the lemma for this case.

Remark 1): Lemma 1 is general, although label 1 has been considered. Let us suppose label ß with $P_{i\beta} \neq 0$ (and $\beta \leq 1$), then we have

$$\begin{aligned} C_{i1}-C_{i\alpha} &= C_{i1}-C_{i\beta} + C_{i\beta}-C_{i\alpha} = C_{i\beta}-C_{i\alpha} \\ C_{i1}-C_{im} &= C_{i1}-C_{i\beta} + C_{i\beta}-C_{im} = C_{i\beta}-C_{im} \end{aligned} \tag{38}$$

Equalities obtained from the above lemma are transformed into others at which label β is concerned instead of label 1, since $C_{i1}-C_{i\beta}=0$ in Lemma 1.

Appendix B

We notice that the function f(P) in Eqs. 22 and 23 is non-linear. Therefore, the stability analysis of a *corner* $\underline{P}^0=(e'_{\alpha1},\ldots,e'_{\alpha l},e'_{m},\ldots,e'_{m})'$ (where, e'_{ti}, $i=1,\ldots,1$, $t_i \leq m-1$ is a unit vector) will be in a local zone of $\underline{P}^0$. The function $\underline{f}(\underline{P})$ is approximated by a linear one in the local zone of $\underline{P}^0$ in order to analyse the stability of the linear function. After applying the linear approximation of $\underline{f}(\underline{P})$ the components of $\underline{f}(\underline{P})$ have the following expressions

$$f_{i\alpha} = 1/V_n \cdot P_{i\alpha} \sum_{j=1}^{l} \{ r_{ij}(\alpha,\alpha_{tj}) - r_{ij}(\alpha_{ti},\alpha_{tj}) \} + \text{T.O.S.},$$

$$\alpha \neq t_i \ (\alpha < m) \qquad (39)$$

$$f_{im} = P_{im} \cdot \sum_{j=1}^{l} \Big[\sum_{k=1}^{l} \{r_{ijk}(m,\alpha_{tj},\alpha_{tk})/V'_n\} - r_{ij}(\alpha_{ti},\alpha_{tj})/V_n \Big]$$

$$+ \text{T.O.S.} \qquad (40)$$

After applying the linear approximation of $\underline{P}^0$, other unit vectors of $\underline{P}^0$, e'_m, have the following expression

$$f_{i\alpha} = P_{i\alpha} \cdot \sum_{j=1}^{l} [r_{ij}(\alpha,\alpha_{tj})/V_n - \sum_{k=1}^{l} \{r_{ijk}(m,\alpha_{tj},\alpha_{tk})/V'_n\}] + \text{T.O.S.} \qquad (41)$$

The following transformation allows us to change the origin to the vector $\underline{P}^0$

$$\tilde{P}_{i\alpha} = P_{i\alpha},\ \alpha \neq t_i$$
$$= 1 - P_{iti},\ \alpha = t_i,\ i = 1,\ldots,l \qquad (42)$$

$$\tilde{P}_{i\alpha} = P_{i\alpha},\ \alpha \neq m$$
$$= 1 - P_{im},\ \alpha = m,\ i = l+1,\ldots,n \qquad (43)$$

After applying the transformation of Eqs. 42 and 43, the components of $\underline{f}$ have the following expressions

$$f_{i\alpha} = 1/V_n \cdot \tilde{P}_{i\alpha} \sum_{j=1}^{l} [r_{ij}(\alpha,\alpha_{tj}) - r_{ij}(\alpha_{ti},\alpha_{tj})] + \text{T.O.S.},\ \alpha < m \qquad (44)$$

$$f_{im} = \tilde{P}_{im} \cdot \sum_{j=1}^{l} \Big[\sum_{k=1}^{l} \{r_{ijk}(m,\alpha_{tj},\alpha_{tk})/V'_n\} - r_{ij}(\alpha_{ti},\alpha_{tj})/V_n \Big] + \text{T.O.S.},$$

$$i = 1,\ldots,l \qquad (45)$$

$$f_{i\alpha} = \tilde{P}_{i\alpha} \cdot \sum_{k=1}^{l} [r_{ij}(\alpha,\alpha_{tj})/V_n - \sum_{j=1}^{l} \{r_{ijk}(m,\alpha_{tj},\alpha_{tk})/V'_n\}] + \text{T.O.S.},$$

$$\forall\ \alpha,\ i = l+1,\ldots,n \qquad (46)$$

1) Let us define the positive definite function $\forall \underset{\sim}{\tilde{P}} \in K$, $\underset{\sim}{\tilde{P}} \neq \underset{\sim}{\tilde{P}}^0$

$$U_1(\underset{\sim}{\tilde{P}}) = \sum_{i=1}^{n} [\sum_{\alpha=1}^{m-1} \tilde{P}_{i\alpha} + \tilde{P}_{im}] \tag{47}$$

Derivative function of $U_1(\underset{\sim}{\tilde{P}}^o)$ in Eq. 47 from the linear function has the expression

$$\dot{U}_1(\underset{\sim}{\tilde{P}}) = \text{diag } (a_{i\beta},a_{im}) \text{ x diag } (P_{i\alpha},P_{im}),\ \alpha < m \tag{48}$$

where

$$a_{i\alpha} = 1/V_n \sum_{j=1}^{l} [r_{ij}(\alpha,\alpha_{tj}) - r_{ij}(\alpha_{ti},\alpha_{tj})],\ \alpha < m \tag{49}$$

$$a_{im} = \sum_{j=1}^{l} [\sum_{k=1}^{l} \{r_{ijk}(m,\alpha_{tj},\alpha_{tk})/V'_n\} - r_{ij}(\alpha_{ti},\alpha_{tj})/V_n]$$

$$i = 1,\ldots,l \tag{50}$$

$$a_{i\alpha} = \sum_{k=1}^{l} [r_{ij}(\alpha,\alpha_{tj})/V_n - \sum_{j=1}^{l} r_{ijk}(m,\alpha_{tj},\alpha_{tk})/V'_n],\ i = l+1,\ldots,n \tag{51}$$

As $\underset{\sim}{\tilde{P}}$ is a strictly consistent labeling, we infer $a_{i\alpha} < 0$, $\forall\ \alpha$, $1=1, \ldots ,n$. Then, $\dot{U}_1(\underset{\sim}{\tilde{P}})$ is negative definite except in the origin.

2) The EDO $\dot{\underset{\sim}{\tilde{P}}} = \underset{\sim}{f}(\underset{\sim}{\tilde{P}})$ can be expressed as $\underset{\sim}{\tilde{P}} = A\underset{\sim}{\tilde{P}}$, where a = diag($a_{i\alpha}$). If $\underset{\sim}{\tilde{P}}$ is a stable stationary point, the eigenvalues have nonpositive real parts and hence $a_{i\alpha} < 0$, $\forall\ \alpha$, $1=1, \ldots ,n$. This implies $\underset{\sim}{\tilde{P}}$ is consistent.

3) Let us suppose that $\underset{\sim}{P}^0$ is a nonconsistent *corner*. Then, at least for one unit vector $e'_{\alpha ti}$ (or e'_m) of $\underset{\sim}{P}^0$, say e', we must have

$$\sum_{j=1}^{l} [r_{ij}(\alpha,\alpha_{tj}) - r_{ij}(\alpha_{ti},\alpha_{tj})] > 0,\ i = 1,\ldots,l \tag{52}$$

$$\sum_{j=1}^{l} [\sum_{k=1}^{l} \{r_{ijk}(m,\alpha_{tj},\alpha_{tk})/V'_n\} - r_{ij}(\alpha_{ti},\alpha_{tj})/V_n] > 0,$$

$$i = l+1,\ldots,n \tag{53}$$

Consider the positive definite function (except in $\underline{P}^{0}$)

$$U_2(\tilde{\underline{P}}) = P_{i\alpha i}\ ,\ i= 1,\ldots,l$$

$$= P_{i\alpha m},\ i=l+1,\ldots,n \qquad (54)$$

Then

$$\dot{U}_2(\tilde{\underline{P}}) = f_{i\alpha i}\ ,\ i= 1,\ldots,l$$

$$= f_{i\alpha m}\ ,\ i=l+1,\ldots,n \qquad (55)$$

Eq. 55 is unstable in $\underline{P}^0$ as Lyapunov, since $\dot{U}_2(\tilde{\underline{P}})$ is positive except in $\underline{P}^0$ (Eqs. 44-46).

Consider $\bar{\underline{P}}=(\bar{\underline{P}}'_1, \ldots ,\bar{\underline{P}}'_n)'$ as an interior zero of $\underline{f}$ with $P_{i\alpha} \neq 0$, and the following transformation of $\bar{\underline{P}}$ to the origin

$$\tilde{P}_{i\alpha} = P_{i\alpha} - \bar{P}_{i\alpha}\ ,\quad \alpha < m$$

$$\tilde{P}_{im} = P_{im} - \bar{P}_{im}\ ,\ i = 1,\ldots,n \qquad (56)$$

We can express the linear transformation of Eq. 56 as $\dot{\tilde{\underline{P}}}=A\tilde{\underline{P}}$, where all the diagonal elements of A are zero. Hence, at least one eigenvalue of A must have a positive real part and P is unstable. Even the rare case of all eigenvalues being purely imaginary can be excluded if A is made to be odd by adding a few extra objects or labels.

Remark 2): As non-linear terms in Eqs. 22 and 23 were not considered in detail, necessary and sufficient conditions are not inferred for asymptotic stability which has been until now a difficult problem.

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