Chapter 1

Introduction

1.1 Geometry and Physics

The use of geometrical techniques in modelling physical problems goes back over two thousand years to the Greek civilisation. But for these people physics was a prisoner of geometry (not to mention of egocentricity) and, in particular, of the circle. So, for example, whilst the Ptolemaic geocentric model of the solar system was a powerful model for its time, the geometrical input was not only restricted by the knowledge and prejudices of the period, but provided only an arena in which physics took place. In this sense geometry was not part of the physics but merely a mathematical convenience for its description. In the scientific renaissance of the 16th century onwards the Greek dominance and prejudices were gradually eroded but even Copernicus still felt the need of the beloved circle and his heliocentric theory still needed epicycles. Not until Kepler came upon the scene was the circle finally dethroned when, by the introduction of eliptical orbits, he was able to explain all, and more, than Copernicus could, without the need of epicycles.

The Keplerian system, precisely stated, turns out to be equivalent to a model of the solar system of a Newtonian type based on the inverse square law attraction of a central force. This was one of the most outstanding achievements of Newtonian physics and the Kepler-Newton system has been the standard theory of the solar system ever since (the modifications arising from general relativity theory, although important theoretically, for example in the solving of the Mercury orbital problem, are minor in practice). But even in Newtonian mechanics the geometry of Euclid was still dominant and provided the unquestioned background in which all else took place.

The geometrical description of physics alluded to above has, however, a formal beauty in that it is perhaps to be considered remarkable that the Greek description of, for example, planetary events using one of the most important constructions in Euclidean geometry, the circle, gives a reasonably accurate picture of the solar system. The other important object in Euclid's geometry, the straight line, can also be thought of as having a physical context in the following simple way. Consider the Euclidean plane $\mathbb{R}^2$ with its usual system of "straight" lines etc. Now let f be a bijective continuous map $f : \mathbb{R}^2 \to \mathbb{R}^2$ (although continuity hardly matters here) and consider the set $\mathbb{R}^2$ now with the straight line structure understood to be the images of the original straight lines under f. Thus the straight lines in the new system are the subsets of $\mathbb{R}^2$ of the form $f(L)$ for all straight lines L in the original system. This set with this new linear structure formed by taking across from the original in an obvious way the concepts of length, angle etc under f is mathematically indistinguishable from the original and constitutes a perfectly good model of Euclidean geometry. However, one can distinguish them physically by, for example, regarding the copy of $\mathbb{R}^2$ as a horizontal surface and appealing to Galileo's law of inertia for free particles traversing this surface or to the paths of light beams crossing it (or, quite simply, to the lines of shortest distance measured by taut pieces of string held on the surface) to determine "straight lines". Each of these would presumably single out, at least in a local sense,the original copy of Euclid's geometry.

These examples may be considered early references to the interplay between geometry and physics but their true significance was presumably not fully realised until Hilbert set Euclidean geometry in proper perspective with his axiomatisation of this system (Gauss, Bolyai and Lobachevski having earlier confirmed the existence of alternative geometries). In this sense, these mathematicians played important roles in the understanding of physics.

Another example of the role of geometry in physics comes from the development in the 18th and 19th centuries of analytical mechanics and Lagrangian techniques. The work of D'Alembert, Euler, Lagrange, Hamilton and others cast the formulation of Newtonian mechanics into the geometry of the configuration space (or extended configuration space). From this, the calculus of variations, and the work of Riemann on differential geometry, Newtonian theory was in some sense "geometrised" by reformulating it in a setting of the Riemannian type. As a simple example, consider an n-particle system under a time independent conservative force described in the usual $3n$-dimensional Euclidean (configuration) space. The Euclidean nature (metric) of this space can be thought of as derivable from the kinetic energy of the unconstrained system in a standard way. Suppose now that a holonomic constraint is imposed on the system by restricting the particle to move on some subspace (submanifold) of the original Euclidean space. One could attempt the solve this problem by working in the submanifold of constraint. In doing so, this submanifold inherits a metric from the original Euclidean metric and which, in general, is no longer Euclidean but is rather a "curved metric" of the general type envisaged by Riemann. An elegant finale to the problem was then given by Routh's reduced Lagrangian (Routhian). The problem has in a sense been geometrised (and the constraint removed). Also the theory assumes a "generally covariant" form since the original inertial frame structure has been replaced by "generalised coordinates". A similar geometrisation occurs when one rewrites Maxwell's equations in generally covariant form by changing the usual Maxwell equations, written in an inertial frame in special relativity, by replacing the Minkowski metric that arises there by some general metric and partial derivatives by covariant derivatives.

However, there is a criticism of the claim that this procedure has progressed further with the geometrising of the problem than the earlier examples discussed above. The geometry in these examples really entered the proceedings with the imposition of the original Euclidean structure (and, in the first example, before the constraints were introduced). Thus the geometrical structure was on the arena initially and is again merely a convenience (albeit a useful one) for the solution of the problem. Similar comments apply to the more sophisticated geometrical approaches to mechanics inaugurated by Jacobi and Cartan (the latter albeit after the advent of Einstein's general relativity). In this sense, the geometry was given a-priori and was not subject to any restrictions ("field equations"). Thus it invites the criticism that it can influence the physics without any reciprocal influence on itself and was subject to unfavourable philosophical scrutiny by amongst others, Berkeley [1] and Mach [2]. Also the introduction of "absolute" variables such as the background Euclidean metric in classical mechanics and Maxwell theory (essentially Newton's absolute space) seems to suggest that making a theory "covariant" is a rather trivial matter. One more or less writes down the theory in its original "Euclidean" form and then allows everything to transform as a tensor when moving to some other coordinate system. This problem was apparently first recognised by Kretchmann [3] and elaborated on by Anderson [4] and Trautmann [5]).

The advantage of general relativity is that it contains the space-time metric as a "dynamical" variable (to be determined by solving field equations) and that it contains no such "absolute" variables (except one must, perhaps, concede that the imposition of zero torsion on the Levi-Civita connection derived from the space-time metric amounts to making the torsion an absolute variable in this theory [5]. In this stronger sense general relativity is "generally covariant". On the other hand, Newtonian classical theory has an absolute space-time splitting into absolute space and absolute time together with privileged (inertial) observers and, in addition, a privileged frame in which the ether is at rest if classical electromagnetic theory is to be accomodated within it. Special relativity has an absolute space-time metric and privileged inertial observers whereas, for example, the "bimetric" and "tetrad" variants of Einstein's theory have absolute variables in the form of a flat metric [6] or a privileged tetrad system [7], respectively. Classical theory with its concept of force has to be able to distinguish between "real" forces (*e.g.* the gravitational attraction of one body on another) and the so called "fictitious" (accelerative) forces which arise in non-inertial frames. The ability to distinguish between these types of force is essentially the ability to distinguish between inertial and non-inertial frames. Newton's theory claims each of these abilities (and thus introduces absolute variables) and Einstein's claims neither. Thus there are no a priori distinguished reference frames (coordinate systems) in general relativity (although there are, of course, convenient ones!) and no concept of force (beyond that required in order to describe situations in relativity theory using Newtonian language!) Such remarks as these go under the name of the principal of covariance and its role in the foundations of general relativity is by no means agreed upon. A related (and also contentious) issue is the principle of equivalence. In its strong form it advocates the (local) indistinguishability of a gravitational field and an appropriately chosen acceleration field. In its weaker form it simply reiterates the results of the Eotvos type experiments thus effectively saying that there is a unique symmetric connection in space-time for determining space-time paths (this latter statement leaving room for curvature coupling terms). These topics will not be discussed any further here except to say that they can be used to suggest that a theory of gravitation could be based on a 4-dimensional manifold and whose dynamical variable is a metric of Lorentz signature which is determined by field equations of a tensorial nature. Such a theory is Einstein' general relativity and it will be accepted, henceforth, without question.

Einstein's general theory of relativity is the most successful theory of the gravitational field so far proposed. It describes the gravitational field by a mathematical object called a space-time and is formalised as a pair (M, g) where M is a certain type of 4-dimensional manifold (see chapter 7) and g is a Lorentz metric on M. The metric g together with its connection and curvature "represent" the gravitational field. The restrictions on g are the Einstein field equations (together with boundary and other initial conditions) and are ten second order partial differential equations for g. Since the theory was first published in 1916, it has progressed through early uncertain beginnings, when it played second fiddle to quantum theory and suffered from the "cosmological time problem" (until the latter's correction), to a renaissance in the last fifty years. The problem of the lack of exact solutions to Einstein's field equations has, to some extent, been overcome. In addition, the theory has been put on a much sounder footing both mathematically and physically.

1.2 Preview of Future Chapters

This book is not a text book on general relativity. There are numerous excellent such texts available and thus no point in further duplication. Rather, it concentrates first on the topics of the connection and curvature structure of space-times and then, later, on the various symmetries that are commonly studied in Einstein's theory. In spite of this, an attempt will be made to make the book in some restricted sense, self-contained and so the chapters on certain prerequisite branches of mathematics will sometimes contain a little more than is strictly demanded by the remainder of the text. But the excess over absolute necessity will be restricted to that required for sensible self-containment. The reader will be assumed familiar with basic mathematics although, as stated below, algebra, topology, manifold theory and Lie groups will be treated *ab-initio*. However, although general relativity will be formally introduced, some basic familiarity with it will inevitably have to be assumed. This will mainly take the form of assuming that the reader has some knowledge of the better known exact solutions of Einstein's equations such as the Schwarzschild, Riessner-Nordstrom, Friedmann-Robertson-Walker and plane wave metrics. These metrics can be found treated in detail in many places and references will be given when appropriate. Also, some elementary knowledge of the symmetries (Killing vector fields) they possess will be desirable but not necessary.

Chapter 2 is a review of elementary group theory, linear algebra and the Jordan canonical form. The classification of matrices using Jordan-Segre theory is rather useful in general relativity and is treated in some detail. It is used in several different forms in chapter 7. Chapter 3 gives, along similar lines, a summary of elementary topology. This latter topic is still, unfortunately, used rather sparingly in certain branches of relativity theory. It is used in this text only in a somewhat primitive way but the advantages gained seem, at least to the author, to make it worthwhile. It consists of mainly point set topology together with a brief discussion of the fundamental group and the "rank" theorem. An attempt is made to ensure that these briefest of introductions are sufficient not only for that which is required later but also for some basic understanding of the subject matter.

Chapter 4 is a lengthy chapter on manifold theory. This starts, not surprisingly, with the definition of a manifold and its topology. Then the various mathematical objects required for the study of general relativity are introduced such as the tensor bundles and tensor fields, vector fields and their integral curves, submanifolds, distributions and metrics together with their associated Levi-Civita connections and curvature and Weyl tensors. Although this chapter will introduce the "coordinate free" approach (and this will be used occasionally in the text where convenient) the use of coordinate expressions will be exploited where appropriate. In this respect, the important thing is to recognise when a coordinate-free (or component) expression is unnecessarily clumsy. But this, in many cases, merely reflects personal preference. The calculations in this text are mostly in coordinate notation. In fact, on occasions one will find both used in the same calculation if, for some reason, this led to an economy of expression. This chapter also contains a discussion of some of the troublesome properties of certain types of submanifolds and which is needed in later chapters.

In chapter 5 the concept of a Lie group and its Lie algebra is introduced. The main aims here are firstly to introduce notation for the idea of a (local and global) transformation group and the associated Lie algebras of vector fields on a manifold and secondly to prepare the way for the discussion of the Lorentz group in chapter 6. In chapter 5 a discussion is given of Palais' theorem on the necessary and sufficient conditions for a Lie algebra of vector fields on a manifold to be regarded as arising from a global Lie group action. Such a global action is usually assumed in the literature without justification. In chapter 6 an attempt is made to promote the usefulness of a reasonable knowledge of the Lorentz group beyond that of merely something which occurs in special relativity. This group is investigated both algebraically and also as a Lie group. Its (connected) Lie subgroups are listed and their properties derived.

In chapter 7, general relativity is finally introduced. Here a brief summary of the properties of a space-time are laid down and Einstein's equations are given. The energy-momentum tensors for the "standard" gravitational fields encountered in general relativity are described. There then follows sections on algebraic classification theory on a space-time. Thus the classification of bivectors, the Petrov classification of the Weyl tensor and the classification of second order symmetric tensors (usually the energy-momentum tensor) are described in some detail. Heavy use will be made of them in later chapters. Here the Jordan-Segre theory developed in chapter 2 is justified. The chapter concludes with some comments on the topological decomposition of a space-time with respect to the algebraic types of the Weyl and energy-momentum tensors and the local and global nature of these classification schemes are described.

Chapter 8 is on holonomy theory. Its introduction can be regarded as two-fold. Firstly, the techniques derived from it are useful elsewhere in later chapters and secondly, it provides a classification of space-times which (unlike the ones in chapter 7) is not pointwise but applies to the space-time as a whole. Unfortunately (and not unlike those of chapter 7) it is somewhat too coarse in places. However, it displays the curvature structure clearly and its usefulness in later descriptions of symmetries justifies its inclusion.

Chapter 9 concentrates on the general relations between the metric and its Levi-Civita connection and associated curvature and sectional curvature functions. Thus the problem of the extent to which the prescription of one of these objects determines any other is studied. Holonomy theory is useful here as is a convenient classification of the curvature tensor which is developed in this chapter. In this section the sectional curvature function is discussed and its (generic) equivalence to the metric tensor displayed. This raises the prospect of regarding the sectional curvature function as an alternative field variable in general relativity, at least for vacuum space-times.

The remaining four chapters of the book are on symmetries in general relativity. The idea here is to present some techniques for studying such symmetries and the presentation will be guided by an attempt to achieve a certain reasonable level of rigour and elegance. These symmetries are defined in terms of local transformations and are then described in terms of certain families of vector fields. The emphasis will be placed on technique rather than a multitude of examples, although the salient points will be exemplified. Here, some elementary knowledge of Killing vector theory in the more well known exact solutions is desirable but not necessary. The essential philosophy behind these sections is the theory and the finding of symmetries. The symmetries treated are those described by Killing, homothetic and affine vector fields (chapter 10), conformal vector fields (chapter 11), projective vector fields (chapter 12) together with symmetries of the curvature tensor (curvature collineations) which will be treated in chapter 13. Of some importance in these matters is the study of the zeros of such vector fields (that is the "fixed points" of the associated local transformations) and the description of the consequent orbit structure that they exhibit. Sections on the zeros of Killing, homothetic and conformal vector fields and on the orbit structure of Killing vector fields are given in the relevant chapters. This is applied, in chapter 11, to the study of the conformal reduction of conformal vector fields to Killing and homothetic vector fields and the "linearisation" problem. Holonomy theory is used significantly in the treatment of proper affine and projective symmetry.

The general approach of this book is geometrical, in keeping with the spirit of general relativity. If a geometrical argument could be found then it was used in place of a (usually more) lengthy calculation. Unfortunately the author was sometimes unable to find an elegant argument and thus in these cases a clumsy calculation must suffice. Also, if a certain result depended on a space-time M satisfying a certain algebraic type (*e.g.* its Petrov type), then rather than assume that M had that Petrov type everywhere, M was topologically decomposed into a union of open subsets in each of which the Petrov type was constant (together with a nowhere dense "leftover" subset) and an attempt was then made to prove that result more generally. The author apologises if some of the proofs seem unnecessarily pedantic. This is simply his way of avoiding "cheating at patience" and, in many cases, is probably nothing more than a statement of his ignorance of a better proof. A similar apology is offered for the terse nature of some of the arguments. Conservation of space prevented more than this but it is hoped that enough is given for the details to be traced. This lack of space also prevents a proper discussion of the history of the topics treated. One unfortunate but inevitable consequence of this is the omission of many references and yet another apology must be offered.

The notation used is a fairly standard one with references given numerically in square brackets. Sections, theorems and equations are numbered within each chapter in an obvious way with equation numbers in round brackets. In chapter 2 vectors are introduced in bold face type as is the usual convention. However, later in the text when many other geometrical objects are brought into play, this procedure is dropped. When the Σ symbol is used for summation and the limits are obvious, they are sometimes omitted (and, of course, for tensor notation, the Einstein summation convention is eventually adopted). The end of a proof is denoted $\Box$.