Preface

Two of the most exciting developments of 20th century physics were general relativity and quantum theory, the latter culminating in the 'standard model' of particle interactions. General relativity treats gravity, while the standard model treats the rest of the forces of nature. Unfortunately, the two theories have not yet been assembled into a single coherent picture of the world. In particular, we do not have a working theory of gravity that takes quantum theory into account. Attempting to 'quantize gravity' has led to many fascinating developments in mathematics and physics, but it remains a challenge for the 21st century.

The early 1980s were a time of tremendous optimism concerning string theory. This theory was very ambitious, taking as its guiding philosophy the idea that gravity could be quantized only by unifying it with all the other forces. As the theory became immersed in ever more complicated technical issues without any sign of an immediate payoff in testable experimental predictions, some of this enthusiasm diminished among physicists. Ironically, at the same time, mathematicians found string theory an ever more fertile source of new ideas. A particularly appealing development to mathematicians was the discovery by Edward Witten in the late 1980s that Chern-Simons theory — a quantum field theory in 3 dimensions that arose as a spin-off of string theory — was intimately related to the invariants of knots and links that had recently been discovered by Vaughan Jones and others. Quantum field theory and 3-dimensional topology have become firmly bound together ever since, although there is much that remains mysterious about the relationship.

While less popular than string theory, a seemingly very different approach to quantum gravity also made dramatic progress in the 1980s. Abhay Ashtekar, Carlo Rovelli, Lee Smolin and others discovered how to rewrite general relativity in terms of 'new variables' so that it more closely resembled the other forces of nature, allowing them to apply a new set of techniques to the problem of quantizing gravity. The philosophy of these researchers was far more conservative than that of the string theorists. Instead of attempting a 'theory of everything' describing all forces and all particles, they attempted to understand quantum gravity *on its own*, following as closely as possible the traditional guiding principles of both general relativity and quantum theory. Interestingly, they too were led to the study of knots and links. Indeed, their approach is often known as the 'loop representation' of quantum gravity. Furthermore, quantum gravity in 4 dimensions turned out to be closely related to Chern-Simons theory in 3 dimensions. Again, there is much that remains mysterious about this. For example, one wonders why Chern-Simons theory shows up so prominently both in string theory and the loop representation of quantum gravity. Perhaps these two approaches are not as different as they seem!

It is the goal of this text to provide an *elementary* introduction to some of these developments. We hope that both physicists who wish to learn more differential geometry and topology, and mathematicians who wish to learn more gauge theory and general relativity, will find this book a useful place to begin. The main prerequisites are some familiarity with electromagnetism, special relativity, linear algebra, and vector calculus, together with some of that undefinable commodity known as 'mathematical sophistication'.

The book is divided into three parts that treat electromagnetism, gauge theory, and general relativity, respectively. Part I of this book introduces the language of modern differential geometry, and shows how Maxwell's equations can be drastically simplified using this language. We stress the coordinate-free approach and the relevance of global topological considerations in understanding such things as the Bohm-Aharonov effect, wormholes, and magnetic monopoles. Part II introduces the mathematics of gauge theory — fiber bundles, connections and curvature — and then introduces the Yang-Mills equation, Chern classes, and Chern-Simons classes. It also includes a brief introduction to knot theory and its relation to Chern-Simons theory. Part

III introduces the basic concepts of Riemannian and semi-Riemannian geometry and then concentrates on topics in general relativity of special importance to quantum gravity: the Einstein-Hilbert and Palatini formulations of the action principle for gravity, the ADM formalism, and canonical quantization. Here we emphasize tensor computations written in the notation used in general relativity. We conclude this part with a sketch of Ashtekar's 'new variables' and the way Chern-Simons theory provides a solution to the Wheeler-DeWitt equation (the basic equation of canonical quantum gravity).

While we attempt to explain everything 'from scratch' in a self-contained manner, we really hope to lure the reader into further study of differential geometry, topology, gauge theory, general relativity and quantum gravity. For this reason, we provide copious notes at the end of each part, listing our favorite reading material on all these subjects. Indeed, the reader who wishes to understand any of these subjects in depth may find it useful to read some of these references in parallel with our book. This is especially true because we have left out many relevant topics in order to keep the book coherent, elementary, and reasonable in size. For example, we have not discussed fermions (or mathematically speaking, spinors) in any detail. Nor have we treated principal bundles. Also, we have not done justice to the experimental aspects of particle physics and general relativity, focusing instead upon their common conceptual foundation in gauge theory. The reader will thus have to turn to other texts to learn about such matters.

One really cannot learn physics or mathematics except by doing it. For this reason, this text contains over 300 exercises. Of course, far more exercises are assigned in texts than are actually done by the readers. At the very least, we urge the reader to read and ponder the exercises, the results of which are often used later on. The text also includes 130 illustrations, since we wish to emphasize the geometrical and topological aspects of modern physics. Terms appear in boldface when they are defined, and all such definitions are referred to in the index.

This book is based on the notes of a seminar on knot theory and quantum gravity taught by J.B. at U. C. Riverside during the school year 1992-1993. The seminar concluded with a conference on the subject, the proceedings of which will appear in a volume entitled *Knots*

and Quantum Gravity.

We would like to thank Louis Kauffman for inviting us to write this book, and also Chris Lee and Ms. H. M. Ho of World Scientific for helping us at every stage of the writing and publication process. We also wish to express our thanks to Edward Heflin and Dardo D. Píriz for reading parts of the manuscript and to Carl Yao for helping us with some LaTeXcomplications. Scott Singer of the Academic Computing Graphics and Visual Imaging Lab of the U. C. Riverside deserves special thanks and recognition for helping us to create the book cover. Some of the graphics used for the design of the cover were generated with *Mathematica*, by Wolfram Research, Inc.; these were kindly given to us by Joe Grohens of WRI.

J.B. is indebted to many mathematicians and physicists for useful discussions and correspondence, but he would particularly like to thank Abhay Ashtekar, whose work has done so much to unify the study of gauge fields, knots and gravity. He would also like to thank the readers of the USENET physics and mathematics newsgroups, who helped in many ways with the preparation of this book. He dedicates this book to his parents, Peter and Phyllis Baez, with profound thanks for their love. He also gives thanks and love to his mathematical muse, Lisa Raphals.

J.M. dedicates this book to his parents, Luis and Crescencia Pérez de Muniáin y Mohedano, for their many years of continued love and support. He is grateful to Eleanor Anderson for being a patient and inspiring companion during the long and hard hours taken to complete this book. He also acknowledges José Wudka for many discussions on quantum field theory.