Introductory Chapters

1. History and background

Soon after the first quantum $\mathcal{W}$-algebras were written down in 1985 (see reprint [**3.1**]), it became clear that a number of results that had already been developed in the mathematical literature were going to be of great help for understanding these algebras. In this chapter we reprint two mathematical papers that have had a major influence on the development of a systematic description of $\mathcal{W}$-algebras.

The papers [**1.1**] and [**1.2**] are both devoted to the study of integrable hierarchies of non-linear differential equations. Let us explain why they are at the same time relevant for the understanding of $\mathcal{W}$-symmetry.

In the work of I.M. Gel'fand and L. Dickey ([**1.1**] and references therein) it is shown that it is often possible to write hierarchies of integrable differential equations in hamiltonian form. Among other things, this involves the specification of a generalized Poisson bracket, called the Gel'fand-Dickey bracket. It has been found [243,168,165,230,14] that, for the special case of the so-called Korteweg-de Vries (KdV) hierarchy, the Gel'fand-Dickey bracket for the second hamiltonian structure gives rise to a Virasoro algebra. In a similar way, the $\mathcal{W}_N$ algebras are related to the second hamiltonian structure of generalized KdV hierarchies, see [**1.1**], [**2.2**] and [1,335,247,15,114]. Multi-component generalizations of the KdV hierarchy have been shown to give rise to non-local analogues of $\mathcal{W}$-algebras [54].

The paper [**2.1**] makes a connection between affine Lie algebras and, again, integrable hierarchies of differential equations. Combining this with the possibility, described above, to associate a $\mathcal{W}$-algebra structure with such integrable hierarchies, we obtain a direct link between affine Lie algebras on the one hand and $\mathcal{W}$-algebras on the other. Once this association, referred to as the *Drinfel'd-Sokolov reduction*, has been understood it can be formulated without reference to integrable hierarchies of differential equations and in that form it is a powerful and elegant tool for the analysis of $\mathcal{W}$-algebras. In Chapters 2 and 4 we reprint a number of papers where this Drinfel'd-Sokolov reduction is worked out in detail.

2. Classical $\mathcal{W}$-algebras and their connection to Toda field theories

The defining relations of a $\mathcal{W}$-algebra express the bracket of any two generators of the algebra in terms of a non-linear expression built from the fundamental generators. If we view the generators as functions on a (classical) phase space, the brackets can be viewed as Poisson brackets. The corresponding algebra is called a *classical* $\mathcal{W}$-algebra. However, in the context of quantum mechanics the generators should rather be viewed as operators acting in a Hilbert space of states, and in that situation the bracket acquires the interpretation of a commutator bracket. In the quantum mechanical case the non-linear expressions appearing on the right hand side of the defining brackets have to be normal ordered. Because of this, a consistent set of structure constants for a *quantum* $\mathcal{W}$-algebra (consistent in the sense of the Jacobi identities) will be different from an analogous set in the classical case. Obviously, this distinction does not arise in the case of (linear) Lie algebra symmetries.

Once the distinction between classical and quantum $\mathcal{W}$-algebras has been made, it is natural to look for relations. One idea is to obtain a classical $\mathcal{W}$-algebra by taking the '$h \to 0$' limit of a quantum $\mathcal{W}$-algebra. On the other hand, one may also try to obtain a quantum $\mathcal{W}$-algebra through the quantization of a classical solution to the Jacobi identities.

A classical limit can be defined for most quantum $\mathcal{W}$-algebras, although there are cases where the resulting classical algebra is highly degenerate (see reprint [**4.6**] for more precise remarks). It was shown in [**3.4**] that a number of quantum $\mathcal{W}$-algebras for which the naive classical limit is degenerate, can be viewed as quantizations of certain classical Poisson bracket algebras which are, however, *infinitely* and *non-freely* generated (see also [107]).

In general, the process of 'quantizing' a classical $\mathcal{W}$-algebra has not been developed very well, and it has proven more effective to construct quantum $\mathcal{W}$-algebras directly at the quantum level (see Chapters 3, 4 and 5). Nevertheless, the interplay between classical and quantum $\mathcal{W}$-algebras has been a strong guide in the search for and construction of quantum $\mathcal{W}$-algebras. In particular, the role of the Gel'fand-Dickey bracket and the idea of Lie algebra reductions à la Drinfel'd-Sokolov have been studied first in the classical context.

An interesting observation, which was first clearly stated in [**2.1**], is that a particular class of classical field theories in $1+1$ dimensions, the so-called Toda field theories, possess a symmetry algebra that is precisely a $\mathcal{W}$-algebra. In the simplest case, the $\mathfrak{sl}_2$ Toda theory, this algebra is simply the Virasoro algebra, which expresses the conformal invariance of that theory. We refer to [57,58,59,289,244,245,324,222,223] for detailed results on $\mathcal{W}$-symmetry in classical and quantum Toda field theory. Supersymmetric extensions were discussed in [121,262,261,229,201].

A closely related occurrence of classical $\mathcal{W}$-symmetry is in so-called constrained or gauged Wess-Zumino-Witten (WZW) models. The presence of the constraints leads to a reduction of the affine current algebra of these models, and in this way the models provide concrete realizations of the Drinfel'd-Sokolov reduction scheme. We reprint **[2.3]** (see also [23,25,211,138]), which discusses these reductions and shows that with a certain choice of gauge (the so-called diagonal gauge) results from Toda field theory are reproduced. A different choice of gauge (Drinfel'd-Sokolov gauge) directly points at the $\mathcal{W}$-algebra generators and through this connection (the so-called Miura transformation) a free field representation of many classical $\mathcal{W}$-algebras can be obtained.

The papers [142,143,144,202,203,200,81,209,153] discuss supersymmetric $\mathcal{W}$-algebras and extended superconformal algebras in relation to supersymmetric integrable hierarchies and hamiltonian reduction.

Important progress in the study of classical $\mathcal{W}$-algebras was made in **[2.4]**, see also [130,267,163]. In **[2.4]** it is pointed out that for each embedding of the algebra $\mathfrak{sl}_2$ into a semisimple Lie algebra $\mathfrak{g}$, one may define a reduction of the current algebra of the (untwisted) affine extension $\widehat{\mathfrak{g}}$ and that this leads to a classical $\mathcal{W}$-algebra. The earlier work on such reductions had mostly dealt with the principal $\mathfrak{sl}_2$ embedding. An excellent review on the general structure of these reductions can be found in [131]. Integrable hierarchies associated with the more general Drinfel'd-Sokolov reductions have been discussed in [86,106,87,315,316,248,127,309]. The paper **[2.5]** (see also [69,225,136,109,154,286,233,282]) discusses general constructions of $\mathcal{W}$-superalgebras, where the relevant data are a Lie superalgebra $\mathfrak{g}$ and an embedding of $\mathfrak{osp}(1|2)$ into $\mathfrak{g}$.

More recent work has focussed on the possibility of reductions that are different from those described in **[2.4]**. An important observation is that, in general, one expects algebras that cannot be obtained in the standard Drinfel'd-Sokolov scheme to be non-freely generated [129]. Examples for this are the $\mathcal{W}_n^\ell$ ($n \geq 4$) algebras [268,42,110,128], and the algebras discussed in **[3.4]**.

3. Quantum $\mathcal{W}$-algebras

The first systematic investigation as well as the first examples of (quantum) $\mathcal{W}$-algebras (with nonlinear defining relations) appeared in a paper by A.B. Zamolodchikov [**3.1**] (for a definition of $\mathcal{W}$-algebras in the context of meromorphic conformal field theory [169] we refer to [**5.4**] and [79]). In this paper Zamolodchikov investigates the possibility of adding additional generators (of spins $\Delta \leq 3$) to the Virasoro algebra in such a way that the resulting algebra closes (albeit, possibly, nonlinearly) and is associative. Associativity is imposed through the crossing symmetry of the four-point functions.

Although the investigation in [**3.1**] was rather limited already an important feature emerged. Namely, it turned out that there are essentially two types of $\mathcal{W}$-algebras, the 'generic' and the 'exotic' ones. Generic $\mathcal{W}$-algebras are those that are associative for all but a finite number of values for the central charge c (*i.e.* excluding those c-values for which the structure constants blow up), while exotic $\mathcal{W}$-algebras are associative for at most a finite number of c-values. An example of the generic type is the algebra which contains, besides the Virasoro generator of spin 2, one additional generator of spin 3. This algebra has become known as the $\mathcal{W}_3$ algebra, and constitutes the prototype $\mathcal{W}$-algebra. An example of the exotic type is the algebra which contains an additional generator of spin $\frac{5}{2}$. This algebra is associative if and only if $c = -\frac{14}{13}$.

The analysis of Zamolodchikov was continued in [**3.2**] where, apart from several exotic cases, a classification of all generic 'rank-2 $\mathcal{W}$-algebras' (*i.e.* those $\mathcal{W}$-algebras which contain, besides the Virasoro generator, only one additional generator of spin Δ) was obtained. Surprisingly, it turned out that (bosonic) generic rank-2 $\mathcal{W}$-algebras only exist for $\Delta = 1, 2, 3, 4$ and 6. The spin-4 and spin-6 algebra were explicitly constructed afterwards in [178,339,147].

Subsequent investigations of possible $\mathcal{W}$-algebras through the 'crossing symmetry method' as well as a further systematization of the method have appeared in [80,185,188]. Similar computations for extended superconformal algebras and $\mathcal{W}$-superalgebras can be found in [41,227,204,205,148,149]. Realizations of $\mathcal{W}$-algebras were constructed in *e.g.* [126,30,288,285,3,232,325] (see Chapters 4 and 5 for additional references). The investigation of the representation theory of some of the newly discovered (exotic) $\mathcal{W}$-algebras was undertaken in [318,119,151,152].

As an alternative for checking crossing symmetry of the four-point functions one can attempt to analyse the Jacobi identities (for the modes) directly. This method was developed and applied simultaneously in [66,221] (see also [80,256]). We have chosen to reprint [66] (reprint [**3.3**]). The analogous approach to $\mathcal{W}$-superalgebras was presented in [62,63,64,65]. Further case-by-case studies of a variety of $\mathcal{W}$-algebras can be found in [150,219,226,27,287,263,184,186,120,189]. A systematic construction of some of the algebras obtained in [221,120,189] is given in the preprint [**3.4**].

There are various ways in which different $\mathcal{W}$-algebras can be related, *c.q.* procedures for constructing new $\mathcal{W}$-algebras out of existing ones. Here we would only like to mention the procedure of 'twisting' (*i.e.* 'orbifolding'), see *e.g.* [180,181,182,183], and the idea of 'factoring out free fields,' see *e.g.* [173,104].

4. Quantum Drinfel'd-Sokolov reduction

A very powerful way of constructing both examples of $\mathcal{W}$-(super)algebras as well as their representation theory is through a quantization of the Drinfel'd-Sokolov reduction (see [**1.2**]). At first, this was attempted through a direct quantization of the outcome of the classical reduction, *i.e.* by replacing the classical free fields in the expressions for the generating currents by their quantum counterparts and normal ordering. See, in particular, the reprints [**4.1**][**4.2**] and the paper [242], where the closure of the quantum algebra $\mathcal{W}A_n$ was established (for a major review on these works, see [124]).

The methods of [124], unfortunately, remained rather 'ad hoc,' and only worked well for the $\mathcal{W}$-algebra $\mathcal{W}A_n$ associated to the Lie algebra $A_n \cong \mathfrak{sl}_{n+1}$. It was realized that, rather than quantizing the classical reduction at the end, one may quantize the reduction from the outset. The basic idea is to impose the constraints on the quantum currents directly through a Becchi-Rouet-Stora-Tyutin (BRST) procedure and obtain the $\mathcal{W}$-algebra as the cohomology of the corresponding BRST operator. These ideas were first put forward in [**4.3**] and [**4.4**] (see also [26,112]). This cohomological approach to $\mathcal{W}$-algebras has become known as the 'quantum Drinfel'd-Sokolov reduction' and was, as such, clearly formulated and further developed in [134,135,159] and in the paper reprinted as [**4.5**].

A big advantage of formulating $\mathcal{W}$-algebras in the context of homological algebra is that not only the $\mathcal{W}$-algebra itself but, in principle, all its properties (such as representation theory) follow from the corresponding properties of the underlying affine Lie algebra. In particular, free field realizations of the $\mathcal{W}$-algebra can be obtained by reducing the free field realizations of the underlying affine Lie algebras, *i.e.* the so-called Wakimoto realizations [319,133,72,73]. A study of the representation theory of $\mathcal{W}$-algebras from this perspective was started in [**4.6**] (see also [75,259]). For other studies on various aspects of the representation theory of quantum $\mathcal{W}$-algebras (*e.g.* Kac determinants) in this context, see [126,124,252,70,320,321,253,254,306,51,52,53,84,258,260].

Recent developments have evolved around the generalization of these reductions to arbitrary $\mathfrak{sl}_2$ embeddings in $\mathfrak{g}$, the quantization of the reductions discussed in Chapter 2 (in particular [**2.4**]). At first, progress was made in the finite-dimensional case where previously unknown 'finite $\mathcal{W}$-algebras' were discovered [313,101] (see also [111]). Subsequently, the methods of [101] were generalized to the infinite-dimensional case [**4.8**]. See also [**4.9**], where the quantization was performed in a somewhat different way.

Some ideas towards a possible classification of $\mathcal{W}$-algebras based on these concepts are contained in the paper reprinted as [**4.7**] (see also [132]).

For other recent developments, extensions to the supersymmetric case and closely related papers we refer the reader to [206,207,208,103,323,324,46,257,224].

5. Coset constructions

The second powerful method to obtain examples of $\mathcal{W}$-algebras is through the so-called coset construction. It is well known that for every affine Lie algebra $\hat{\mathfrak{g}}$ there is an associated conformal algebra for which the Virasoro generator is expressed as a bilinear in the affine Lie algebra currents, the so-called Sugawara construction [310] (for generalizations see *e.g.* [177,105]). It turns out that this Virasoro generator commutes with the finite-dimensional Lie algebra $\mathfrak{g}$ underlying $\hat{\mathfrak{g}}$, and can thus be interpreted as belonging to the coset pair $(\hat{\mathfrak{g}}, \mathfrak{g})$. It was found some time ago [170,171] that this construction could be generalized to arbitrary coset pairs $(\hat{\mathfrak{g}}, \hat{\mathfrak{g}}')$ where $\hat{\mathfrak{g}}'$ is a (finite-dimensional or affine) subalgebra of $\hat{\mathfrak{g}}$. The corresponding Virasoro generator has a central charge given by the differences of central charges associated to $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}}'$ and commutes with $\hat{\mathfrak{g}}'$.

It was observed in concrete examples that the spectrum of a coset model very often contains chiral fields of higher integer spin (as an example, the spectrum of the $c = \frac{4}{5}$ three-state Potts model contains a spin 3 field), and thus the question arose whether coset models generally contain higher spin generators that commute with the subalgebra $\hat{\mathfrak{g}}'$. This question was first studied, and answered affirmatively, in **[5.1]** and **[5.2]** (see also [162,312]) for the diagonal coset pair $(\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}, \hat{\mathfrak{g}})$ (and, worked out in detail for $\mathfrak{g} \cong \mathfrak{sl}_3$). [For some closely related results in the mathematical literature we refer to [175,179].] It turns out, miraculously, that for simply-laced Lie algebras $\mathfrak{g}$ the coset $\mathcal{W}$-algebra is isomorphic to the $\mathcal{W}$-algebra obtained by the quantum Drinfel'd-Sokolov reduction based on the principal $\mathfrak{sl}_2$ embedding in $\mathfrak{g}$. This can be concluded from the explicit construction of the algebra (in the case of $\mathcal{W}_3$, see **[5.1]** and **[5.2]**) or by comparing the resulting characters [79]. We refer to **[5.4]** for a discussion of the independent generators of the $\mathcal{W}$-algebra of a diagonal coset model. One of the outstanding problems to date is to gain an a priori insight into which coset $\mathcal{W}$-algebras are isomorphic to which Drinfel'd-Sokolov $\mathcal{W}$-algebras.

Other coset-algebras that have been examined in detail are the ones corresponding to the coset pairs $(\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}', \hat{\mathfrak{g}}')$ where $\hat{\mathfrak{g}}'$ is a conformal subalgebra of $\hat{\mathfrak{g}}$ (see [290,9] for a definition and classification of conformal subalgebras). We reprint **[5.3]** (see also [172]).

For a discussion of other cosets and supersymmetric extensions of the above constructions the reader may want to consult [116,117,61,93,47,190,4,292].

Not all coset pairs yield different $\mathcal{W}$-algebras. Coset pairs which do yield isomorphic $\mathcal{W}$-algebras are termed 'dual' [213]. The reprinted paper **[5.3]** contains a discussion and classification of so-called T-equivalent coset pairs, which are precursors of the aforementioned dual coset pairs (see also [6]).

Any coset $\mathcal{W}$-algebra comes naturally equipped with a set of (not necessarily irreducible) representations, whose characters are given by the branching functions [214,216]. Conversely, a priori insight into *e.g.* the generators of a certain coset $\mathcal{W}$-algebra can be obtained by careful examination of these branching functions, *i.e.* by means of the so-called ***character technique*** (see *e.g.* [70] and [**5.4**]). We refer to [216,218,89,74] for the computation of branching functions of several coset pairs $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}')$. Finally, we would like to mention that one important advantage of the coset construction over the quantum Drinfel'd-Sokolov reduction is that the question of unitarity of $\mathcal{W}$-algebra representations is easier to address in the context of the coset construction (see *e.g.* [252,253]).

6. $\mathcal{W}_\infty$-type algebras

$\mathcal{W}$-algebras with an infinite number of independent generating currents, with spins increasing all the way to infinity, form a very special class. We will generically denote them as $\mathcal{W}_\infty$-type algebras. It is sometimes possible to write these algebras in a basis where all the defining bracket relations are linear. Due to this, those algebras are more tractable than their finitely generated counterparts, and in some cases have a clear geometrical meaning. Linear $\mathcal{W}_\infty$-type algebras have found applications in a variety of fields including string theory and the study of the quantum Hall effect. However, some of the most interesting $\mathcal{W}_\infty$-type algebras are non-linear.

In this chapter we reprint some of the earlier papers on $\mathcal{W}_\infty$-type algebras: **[6.1]** on the w_∞ algebra, **[6.2]** and **[6.3]** discussing the algebras $\mathcal{W}_\infty$ and $\mathcal{W}_{1+\infty}$, respectively, and **[6.4]**, which is one of the first papers on the so-called $\widehat{\mathcal{W}}_\infty(k)$ algebra. The paper **[6.5]** discusses relations between various $\mathcal{W}_\infty$-type algebras.

One of the original motivations for considering $\mathcal{W}_\infty$-type algebras was to study the 'large-N limit' (**[6.1]**, [48,255]) of the $\mathcal{W}_N$ symmetries, which were already known at the time. This limit is by no means unique, and a number of different $\mathcal{W}_\infty$-type algebras have been proposed. The simplest one, called w_∞ **[6.1]**, has generators of conformal spins $2, 3, \ldots$. The standard central extension of the Virasoro algebra cannot be extended to the full w_∞ algebra, and this prevents the algebra from playing a direct role in conformal field theory. The algebra $\mathcal{W}_\infty$, which was first proposed by C. Pope, L. Romans and S. Shen in [272], **[6.2]** (see also [18]), is a linear deformation of w_∞ that is precisely such that a non-trivial central extension can be defined on the full algebra.

The algebra w_∞ can be extended by an additional current of spin 1, giving rise to $w_{1+\infty}$. This algebra can be viewed as the algebra of area preserving diffeomorphisms of a cylinder. It admits a universal central extension **[6.3]**, which has been called $\mathcal{W}_{1+\infty}$. [In the mathematical literature, $\mathcal{W}_{1+\infty}$ is known as $\widehat{\mathcal{D}}$ [215,217].]

Matrix extensions of $\mathcal{W}_\infty$-type algebras were considered in [20,265]. The papers [38,35,36], see also [280], discuss an algebra called super-$\mathcal{W}_\infty$, which is an $N = 2$ supersymmetric extension of $\mathcal{W}_\infty$. Various field theoretical realizations of $\mathcal{W}_\infty$-type algebras have been discussed in [19,20,22,38,305,336]. A systematic discussion of the representation theory of some of the $\mathcal{W}_\infty$-type algebras can be found in [264]. The unitary representations of $\mathcal{W}_{1+\infty}$ have been classified in [217]. Useful reviews on the linear $\mathcal{W}_\infty$-type algebras can be found in [275,304].

Non-linear $\mathcal{W}_\infty$-type algebras were first studied through their connection with integrable hierarchies of differential equations. In Chapter 1 we mentioned that the (classical) $\mathcal{W}_N$ algebra arises from the second hamiltonian structure of a generalized Korteweg-de Vries (KdV) hierarchy. This, in combination with the fact that the generalized KdV hierarchies can be viewed as reductions of the Kadomtsev-Petviashvili (KP) hierarchy, suggests that the (second) hamiltonian structure of the KP hierarchy can lead to a $\mathcal{W}_\infty$-type algebra that plays the role of 'universal $\mathcal{W}$-algebra' for the $\mathcal{W}_N$ series [146].

The bi-hamiltonian structure of the KP hierarchy was unraveled in a number of papers: in [337,328] it was found that the first hamiltonian structure corresponds to the algebra $\mathcal{W}_{1+\infty}$, and in [113,139] a (classical) $\mathcal{W}_\infty$-type algebra associated to the second hamiltonian structure, called $\mathcal{W}_{KP}$ was obtained. The algebra $\mathcal{W}_{KP}$ has non-linear defining relations and it has zero central charge. The same algebra (with the spin-1 generators omitted) was also obtained in [329], where it was called $\widehat{\mathcal{W}}_\infty$.

In an independent study, I. Bakas and E. Kiritsis [**6.4**] studied the chiral symmetry algebra of the conformal field theory based on the non-compact coset $SL(2,\mathbf{R})_k/U(1)$. They obtained a quantum $\mathcal{W}_\infty$-type algebra which they called $\widehat{\mathcal{W}}_\infty(k)$, and which has a non-zero central charge given by $c_k = 2(k+1)/(k-2)$. They suggested that this algebra is the quantum version of a suitable deformation of the algebra $\mathcal{W}_{KP}$. In [141], a one-parameter family of deformations of $\mathcal{W}_{KP}$, called $\mathcal{W}_{KP}^{(q)}$, was explicitly constructed. The relations between the KP hierarchy, the $SL(2,\mathbf{R})_k/U(1)$ coset conformal field theory and the nonlinear $\mathcal{W}_\infty$-type algebras were further worked out by Y.-S. Wu and F. Yu [330]-[334], who used a two-boson realization to connect the classical and quantum constructions.

The quantum algebra $\widehat{\mathcal{W}}_\infty(k)$ truncates on the $\mathcal{W}_N$ algebra for $k=-N$, and this confirms its role as the universal $\mathcal{W}$-algebra for the $\mathcal{W}_N$ series. At the classical level $\mathcal{W}_{KP}^{(q)}$ plays a similar role; we reprint paper [**6.5**] where the relation of $\mathcal{W}_{KP}^{(q)}$ with other $\mathcal{W}_\infty$-type algebras is reviewed. (Notice that [**6.5**] uses the notion of the 'classical limit' of a Poisson bracket algebra [145], which is not to be confused with the more standard notion of the classical limit of a quantum $\mathcal{W}$-algebra.)

7. $\mathcal{W}$-gravity and $\mathcal{W}$-strings

Conformal symmetry plays a natural role in string theory, where it arises as a residual symmetry associated with the reparametrization invariance of the coordinate maps from the world-sheet into space-time. The fact that, at the algebraic level, conformal symmetry can be extended by including higher spin generators strongly suggests that string models based on extended conformal symmetry can be constructed. This motivation has led to extensive studies of higher spin extensions of two dimensional gravity ($\mathcal{W}$-gravity) and of higher-spin string theories ($\mathcal{W}$-strings).

There are two distinct approaches to the subject of $\mathcal{W}$-strings, which are usually referred to as the *critical* and the *non-critical* approach. The latter requires a detailed understanding of a fully interacting $\mathcal{W}$-gravity sector as part of the $\mathcal{W}$-string theory. We shall first focus on these $\mathcal{W}$-gravity theories and after that make further comments on $\mathcal{W}$-strings.

The study of $\mathcal{W}$-gravity has been guided by the analogy with ordinary gravity in two dimensions, which has been studied in great detail in recent years. However, some features of $\mathcal{W}$-gravity are essentially different from those of ordinary gravity. Most of these have been addressed in the context of $\mathcal{W}_3$ gravity. Those features that can be understood from the point of view of affine Lie algebra reductions (see below) can usually be treated for many other examples as well.

One of the first issues addressed in $\mathcal{W}$-gravity was the construction of classical $\mathcal{W}$-invariant matter couplings. We reprint the paper by C. Hull [**7.2**] where the coupling of chiral gauge $\mathcal{W}_3$ gravity to scalar matter fields is described. In [294,295] it was found that the matter couplings in more general gauges contain the fundamental $\mathcal{W}$-gravity fields in non-polynomial form. These couplings can be described in closed form by using so-called nested covariant derivatives. Paper [**7.3**] reviews the construction of the most general matter couplings in classical $\mathcal{W}_3$ gravity (see also [192,251,195,269]). These couplings, which were extended to w_∞ in [39,296,34], are covariant in the sense that they employ covariant vielbein fields for the $\mathcal{W}$-gravity degrees of freedom.

The geometrical structure behind the highly non-linear structure of the classical $\mathcal{W}$-gravities has been investigated in a series of papers by C. Hull [193,196,197,199]. Other approaches to the notion of '$\mathcal{W}$-geometry' can be found in [49,55,307,308,166,167,250,99].

At the quantum level, $\mathcal{W}$-gravity is defined by an induced action, which is obtained by integrating out the matter fields. The induced action for $\mathcal{W}_3$ gravity shows a number of interesting features. In the chiral gauge, one finds that the dependence on the central charge c of the matter system is not through an overall constant (as it is for ordinary gravity); the result can be written in a $1/c$ expansion which can be worked out in perturbation theory [249,297,194] (see also [37,88]). Upon quantization of the $\mathcal{W}_3$ gravity fields, which leads to an effective action, the $1/c$ corrections of the induced action interfere with contributions from $\mathcal{W}_3$ gravity loop diagrams. Keeping track of all contributions allows one to compute order by order in $1/c$ physical quantities such as the string susceptibility [299] (see also [176]). These perturbative computations are reviewed in [**7.5**].

In the conformal gauge both the induced and the effective action for $\mathcal{W}_3$ gravity take the form of a Toda action. Correspondingly, the $\mathcal{W}_3$ currents take the free field form first given in [126]. We reprint the paper [**7.4**] (see also [91]), which discusses the physical states for conformal gauge $\mathcal{W}_3$ gravity. It makes essential use of the explicit form of the Becchi-Rouet-Stora-Tyutin (BRST) charge for the $c = 100$ quantum $\mathcal{W}_3$ algebra, which was first obtained by J. Thierry-Mieg in [**7.1**] (see also [293]). (Recently, analogous BRST charges for various higher rank $\mathcal{W}$-algebras have been constructed [240,187,340]; their cohomology was studied in [239,67]).

In a somewhat different approach to quantum $\mathcal{W}_3$ gravity, one starts from the covariant induced action (without $1/c$ corrections), which can be quantized by using a BRST procedure. This covariant induced action (in the large c limit) was first given in [96]; we reprint the paper [**7.6**], where a systematic derivation is presented. The quantization of this action was discussed in [100]. If one goes to a chiral gauge, the nilpotency of the relevant BRST charge leads to a so-called Knizhnik-Polyakov-Zamolodchikov (KPZ) formula for the critical exponents of $\mathcal{W}_3$, which agrees with the result obtained from direct quantization in the chiral gauge. In the conformal gauge the action reduces to a Toda action and the BRST charge takes the form that was first proposed in [**7.8**] (see also [44,40]). It should be stressed that this BRST charge, which is appropriate for the coupling of $\mathcal{W}_3$ matter to $\mathcal{W}_3$ gravity, is a non-trivial extension of the BRST charge of [**7.1**].

Many of the results for $\mathcal{W}$-gravity cited above can be understood and extended by using the representation (of both $\mathcal{W}$-matter and $\mathcal{W}$-gravity) as a constrained Wess-Zumino-Witten model, and using the ideas of Drinfel'd-Sokolov reduction. At the classical level this allows one to construct exact expressions for the large-c limit of the chiral and the covariant induced actions [266,96]. Furthermore, using quantum Drinfel'd-Sokolov reduction leads to exact results (such as the KPZ formula) for the effective (*i.e.* quantized) $\mathcal{W}$-gravities [5,45,98,303,246], [**4.9**], and explains some of the cancellations that had been observed in perturbation theory.

We already mentioned that the construction of $\mathcal{W}$-strings (mostly $\mathcal{W}_3$ strings) has proceeded along two rather different roads. The first road followed has been that of *critical* $\mathcal{W}_3$ strings [60], which are constructed as follows. One first constructs (using the results of [126,285]) a $c=100$ realization of the $\mathcal{W}_3$ algebra in terms of scalar matter fields (such constructions necessarily involve background charges). For such a matter system an anomaly-free coupling to $\mathcal{W}_3$ gravity can be constructed [277] and this then forms the starting point for a critical $\mathcal{W}_3$ string theory. In this theory, physical states are selected by the cohomology of the BRST operator of [**7.1**]. We refer to [**7.4**], [91,283,271,278,234,238,326,231,279] for results on the spectrum of $\mathcal{W}$-strings. We reprint [**7.10**], in which the complete cohomology of this $\mathcal{W}_3$ string is presented. Interacting $\mathcal{W}_3$ strings were discussed in [155,236,156] and in the reprint [**7.7**]. Two recent reviews [270,327] provide a more complete guide to the literature on this subject.

Non-critical $\mathcal{W}_3$ strings are constructed by starting from a $\mathcal{W}_3$ matter system (which should be such that it allows some kind of target space interpretation) and coupling to $\mathcal{W}_3$ gravity by using the BRST operator of [**7.8**]. (Note that these theories cannot be viewed as critical $\mathcal{W}_3$ strings with a specific choice of matter system, as is the case for the bosonic string.) The cohomology problem for non-critical $\mathcal{W}$-strings has been studied in [**7.9**], [44,31,32,33,78]. Further results for $\mathcal{W}_3$ gravity coupled to a $c=2$ matter system can be found in [77].

There exist intriguing properties (see, *e.g.* [33]) of the cohomologies of the non-critical $\mathcal{W}$-strings where the matter system is a $\mathcal{W}$-minimal model. These have first been explored for the critical $\mathcal{W}$-strings where the minimal model is a trivial $c=0$ theory, see for example [**7.4**] and [91,198,157,158,239]. A typical example is the fact that the cohomology of the critical $\mathcal{W}_3$ string is similar (but not isomorphic!) to the cohomology of an ordinary ($\mathcal{W}_2$) string, where the matter system contains an additional $c=1/2$ Ising conformal field theory.