1

Introduction

In this book we describe numerous connections of multidimensional hypergeometric functions with Kac–Moody Lie algebras and their quantum deformations. The hypergeometric functions have the following form:

$$I(t_1,\dots,t_n;\gamma;P;\{a_{ij}\};\kappa)$$
$$= \int_{\gamma(t_1,\dots,t_n;\{a_{ij}\};\kappa)} \prod_{1\le i<j\le N} (t_i - t_j)^{a_{ij}/\kappa} P(t_1,\dots,t_N)\, dt_{n+1} \wedge \cdots \wedge dt_N \,.$$

Here $\{a_{ij}\}$, κ are complex parameters, P runs through a suitable space of rational functions, and γ runs through a suitable space of cycles.

There are two main themes in this book.

The first theme: Studying representations of Kac–Moody algebras and the corresponding quantum groups is essentially equivalent to studying the function $\Pi(t_i - t_j)^{a_{ij}/\kappa}$.

The second theme: The hypergeometric functions provide a new and powerful way to compare the representation theory of Kac–Moody algebras and the representation theory of quantum groups.

The differential-geometrical side of the function $\Pi(t_i - t_j)^{a_{ij}/\kappa}$ is connected with the theory of Kac–Moody algebras. Complexes of multivalued differential forms associated with this function have a natural description in terms of Kac–Moody algebras. The differential equation for associated hypergeometric functions has a description in terms of Kac–Moody algebras, and this differential equation is the famous Knizhnik–Zamolodchikov differential equation discovered by physicists in conformal field theory.

The topological side of the function $\Pi(t_i - t_j)^{a_{ij}/\kappa}$ is connected with the theory of quantum groups. Univalued branches of this function define a local system of coefficients over the complement to the union of diagonal hyperplanes. Complexes of chains with coefficients in this local system have a natural description in terms of quantum groups.

It is well-known that the representation theories of Kac–Moody algebras and their quantum deformations are similar and have striking differences at the same time. Studying their interrelations is a fascinating subject.

The hypergeometric functions provide a new approach to this problem. Namely, integration of multivalued differential forms over chains gives a correspondence between the corresponding objects of the representation theory of Kac–Moody algebras and the representation theory of quantum groups.

1.1. Example. Three Points on the Line

Let z_1, z_2, z_3 be points on the complex line. Fix numbers $m_1, m_2, m_3, \kappa \in \mathbb{C}$, and consider a multivalued holomorphic function

$$\ell = \prod_{1 \le i < j \le 3} (z_i - z_j)^{m_i m_j / 2\kappa} \prod_{j=1}^{3} (t - z_j)^{-m_j/\kappa}$$

on the complement to the union of the three points, here t is a coordinate on the line.

Set

$$\eta_j = \ell \, dt/(t - z_j)$$

for $j = 1, 2, 3$. $\{\eta_j\}$ are multivalued holomorphic forms on the complex line. They are closed and cohomologically dependent:

$$m_1\eta_1 + m_2\eta_2 + m_3\eta_3 = -\kappa d\ell \,. \tag{1.1.1}$$

Consider a vector

$$I = (I_1, I_2, I_3) = \Big(\int_\gamma \eta_1 \,, \int_\gamma \eta_2 \,, \int_\gamma \eta_3 \Big) \,, \tag{1.1.2}$$

where γ is a curve shown in Fig. 1.1. Here z_a and z_b are any two of the three points z_1, z_2, z_3. We fix a univalued branch of ℓ over γ, and hence the integrals are well-defined. We have

$$m_1 I_1 + m_2 I_2 + m_3 I_3 = 0 \,. \tag{1.1.3}$$

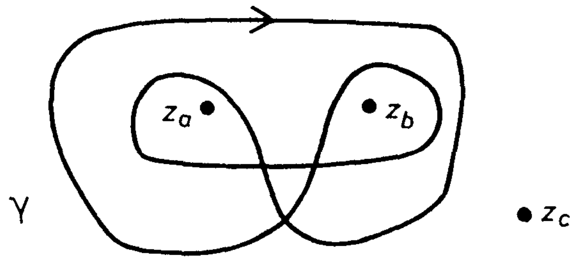

Fig. 1.1

The integrals do not depend on continuous deformations of the curve. Therefore, the vector is a function of z_1, z_2, z_3. The function $I(z_1, z_2, z_3)$ is holomorphic and satisfies the differential equation

$$\frac{\partial I}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} I \qquad \text{for } i = 1, 2, 3\,, \tag{1.1.4}$$

where the matrix Ω_{ij} has zero elements with the following exceptions:

$$\begin{aligned}
(\Omega_{ij})_{pp} &= m_i m_j/2 \quad \text{if } p \notin [i, j]\,, \\
(\Omega_{ij})_{ii} &= m_i m_j/2 - m_j\,, \\
(\Omega_{ij})_{jj} &= m_i m_j/2 - m_i\,, \\
(\Omega_{ij})_{ij} &= m_j, \quad (\Omega_{ij})_{ji} = m_i\,.
\end{aligned}$$

This differential equation does not depend on the choice of the curve γ.

Verification of the differential equation is purely combinatorial. For example,

$$\begin{aligned}
\frac{\partial \eta_2}{\partial z_1} &= \Big[\frac{m_1 m_2}{2\kappa} \frac{1}{z_1 - z_2} + \frac{m_1 m_3}{2\kappa} \frac{1}{z_1 - z_3} + \frac{m_1}{\kappa} \frac{1}{t - z_1}\Big] \ell \frac{dt}{t - z_2} \\
&= \Big[\frac{m_1 m_2}{2\kappa} \frac{1}{z_1 - z_2} + \frac{m_1 m_2}{2\kappa} \frac{1}{z_1 - z_2}\Big] \eta_2 + \frac{m_1}{\kappa} \frac{1}{z_1 - z_2} (\eta_1 - \eta_2)\,.
\end{aligned}$$

The key step in the verification is the equality

$$\frac{1}{t - z_1} \frac{1}{t - z_2} = \frac{1}{z_1 - z_2} \Big[\frac{1}{t - z_1} - \frac{1}{t - z_2}\Big]$$

having deep connections with the Jacobi identity in the theory of Lie algebras.

The differential forms $\{\eta_i\}$, identity (1.1.1), and the differential equation (1.1.4) have the following Lie–algebraic interpretation:

Consider the Lie algebra $\mathfrak{g} = s\ell_2$ of complex 2×2-matrices with the zero trace. $\mathfrak{g}$ is generated by the standard generators e, f, h subject to the relations

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

The tensor

$$\Omega = \frac{1}{2} h \otimes h + e \otimes f + f \otimes e \in \mathfrak{g} \otimes \mathfrak{g}$$

is called the Casimir operator. The Casimir operator is the tensor corresponding to an invariant scalar product on $\mathfrak{g}$.

Let $L_1, \ldots, L_n$ be representations of $\mathfrak{g}$, $L = L_1 \otimes \ldots \otimes L_n$. Let Ω_{ij} be the linear operator on L acting as the Casimir operator on $L_i \otimes L_j$ and as the identity operator on the other factors. The Knizhnik–Zamolodchikov (KZ) equation on an L-valued function $\varphi(z_1, \cdots, z_n)$ is the differential equation

$$\frac{\partial \varphi}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \varphi \quad \text{for } i = 1, \ldots, n,$$

κ is a parameter of the equation.

The KZ equation was discovered in the conformal field theory [**KZ**]. This equation describes conformal blocks in the Wess–Zumino–Witten model of the conformal field theory.

The KZ equation defines an integrable connection on the trivial bundle $L \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$ with singularities over diagonals. Parallel translations for the KZ connection commute with the $\mathfrak{g}$–action on L. Therefore, the bundle $L \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$ has a lot of subbundles invariant with respect to the KZ connection. For example, eigenspaces of the generators of $\mathfrak{g}$ form invariant subbundles.

Consider the following example of the KZ equation.

For $m \in \mathbb{C}$, denote by $M(m)$ the Verma module over $\mathfrak{g}$ with highest weight m. $M(m)$ is the infinite dimensional module generated by one vector v with properties $ev = 0$, $hv = mv$. Set $M = M(m_1) \otimes M(m_2) \otimes M(m_3)$, $m = m_1 + m_2 + m_3$, $\mathrm{Vac}\, M_{m-2} = \{x \in M \,|\, ex = 0, hx = (m-2)x\}$. Let $v_i \in M(m_i)$ be a generator vector, $i = 1, 2, 3$. Then

$$\begin{aligned} \mathrm{Vac}\, M_{m-2} = \big\{x \,=\, & I_1 f v_1 \otimes v_2 \otimes v_3 + I_2 v_1 \otimes f v_2 \otimes v_3 \\ & + I_3 v_1 \otimes v_2 \otimes f v_3 \,|\, m_1 I_1 + m_2 I_2 + m_3 I_3 = 0\big\}. \end{aligned} \tag{1.1.5}$$

Consider the KZ equation with values in $\mathrm{Vac}\, M_{m-2} \subset M$.

The KZ equation on the coordinates I_1, I_2, I_3 in (1.1.5) coincides with the hypergeometric differential equation (1.1.4). (1.1.6)

Therefore, an M-valued function

$$\int_{\gamma(z_1,z_2,z_3)} \eta_1 f v_1 \otimes v_2 \otimes v_3 + \int_{\gamma(z_1,z_2,z_3)} \eta_2 v_1 \otimes f v_2 \otimes v_3 + \int_{\gamma(z_1,z_2,z_3)} \eta_3 v_1 \otimes v_2 \otimes f v_3$$

takes values in $\mathrm{Vac}\, M_{m-2} \subset M$ and is a solution of the KZ equation.

Set

$$\mathcal{H}^1 = (\mathbb{C}\,\eta_1 \oplus \mathbb{C}\,\eta_2 \oplus \mathbb{C}\eta_3)/\mathbb{C}\, d\ell\,,$$

cf. (1.1.1).

$$\mathcal{H}^1 \textit{ is canonically isomorphic to } (\mathrm{Vac}\, M_{m-2})^*. \tag{1.1.7}$$

Let us return to the hypergeometric differential equation (1.1.4). Its solutions are parametrized by linear combinations of cycles γ shown in Fig. 1.1. Such linear combinations form a suitable homology group of the complement to the three points.

The function ℓ defines the following complex one dimensional local system $\mathcal{S}$ over $\mathbb{C} - \{z_1, z_2, z_3\}$. The sections of $\mathcal{S}$ are complex linear combinations of univalued branches of ℓ. The local system has monodromy around the points z_1, z_2, z_3. The monodromy around the point z_a is multiplication by $\exp(-2\pi i m_a/\kappa)$. Set

$$q = \exp(2\pi i/\kappa)\,,$$

then the monodromy is q^{-m_a}.

Integration defines a pairing

$$I[z_1, z_2, z_3] \,:\, \mathcal{H}^1 \otimes H_1(\mathbb{C} - \{z_1, z_2, z_3\}, \mathcal{S}) \longrightarrow \mathbb{C}\,. \tag{1.1.8}$$

Each element $\gamma \in H_1(\mathbb{C} - \{z_1, z_2, z_3\},\, \mathcal{S})$ generates a solution of the hypergeometric equation (1.1.4).

Now we calculate H_1. Assume that $z_1, z_2, z_3 \in \mathbb{R}$ and $z_1 < z_2 < z_3$. Fix a point b_0 in the upper half plane and three oriented loops b_1, b_2, b_3 as shown in Fig. 1.2.

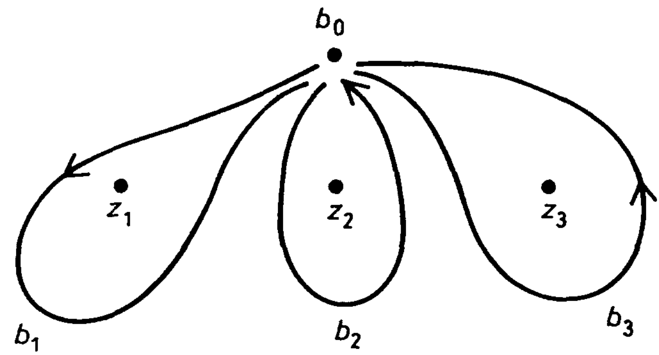

Fig. 1.2

Fix sections s_0, s_1, s_2, s_3 of $\mathcal{S}$ over b_0, b_1, b_2, b_3. For $j = 0, \ldots, 3$, the pair (b_j, s_j) is a singular chain with a coefficient in $\mathcal{S}$. Let

$$d : \bigoplus_{j=1} \mathbb{C}(b_j, s_j) \longrightarrow \mathbb{C}(b_0, s_0) \tag{1.1.9}$$

be the boundary operator.

$$\textit{Complex (1.1.9) computes } H.(\mathbb{C} - \{z_1, z_2, z_3\}, \mathcal{S}). \tag{1.1.10}$$

There is a natural choice of sections s_0, s_1, s_2, s_3 such that the boundary operator has the following form:

$$\begin{aligned}
(b_1, s_1) &\longmapsto (q^{m_1/2} - q^{-m_1/2}) q^{(m_2+m_3)/4} (b_0, s_0)\,, \\
(b_2, s_2) &\longmapsto (q^{m_2/2} - q^{-m_2/2}) q^{(-m_1+m_3)/4} (b_0, s_0)\,, \\
(b_3, s_3) &\longmapsto (q^{m_3/2} - q^{-m_3/2}) q^{(-m_1-m_2)/4} (b_0, s_0)\,.
\end{aligned} \tag{1.1.11}$$

Therefore, $H_1(\mathbb{C} - \{z_1, z_3, z_3\}, \mathcal{S})$ is naturally isomorphic to the kernel of the operator d given by formulas (1.1.11).

This computation has the following algebraic interpretation.

The universal enveloping algebra $U\mathfrak{g} = Us\ell_2$ has a one parameter deformation $U_q\mathfrak{g}$, $q = \exp(2\pi i/\kappa) \in \mathbb{C}^*$, called the quantum group. $U_q\mathfrak{g}$ is a $\mathbb{C}$–algebra generated by elements e, f, h subject to the relations

$$\begin{aligned}
[e, f] &= q^{h/2} - q^{-h/2}\,, \\
[h, e] &= 2e, \quad [h, f] = -2f\,.
\end{aligned} \tag{1.1.12}$$

By definition $q^{ah} = \exp(ah \times 2\pi i/\kappa)$.

In the standard definition of $U_q s\ell_2$, the first relation is replaced by

$$[e, f] = (q^{h/2} - q^{-h/2})/(q^{1/2} - q^{-1/2}). \tag{1.1.13}$$

The substitution of $f \mapsto f/(q^{1/2} - q^{-1/2})$ transforms (1.1.13) into (1.1.12).

The quantum group is a Hopf algebra. The comultiplication $\Delta : U_q\mathfrak{g} \to U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ is defined by the formula

$$\Delta(h) = h \otimes 1 + 1 \otimes h\,,$$
$$\Delta(f) = f \otimes q^{h/4} + q^{-h/4} \otimes f\,,$$
$$\Delta(e) = e \otimes q^{h/4} + q^{-h/4} \otimes e\,.$$

For $m \in \mathbb{C}$, denote by $M(m, q)$ the Verma module over $U_q\mathfrak{g}$ with highest weight m. $M(m, q)$ is the infinite dimensional module generated by one vector v with properties $ev = 0$, $hv = mv$. Set $M(q) = M(m_1, q) \otimes M(m_2, q) \otimes M(m_3, q)$, $m = m_1 + m_2 + m_3$, $\operatorname{Vac} M(q)_{m-2} = \{x \in M(q) | ex = 0, hx = (m-2)x\}$. Let $v_i \in M(m_i, q)$ be a generating vector, $i = 1, 2, 3$. Then

$$\begin{aligned}\operatorname{Vac} M(q)_{m-2} = \{x = {} & I_1 f v_1 \otimes v_2 \otimes v_3 + I_2 v_1 \otimes f v_2 \otimes v_3 \\ & + I_3 v_1 \otimes v_2 \otimes f v_3 \,|\, (q^{m_1/2} - q^{-m_1/2})\, q^{(m_2+m_3)/4} I_1 \\ & + (q^{m_2/2} - q^{-m_2/2})\, q^{(-m_1+m_3)/4} I_2 \\ & + (q^{m_3/2} - q^{-m_3/2})\, q^{(-m_1-m_2)/4} I_3 = 0\}\,.\end{aligned} \tag{1.1.14}$$

The map

$$\begin{aligned}(b_1, s_1) &\longmapsto f v_1 \otimes v_2 \otimes v_3\,, \\ (b_2, s_2) &\longmapsto v_1 \otimes f v_2 \otimes v_3\,, \\ (b_3, s_3) &\longmapsto v_1 \otimes v_2 \otimes f v_3\,,\end{aligned} \tag{1.1.15}$$

defines natural isomorphisms

$$\bigoplus_{j=1}^{3} \mathbb{C}(b_j, s_j) \simeq M(q)_{m-2} := \{x \in M(q) | hx = (m-2)x\}$$

$$H_1(\mathbb{C} - \{z_1, z_2, z_3\}, \mathcal{S}) \simeq \operatorname{Vac} M(q)_{m-2}\,.$$

(1.1.16) *Isomorphisms (1.1.7) and (1.1.15) and integration (1.1.8) define the canonical hypergeometric pairing*

$$I[z_1, z_2, z_3] : (\operatorname{Vac} M_{m-2})^* \otimes \operatorname{Vac} M(q)_{m-2} \longrightarrow \mathbb{C}\,,$$

that is, the canonical map

$$J[z_1, z_2, z_3] : \mathrm{Vac}\, M(q)_{m-2} \longrightarrow \mathrm{Vac}\, M_{m-2}\,.$$

For any $x \in \mathrm{Vac}\, M(q)_{m-2}$, *the function*

$$(z_1, z_2, z_3) \longmapsto J[z_1, z_2, z_3](x)$$

is a solution of the KZ equation with values in $\mathrm{Vac}\, M_{m-2}$.

The hypergeometric pairing is nondegenerate for generic κ. It is an interesting problem to describe degenerations of the hypergeometric pairing.

Isomorphism (1.1.15) gives an algebraic description of the monodromy representation of the KZ equation with values in $\mathrm{Vac}\, M_{m-2}$. To formulate the result we need some definitions.

Let V_1 and V_2 be $U_q\mathfrak{g}$-modules. The comultiplication induces a $U_q\mathfrak{g}$-module structure on $V_1 \otimes V_2$ and $V_2 \otimes V_1$. The modules $V_1 \otimes V_2$ and $V_2 \otimes V_1$ are isomorphic. The isomorphism is given by the formula

$$V_1 \otimes V_2 \xrightarrow{R} V_1 \otimes V_2 \xrightarrow{P} V_2 \otimes V_1\,,$$

where P is the transposition of the factors and $R \in U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ is the element called the universal R-matrix of the quantum group:

$$R = q^{h\otimes h/4} \sum_{k\geq 0} q^{-k(k+1)/4} \frac{(1)_q^k}{(k)_q!} q^{kh/4} e^k \otimes q^{-kh/4} f^k\,.$$

Here $(a)_q = q^{a/2} - q^{-a/2}$, $(a)_q! = (a)_q (a-1)_q \ldots (1)_q$.

Let $V_1, \ldots, V_n$ be representations of the quantum group. Then the pure braid group P_n acts on their tensor product. Namely, let $\sigma_1, \ldots, \sigma_{n-1}$ be the elementary braids shown in Fig. 1.3.

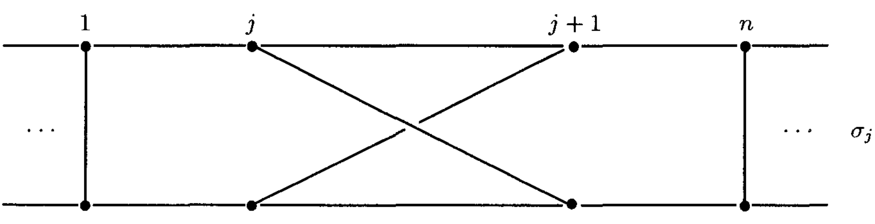

Fig. 1.3

To any braid σ_j, assign a linear operator

$$R_j : V_1 \otimes \ldots V_j \otimes V_{j+1} \cdots \otimes V_n \longrightarrow V_1 \otimes \ldots V_{j+1} \otimes V_j \cdots \otimes V_n$$

acting as PR on the j-th and $(j+1)$-th factors and as the identity on the other factors. These operators define an action of the pure braid group on n strings on $V_1 \otimes \cdots \otimes V_n$. The action of the pure braid group commutes with the action of the quantum group on the tensor product. Therefore, the action of P_n has a lot of invariant subspaces. For example, eigenspaces of the generators of the quantum group form P_n-invariant subspaces.

Let

$$\rho_q : P_3 \longrightarrow \operatorname{Aut}(\operatorname{Vac} M(q)_{m-2})$$

be the R-matrix representation. Let

$$\tau_\kappa : P_3 \longrightarrow \operatorname{Aut}(H_1(\mathbb{C} - \{z_1, z_2, z_3\}, \mathcal{S}))$$

be the natural representation induced by deformations of points z_1, z_2, z_3 on the complex line.

The representations ρ_q and τ_κ are canonically isomorphic for $q = \exp(2\pi i/\ \kappa)$, the isomorphism is induced by (1.1.15). (1.1.17)

Let $\gamma_\kappa : P_3 \longrightarrow \operatorname{Aut}(\operatorname{Vac} M(q)_{m-2})$ be the monodromy representation of the KZ equation.

1.1.18. *Corollary.* *The monodromy representation γ_κ of the KZ equation with values in $\operatorname{Vac} M_{m-2}$ is canonically isomorphic to the R-matrix representation ρ_q for $q = \exp(2\pi i/\kappa)$ and generic κ.*

In fact, for generic values of κ, the hypergeometric pairing is nondegenerate, all solutions of the KZ equation are given by hypergeometric integrals.

Remark. The vector valued function $I(z_1, z_2, z_3)$ defined in (1.1.2) is an object of classical mathematics. Each of its components is a classical hypergeometric function. According to one of the numerous definitions [**WW**], the classical hypergeometric function is an integral of a product of powers of three linear functions on the line. Hence, in this section, we have discussed connections between the theory of the classical hypergeometric function and the representation theory of the Lie algebra $s\ell_2$ and the quantum group $U_q s\ell_2$.

1.2. Brief Description of the Contents

The main subject of this book is the comparison between the geometric and analytic constructions in the theory of hypergeometric functions and algebraic constructions in the theory of Lie algebras and quantum groups.

In Secs. 2–5 we discuss the correspondence between the representation theory of quantum groups and geometry of configurations of hyperplanes. Secs. 6–9 are devoted to the correspondence between the universal R-matrix in the theory of quantum groups and the monodromy of configurations of hyperplanes depending on parameters. Secs. 10–12.1, 12.3 are on the connections between the representation theory of Kac–Moody Lie algebras and the geometry of configurations of hyperplanes. In Secs. 12.2, 12.4, 13, 14 we discuss the interrelations between the representation theory of Kac–Moody algebras and the representation theory of quantum groups coming from hypergeometric functions. In Sec. 15 we discuss possible generalizations.

There is another division of this book into parts: geometric, algebraic, and analytic. Secs. 2, 3, 6, 8, 13.1–13.4 are mainly geometric, Secs. 4, 5, 7, 9, 10.1–10.5, 11, 13.8, 15 algebraic, and Secs. 10.6–10.10, 12, 13.5–13.7 are mainly analytic.

In Sec. 2 we consider a configuration of hyperplanes in a complex affine space and a complex one-dimensional local system of coefficients over the complement to the configuration. We assume that the configuration is the complexification of a configuration in a real space and is weighted, that is a number is assigned to any hyperplane of the configuration. We give a combinatorial construction of two finite dimensional complexes and a homomorphism between them. The first complex computes the homology groups of the complement to the configuration with coefficients in the local system. The second complex computes the homology groups of the affine space modulo the configuration. The homomorphism of the first complex to the second induces the natural homomorphism of the homology of the complement of the configuration to the homology of the space modulo the configuration.

This construction can be considered as an analog of the Orlik–Solomon combinatorial description [OS] of the integral cohomology of the complement to the configuration.

In Sec. 3 we introduce discriminantal configurations. These are configurations connected with the representation theory of Lie algebras and quantum groups and the conformal field theory. Consider the configuration of all diagonal hyperplanes in a complex coordinate space and a projection of the space onto another coordinate space along a part of coordinates. Then the configuration of hyperplanes induced in a generic fiber is called a discriminantal configuration. In Sec. 3 we apply the constructions of Sec. 2 to the discriminantal configurations.

In Sec. 4 we introduce the quantum group $U_q\mathfrak{g}$ associated with an arbitrary complex symmetric matrix. Having a collection of highest weights we construct two complexes and a homomorphism between them. The first complex is a two-sided Hochschild complex of the nilpotent subalgebra $U_q\mathfrak{n}_- \subset U_q\mathfrak{g}$ with coefficients in the tensor product of two-sided Verma modules. The second complex is a two-sided Hochschild complex of the dual to $U_q\mathfrak{n}_-$ with coefficients in the tensor product of the contragradient modules corresponding to the Verma modules. The homomorphism between the complexes is given by a suitable contravariant form. We show that these two-sided Hochschild complexes and the homomorphism between them coincide with the combinatorial complexes and the homomorphism between them constructed for a discriminantal configuration in Secs. 2 and 3.

The two-sided Hochschild complexes constructed in Sec. 4 are non-standard for homological algebra. Their spaces are unusually big. Thus, in Sec. 5 we introduce a standard (one-sided) Hochschild complex of $U_q\mathfrak{n}_-$ with coefficients in the tensor product of the standard (one-sided) Verma modules and a standard Hochschild complex of the dual to $U_q\mathfrak{n}_-$ with coefficients in the tensor product of the contragradient representations. We introduce a homomorphism of the first standard complex to the second in terms of a suitable contravariant form. We construct a monomorphism of each of the standard complexes to the corresponding two-sided Hochschild complex and prove that these monomorphisms are quasi-isomorphisms, i.e., they induce isomorphisms of homology groups.

Thus, as a result of Secs. 2–5 we show that the homology groups of a discriminantal configuration with coefficients in a one-dimensional complex local system can be interpreted as the standard Hochschild cohomology of a suitable quantum group. This allows us to identify constructions in representation theory of quantum groups with geometric constructions for discriminantal configurations.

The theorem on quasi-isomorphism suggests that there might be a general construction of a standard (one-sided) Hochschild complex which is quasi-isomorphic to a given two-sided Hochschild complex with coefficients in the tensor product of two-sided modules.

The main body of Sec. 5 is devoted to the proof of the theorem on quasi-isomorphism and could be skipped under first reading. In Sec. 5.11 we give a review of the results of Secs. 2–5.

A discriminantal configuration depends on parameters. The space of parameters has the form of a complex space with deleted diagonal hyperplanes. Thus, the fundamental group of the parameter space is a pure braid group. The fundamental group of the parameter space has the monodromy representation in the homology groups of the discriminantal configuration. The homology groups of a discriminantal configuration were identified with the Hochschild cohomology of a quantum group in Secs. 2–5. We show that under this identification the monodromy representation is identified with the universal R-matrix representation of the braid group in the tensor product of representations of a quantum group. This theorem is proved in Sec. 8. In Sec. 7 we define R-matrix operators and the R-matrix representation of a braid group. In Sec. 6 we prove an auxiliary property of a discriminantal configuration which is basic for the proof of the monodromy theorem in Sec. 8. In Sec. 9 we discuss algebraic properties of the R-matrix, in particular, if the parameter of the quantum group is a root of unity.

In Sec. 10.1–10.3 we consider a weighted configuration of hyperplanes in a complex affine space and give a combinatorial construction of two finite dimensional complexes and a homomorphism between them. The first complex is defined in terms of the flags of the configuration and the spaces of the second are the graded pieces of the Orlik–Solomon algebra. We realize the complexes as subcomplexes of the holomorphic de Rham complex of the complement to the configuration with values in a suitable trivial line bundle with a holomorphic flat connection. The differential forms of these subcomplexes are called the hypergeometric differential forms associated with a configuration.

If the weights of a configuration are in general position, then these finite dimensional complexes compute the cohomology groups of the complement with coefficients in the sheaf of the horizontal sections of the connection.

Assume that a configuration of hyperplanes in a complex affine space is the complexification of a configuration in a real space. Then there are four finite dimensional complexes associated with this configuration. The first two are the combinatorial cell complexes constructed in Sec. 2, the second two are the complexes of hypergeometric forms constructed in Secs. 10.1–10.3. The integration of the differential forms over the cells defines a pairing of the first pair of complexes with the second pair of complexes. This pairing is called the hypergeometric pairing associated with a configuration. We study this pairing in Secs. 10.6–10.9.

Assume that we have a weighted configuration in an affine space and a projection of the space onto another affine space. Then each fiber has a weighted configuration induced by the initial one. The combinatorial structure of the fiber configuration is the same for almost all fibers. The exceptions lie over the discriminant, which is a collection of hyperplanes in the base of the projection. The complexes of hypergeometric forms of a fiber are defined combinatorially and do not depend on a fiber over the complement to the discriminant. In Sec. 10.5 we combinatorially construct a flat connection in the trivial bundle over the complement to the discriminant with the complex of hypergeometric forms as its fiber. We show in Sec. 10.10 that this combinatorial flat connection is realized as the differential equation describing the dependence of the hypergeometric integrals in the fiber on the parameters in the base.

In Sec. 11 we consider a Kac–Moody algebra $\mathfrak{g}$ associated with a complex symmetric matrix. Having a collection of highest weights we construct two complexes and a homomorphism between them. The first complex is the standard Lie algebra chain complex of the nilpotent subalgebra $\mathfrak{n}_- \subset \mathfrak{g}$ with coefficients in the tensor product of Verma modules. We introduce a Lie algebra structure on the dual to $\mathfrak{n}_-$ and an $\mathfrak{n}_-^*$-module structure on the tensor product of the contragradient modules to the Verma modules. The second complex is the standard Lie algebra chain complex of $\mathfrak{n}_-^*$ with coefficients in the tensor product of the contragradient modules. The homomorphism of the first complex to the second is defined by a suitable contravariant form. We introduce the Knizhnik–Zamolodchikov equation with values in these complexes.

We show in Sec. 12 that the two complexes of the hypergeometric differential forms of a weighted discriminantal configuration and a homomorphism between them constructed in Sec. 10 are identical with the two Lie algebra complexes and a homomorphism between them constructed in Sec. 11 for a suitable Kac–Moody Lie algebra. Moreover, under this identification the hypergeometric differential equation is identified with the Knizhnik–Zamolodchikov differential equation.

Thus, results of Secs. 10–12 allow us to identify constructions in the representation theory of Kac–Moody algebras with geometric constructions for discriminantal configurations.

In Secs. 12.2, 12.4, and 12.5 we discuss general properties of the hypergeometric pairing for discriminantal configurations as a pairing between objects of the representation theory of a Kac–Moody algebra and objects of the representation theory of the corresponding quantum group. In particular, we show the nondegeneracy of the pairing for generic values q of the parameter of the quantum group.

Secs. 13 and 14 are devoted to the hypergeometric pairing corresponding to the case of $U_q s\ell_2$ where $q = e^{2\pi i/\kappa}$ and κ is a natural number. The main results are Theorems 13.7.19 and 13.7.27 claiming that the hypergeometric pairing in this case is reduced to a nondegenerate pairing between the space, called the space of conformal blocks in the conformal field theory, and the "path subspace" in the tensor product of the corresponding $U_q s\ell_2$-modules. In particular, the nondegeneracy of this pairing shows that the monodromy representation of the Knizhnik–Zamolodchikov equation in the space of conformal blocks is the R-matrix representation in the "path subspace". To prove the result we discuss in Sec. 13.8 elements of the representation theory of the quantum double of $U_q\mathfrak{n}_- \subset U_q s\ell_2$.

In Sec. 15 we discuss how the constructions of the previous sections could be applied to studying homology groups of configurations with coefficients more general than complex one-dimensional and to studying homology groups of braid groups.

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