CHAPTER 1

INTRODUCTION

1.1. General

Evolution of physical systems, subject to suitable (constraint-free) internal and external forces and appropriate initial conditions, is often expected to be completely and uniquely determined by Newton's equations of motion (Ref. [1]). For an N-particle system with masses m_i ($i = 1, 2, \ldots, N$) and forces $\mathbf{F}_i$ acting on them, the dynamics is in general described by the set of second order ordinary differential equations

$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = \mathbf{F}_i \left(t, \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n, \frac{d\mathbf{r}_1}{dt}, \frac{d\mathbf{r}_2}{dt}, \ldots, \frac{d\mathbf{r}_n}{dt} \right), \quad i = 1, 2, \ldots, n. \quad (1.1)$$

Here $\mathbf{r}_i$ is the position vector of the ith particle in an inertial frame of reference and Eq. (1.1) is subjected to the prescribed $6N$ initial conditions $\mathbf{r}_i(0)$, $\frac{d\mathbf{r}_i}{dt}|_{t=0}$. It is implicitly assumed here that the initial position and velocity vectors of each particle can be accurately and simultaneously provided. By solving the system of $3N$-second order coupled ordinary differential equations (1.1) along with the initial conditions, one can expect that the future of the system can be completely predicted with any required precision. Such a possibility, in fact, led Laplace to imagine that for a super-intelligence '*nothing could be uncertain and the future, as the past, would be present to its eyes*' (Ref. [2]).

1.2. Nonlinearity and Chaotic Motions

In spite of the impressive conceptual foundation, there are obvious limitations in Newton's description and so in Laplace's dictum:

(i) Presence of external random forces/fluctuations can always introduce a kind of indeterminacy, which is a statistical phenomenon.
(ii) Quantum effects can often lead to indeterminacy, dictated by the Heisenberg's uncertainty relations, due to our limitations in the simultaneous physical measurement of canonically conjugate dynamical variables such as position and momentum.

During recent times, it has been realized that a third kind of limitation can occur in Newton's description of evolution of even simple dynamical systems *when nonlinearity is present* (Refs. [2–10]) in a suitable form. It is true that in order to predict the future behaviour of a physical system accurately, leaving aside the limitations posed by statistical and quantum effects, one has only to solve the initial value problem of a system of deterministic differential equations of the form (1.1).

However, when nonlinear forces are present, the system can in general admit very complex motions and the associated equation of motion cannot in general be exactly integrated. As a result often one has to take recourse to numerical integration of the underlying differential equations. Then any small inaccuracy in the prescription of the initial state or round-off errors at any point or stage of the numerical calculation can build up exponentially fast to make the system deviate appreciably from the actual intended state in a finite time interval. One says that there is an *exponential divergence* of nearby trajectories. There is nothing much we can do about this indeterminism, because however accurate and fast calculating machines we are able to produce, there can still be some small error at some stage of the calculation which will multiply fast in a finite amount of time. This is a fact which we have to live with when nonlinearity is present in an appropriate form.

One might wonder whether the above effect is a mere mathematical or computational artifact or whether it has anything to do with the physical behaviour of the system at all. In fact, one knows now very well that an immediate physical realization of the above exponential divergence of nearby trajectories is the extreme *sensitiveness* of the behaviour of the system on initial conditions. This fact was after all anticipated by H. Poincaré in his celebrated analysis of celestial mechanics (Ref. [2]) itself. Any infinitesimal fluctuation at any time during the evolution of a system can in a finite time lead to a physically realizable effect, the so called 'butterfly effect' as termed by Lorenz (Refs. [2, 4]): "As small a perturbation as a butterfly fluttering its wings somewhere in the Amazons can in a few days time grow into a tornado in Texas".

The above type of complex behaviour admitted by appropriate nonlinear systems, exhibiting extreme sensitiveness to initial conditions, is termed as *chaotic motion* or simply *chaos*, which is a pure manifestation of nonlinearity. Of all possible nonlinear systems, especially of importance are dissipative and conservative systems. There are characteristic differences between the chaos exhibited by these two categories.

1.3. Dissipative and Conservative Nonlinear Systems

(i) *Dissipative systems*: The time evolution of these systems contracts volume in the phase-space (the abstract space of state variables) and consequently the trajectories approach asymptotically either a chaotic or a non-chaotic attractor. The latter may be a fixed point, a periodic limit cycle or a quasiperiodic attractor. These and the chaotic attractors are bounded regions of phase-space towards which the trajectory of the system, represented as a curve, converges in the course of long-time evolution (Refs. [5–10])). Bifurcation or qualitative changes of periodic attractors can occur leading to more complicated and chaotic structures, as a control parameter is varied.

The chaotic attractor is, typically, neither a point nor a curve but a geometrical structure having a self-similar and fractal (often multifractal) nature. Such chaotic attractors are called *strange attractors*. Many physically and biologically important nonlinear dissipative systems, both in low and high dimensions, exhibit strange attractors and chaotic motions. Typical examples are the various damped and driven nonlinear oscillators (Refs. [5–16]), the Lorenz system (Ref. [4]), the Brusselator model (Refs. [13, 14]), the Bonhoeffer–van der Pol oscillator (Ref. [17]), the piecewise linear electronic circuits (Refs. [18–20]), and so on.

(ii) *Conservative or Hamiltonian systems*: Nonlinear systems of *conservative* or *Hamiltonian type* also often exhibit chaotic motions (Refs. [21–23]). But here the phase-space volume is conserved and so no strange attractor is exhibited. Instead, chaotic orbits tend to visit all parts of a subspace of the phase-space uniformly. The dynamics of a nonintegrable conservative system is typically neither entirely regular nor entirely irregular, but the phase-space consists of a complicated mixture of regular and irregular components. In the regular region the motion is quasiperiodic and the orbits lie on tori while in the irregular regions the motion appears to be chaotic but they are not attractive in nature. Typical examples include coupled nonlinear oscillators, the Henon–Heiles system, the anisotropic Kepler problem, and so on. Similarly, the quantum manifestations of such Hamiltonian chaos, namely quantum chaos (Refs. [21–24]), are also of great physical relevance. However, this book does not deal with the Hamiltonian chaos aspects but concentrates only on dissipative systems.

It should be emphasized here that not every nonlinear dynamical system as a rule exhibits chaotic motions. Even very complicated nonlinear systems can sometimes exhibit very coherent and ordered structures such as solitons, dromions, instantons, and so on (Refs. [25, 26]). When a given nonlinear dynamical system will exhibit chaotic behaviour and when it will admit coherent and ordered behaviour are intricate mathematical problems, the understanding of which will constitute an important area of future investigations in the field. Some possible lines of thinking include the Painlevé singularity structure analysis (Refs. [27, 28]), investigation of generalized symmetries (Refs. [28–31]), Melnikov analysis (Refs. [10, 32]) and so on.

1.4. Bifurcations and Chaos-Controlling and Synchronization

In this book we will concentrate mainly on the chaotic motions exhibited by damped and driven nonlinear oscillator systems of interest in different fields of research and will illustrate the rich variety of bifurcations and chaos phenomenon exhibited by them. We will then also discuss how chaos can be controlled to regular motion by minimal efforts and finally the possible technological applications of it in secure communications through the concept of chaos synchronization. As a prelude to these developments we will first consider the oscillations of simple linear and nonlinear systems in the next Chapter.