Chapter 1

Classical Theory of Absolute Stability

1.1 Feedback Control Equation and its Transfer Function

In this chapter we shall consider the differential equation

$$\dot{x} = Px + q\varphi(t, r^*x), \qquad x \in \mathbf{R}^n, \tag{1.1.1}$$

where P, q, r are constant real matrices of orders $n\times n$, $n\times m$ and $n\times l$ respectively and $\varphi : \mathbf{R}_+ \times \mathbf{R}^l \to \mathbf{R}^m$ is continuous and locally Lipshitz continuous in the second argument. The asterisk denotes Hermitian conjugation (in particular, transposition in the case of real matrices and complex conjugation in the case of numbers). Such an equation in the system theory is called a *feedback control equation*. It may be considered as an interconnection of a linear unit [block] (1.1.2) (with input ξ and output $(-\sigma)$) and a nonlinear block (1.1.3):

$$\dot{x} = Px + q\xi, \qquad \sigma = r^*x, \tag{1.1.2}$$

$$\xi = \varphi(t, \sigma). \tag{1.1.3}$$

We shall call n-vector x the *state of the nonlinear system*.

The universal characteristic of the linear part is its *transfer function* from the input ξ to the output $(-\sigma)$

$$\chi(p) = r^*(P - pI_n)^{-1}q \tag{1.1.4}$$

where $p = \tau + i\omega$ ($i = \sqrt{-1}$) is a complex variable. Note that the definition of χ is coordinate free, i.e. after any change of coordinates $y = Sx$ in (1.1.1) (S is a non-singular $n\times n$-matrix) χ remains the same function. Note also that $\chi(p)$ may be received by means of Laplace transform with complex parameter p. Indeed, if we assume that $x(0) = 0$ and take the formal Laplace transform of both sides of the equations of (1.1.2) we receive for the Laplace transforms $\widetilde{\sigma}$ and $\widetilde{\xi}$ of the functions σ and ξ respectively the relation

$$\widetilde{\sigma}(p) = -\chi(p)\widetilde{\xi}(p).$$

The function $\omega \mapsto \chi(i\omega)$ ($\omega \in \mathbf{R}$) is the *frequency response* of (1.1.2).

Let us now consider the scalar case $m = l = 1$. In this situation $\chi(p)$ is scalar rational function

$$\chi(p) = \frac{\alpha(p)}{\delta(p)}, \tag{1.1.5}$$

where $\delta(p) = \det(pI_n - P)$ and $\alpha(p)$ is a polynomial of degree less than n. In this case the geometric image of frequency response is often used. It is depicted on the complex plane of $z = \zeta + i\nu$. For every $\omega \in \mathbf{R}$ a dot with the coordinates $\zeta = \operatorname{Re}\chi(i\omega)$, $\nu = \operatorname{Im}\chi(i\omega)$ is plotted on complex plane $\{z\}$. The set of these dots (provided that ω takes all the values from $-\infty$ to $+\infty$) is the *hodograph of frequency response*. In Fig. 1.1.1 the hodograph of

$$\chi(i\omega) = \frac{1}{a(i\omega)^2 + b(i\omega) + 1} \qquad (a,\, b > 0)$$

is shown.

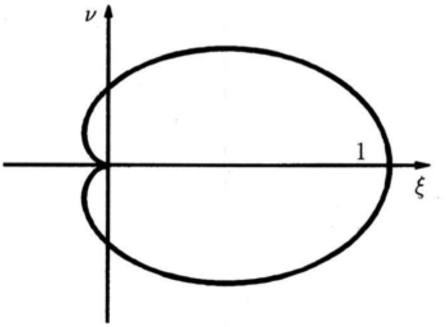


Fig. 1.1.1. The hodograph of $\chi(i\omega) = \{a(i\omega)^2 + b(i\omega) + 1\}^{-1}$, $(a,\, b > 0)$.

Stability theorems for feedback control equation are often formulated in terms of the frequency responce of its linear part. Such assertions are called frequency-domain theorems or frequency-domain criteria. This chapter presents the tool which gives the opportunity to obtain frequency-domain stability criteria.

1.2 Controllability. Observability. Kalman Duality Principle

The notions of controllability and observability are considered in detail in the books [Voronov 1979, Kalman *et al.* 1971, Gelig *et al.* 1978, Popov 1966]. The exposition of this section is close to [Gelig *et al.* 1978].

Consider a linear stationary block

$$\dot{x} = Px + q\xi, \quad x \in \mathbf{R}^n \tag{1.2.1}$$

$$\sigma = r^* x. \tag{1.2.2}$$

Here constant matrices P, q and r (of oder $n \times n$, $n \times m$ and $n \times l$ respectively) may be both real and complex. The input $\xi = \xi(t)$ of the linear block is supposed to be continuous.

Definition 1.2.1 *The linear block (1.2.1), (1.2.2) (resp. the pair (P, q)) is called* controllable *if for any fixed moments $t_1 < t_2$ and any fixed x', $x'' \in \mathbf{R}^n$ there exists an input $\xi(t)$ which transfers $x(t_1) = x'$ into $x(t_2) = x''$.*

In other words the linear block is controllable if by means of appropriate input it may be transferred from any given state into any other state at a given time period.

There are several equivalent assertions about controllability. We shall present here three of them. We shall need a designation

$$Q(p) = \delta(p)(pI_n - P)^{-1}q = Q_1 p^{n-1} + \cdots + Q_{n-1}p + Q_n,$$

where $\delta(p) = \det(pI_n - P)$ and Q_k $(k = 1, \ldots, n)$ are $n \times m$-matrices.

Theorem 1.2.1 *The following assertions are equivalent and each of them is equivalent to controllability of the liner block:*

1. *If S is a non-singular $n \times n$-matrix then matrices $\widehat{P} = S^{-1}PS$ and $\widehat{q} = S^{-1}q$ could not be of the form*

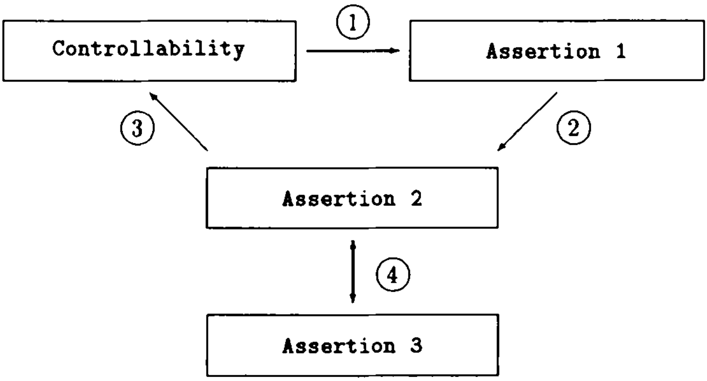
$$\widehat{P} = \left\| \begin{matrix} P_{11} & P_{12} \\ 0 & P_{22} \end{matrix} \right\|, \qquad \widehat{q} = \left\| \begin{matrix} q_1 \\ 0 \end{matrix} \right\|$$

with P_{11} being a $k \times k$-matrix $(0 \leqslant k < n)$ and q_1 being $k \times m$-matrix.

2. *The $n \times mn$-matrix $\|q, Pq, \ldots, P^{n-1}q\|$ has the rank of n.*

3. *The $n \times mn$-matrix $\|Q_1, Q_2, \ldots, Q_n\|$ has the rank of n.*

Proof. We shall confine ourselves to the case $m = 1$ and carry out the following scheme



where "$A \to B$" means "the statement A implies the statement B". According to this scheme we divide the proof into four parts.

1. Let the linear block be controllable. Suppose, Assertion 1 is false. Then by means of the change $x = S\widehat{x}$ with $\widehat{x} = \mathrm{col}(y, z)$ ($y \in \mathbf{R}^k$, $z \in \mathbf{R}^{n-k}$) equation (1.2.1) may be represented in the form

$$\begin{cases} \dot{y} = P_{11}y + P_{12}z + q_1\xi(t), & y \in \mathbf{R}^k,\ z \in \mathbf{R}^{n-k} \\ \dot{z} = P_{22}z. \end{cases} \tag{1.2.3}$$

From the second equation of (1.2.3) we have $z(t) = \exp(P_{22}t)z(0)$. So if $z(0) = 0$ then $z(t) \equiv 0$. It is clear now that we cannot receive any pre-assigned vector $\widehat{x}(t)$ at the expense of the input $\xi(t)$. Thus the pair (P, q) can not be controllable. This contradiction proves that Assertion 1 is true for a controllable block.

2. Let Assertion 1 be true. Suppose that Assertion 2 is false. Then

$$\mathrm{rank}\, \|q, Pq, \dots, P^{n-1}q\| = k < n.$$

Consequently there exists a non-singular $n \times n$-matrix S such that

$$S^{-1}\|q, Pq, \dots, P^{n-1}q\| = \left\| \begin{matrix} l_0 & \dots & l_{n-1} \\ 0 & \dots & 0 \end{matrix} \right\|$$

where l_j ($j = 0, \dots, n-1$) are $k \times 1$-matrices, k of them being linearly independent. Consider then $\widehat{P} = S^{-1}PS$ and $\widehat{q} = S^{-1}q$. The following equalities hold

$$\begin{aligned} \widehat{P}\widehat{q} &= S^{-1}PSS^{-1}q = S^{-1}Pq, \\ \widehat{P}^2\widehat{q} &= \widehat{P}(\widehat{P}\widehat{q}) = S^{-1}PSS^{-1}Pq = S^{-1}P^2q, \\ \dots & \quad \dots \\ \widehat{P}^{n-1}\widehat{q} &= S^{-1}P^{n-1}q. \end{aligned}$$

Hence

$$\|\widehat{q}, \widehat{P}\widehat{q}, \dots, \widehat{P}^{n-1}\widehat{q}\| = S^{-1}\|q, Pq, \dots, P^{n-1}q\| = \left\| \begin{matrix} l_0 & \dots & l_{n-1} \\ 0 & \dots & 0 \end{matrix} \right\|. \tag{1.2.4}$$

The matrices $\widehat{P}$ and $\widehat{q}$ can be represented as follows

$$\widehat{P} = \left\| \begin{matrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{matrix} \right\|, \qquad \widehat{q} = \left\| \begin{matrix} q_1 \\ q_2 \end{matrix} \right\|,$$

where P_{11} and q_1 are $k \times k$ and $k \times 1$-matrices respectively. It follows from (1.2.4) that $q_1 = l_0$, $q_2 = 0$. Now let us equate the corresponding columns of matrices from the left side and the right side of (1.2.4). We have

$$\begin{aligned}
\widehat{P}\widehat{q} &= \begin{Vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{Vmatrix} \begin{Vmatrix} l_0 \\ 0 \end{Vmatrix} = \begin{Vmatrix} P_{11}l_0 \\ P_{21}l_0 \end{Vmatrix} = \begin{Vmatrix} l_1 \\ 0 \end{Vmatrix}, \\
\widehat{P}^2\widehat{q} &= \widehat{P}(\widehat{P}\widehat{q}) = \begin{Vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{Vmatrix} \begin{Vmatrix} l_1 \\ 0 \end{Vmatrix} = \begin{Vmatrix} P_{11}l_1 \\ P_{21}l_1 \end{Vmatrix} \\
&= \begin{Vmatrix} P_{11}{}^2 l_0 \\ P_{21}P_{11}l_0 \end{Vmatrix} = \begin{Vmatrix} l_2 \\ 0 \end{Vmatrix}, \\
\dots & \quad \dots \\
\widehat{P}^{n-1}\widehat{q} &= \widehat{P}(\widehat{P}^{n-2}\widehat{q}) = \begin{Vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{Vmatrix} \begin{Vmatrix} l_{n-2} \\ 0 \end{Vmatrix} = \begin{Vmatrix} P_{11}l_{n-2} \\ P_{21}l_{n-2} \end{Vmatrix} \\
&= \begin{Vmatrix} P_{11}{}^{n-1} l_0 \\ P_{21}P_{11}{}^{n-2}l_0 \end{Vmatrix} = \begin{Vmatrix} l_{n-1} \\ 0 \end{Vmatrix}.
\end{aligned}$$

Hence

$$P_{21}l_0 = 0, \quad P_{21}l_1 = 0, \quad \dots, \quad P_{21}l_{n-2} = 0; \tag{1.2.5}$$

$$P_{11}l_0 = l_1, \quad P_{11}{}^2 l_0 = l_2, \quad \dots, \quad P_{11}{}^{n-1}l_0 = l_{n-1}; \tag{1.2.6}$$

$$P_{21}P_{11}l_0 = 0, \quad P_{21}P_{11}{}^2 l_0 = 0, \quad \dots, \quad P_{21}P_{11}{}^{n-2}l_0 = 0. \tag{1.2.7}$$

It is clear from (1.2.5) that matrix P_{21} annuls vectors $l_0, \dots, l_{n-2}$. Let us prove that P_{21} annuls l_{n-1} as well. Note firstly that any matrix satisfies its characteristic equation. So for $k \times k$-matrix P_{11} we have

$$P_{11}^k = \sum_{i=0}^{k-1} \alpha_i P_{11}^i,$$

where α_i are coefficients of the characteristic polynomial of P_{11}. From (1.2.6) and (1.2.7) it follows that

$$P_{21}l_{n-1} = P_{21}P_{11}^{n-1}l_0 = \sum_{i=0}^{k-1} \alpha_i P_{21}P_{11}^{r_i}l_0 \qquad (r_i = i + n - 1 - k \leqslant n - 2).$$

Hence $P_{21}l_{n-1} = 0$. So P_{21} annuls all vectors l_i $(i = 0, \dots, n-1)$. As k of them are linear independent it follows that P_{21} is a null matrix, which contradicts assumption that Assertion 1 is true. This contradiction proves that Assertion 2 is true.

3. Suppose Assertion 2 is true. Let us represent the solution of (1.2.1) in the Cauchy form

$$x(t_2) = e^{P(t_2-t_1)}x(t_1) + \int_{t_1}^{t_2} e^{P(t_2-\tau)}q\xi(\tau)\,d\tau. \tag{1.2.8}$$

Let $x(t_1) = x'$. Note that if the equation

$$\int_{t_1}^{t_2} e^{P(t_2-\tau)}q\xi(\tau)\,d\tau = d \tag{1.2.9}$$

is resolvable with respect to $\xi(t)$ for an arbitrary n-vector d then there always exits the input $\xi(t)$ such that $x(t_2) = x'' \in \mathbf{R}^n$. Let us consider equation (1.2.9). We shall seek $\xi(t)$ in the form

$$\xi(t) = \left(e^{P(t_2-t)}q\right)^* z = q^* e^{P^*(t_2-t)} z$$

with a certain n-vector z. Then (1.2.9) will take the form

$$Rz = d \tag{1.2.10}$$

where

$$R = \int_{t_1}^{t_2} \left(e^{P(t_2-\tau)}q\right)\left(e^{P(t_2-\tau)}q\right)^* \mathrm{d}\tau. \tag{1.2.11}$$

It is enough to prove now that $\det R \neq 0$. Suppose the opposite, i.e. $\det R = 0$. Then there exists an n-vector $c \neq 0$, such that $Rc = 0$. Consequently, $c^* Rc = 0$. From (1.2.11) we have

$$c^* Rc = \int_{t_1}^{t_2} c^* \left(e^{P(t_2-\tau)}q\right)\left(e^{P(t_2-\tau)}q\right)^* c\,\mathrm{d}\tau = \int_{t_1}^{t_2} \left|c^* e^{P(t_2-\tau)}q\right|^2 \mathrm{d}\tau.$$

Hence

$$c^* e^{P(t_2-t)} q \equiv 0 \quad \text{for} \quad t \in [t_1, t_2]. \tag{1.2.12}$$

Let us consecutively differentiate (1.2.12) with respect to t. We shall receive

$$c^* e^{P(t_2-t)} P^k q \equiv 0 \qquad (k = 0, 1, 2, \ldots;\ t \in [t_1, t_2]).$$

For $t = t_2$ we have

$$c^* q = 0, \quad c^* P q = 0, \quad \ldots, \quad c^* P^{k-1} q = 0,$$

i.e. there exits an n-vector c which is orthogonal to all columns of the matrix $\|q, Pq, \ldots, P^{n-1}q\|$, which contradicts the assumption that Assertion 2 is true. This contradiction proves that $\det R \neq 0$, and consequently (1.2.9) is resolvable. Hence, the pair (P, q) is controllable.

4. Let us now prove that Assertion 2 and Assertion 3 are equivalent.

a). Suppose that Assertion 3 is not true. Then there exists a vector $z \in \mathbf{R}^n$, $z \neq 0$ such that $z^* Q_1 = 0, \ldots, z^* Q_n = 0$, i.e. $z^* Q(p) = 0$ whence $\delta(p) z^* (pI_n - P)^{-1} q \equiv 0$. As $\delta(p) \not\equiv 0$ we receive that $z^*(pI_n - P)^{-1} q \equiv 0$. Let us expand $(pI_n - P)^{-1}$. Then

$$\begin{aligned} z^*(pI_n - P)^{-1} q &= \frac{z^*}{p}\left(I_n + \frac{P}{p} + \cdots + \frac{P^{n-1}}{p^{n-1}} + \cdots\right) q = \\ &= \frac{z^* q}{p} + \cdots + \frac{z^* P^{n-1} q}{p^{n-1}} + \cdots \qquad (|p| \text{ is sufficiently large}). \end{aligned} \tag{1.2.13}$$

As the left side of (1.2.13) is equal to zero for all p then all the coefficients of the series in the right side are equal to zero as well. Consequently Assertion 2 is not true.

b). Suppose now that Assertion 2 is false. Then

$$z^*q = 0, \quad z^*Pq = 0, \quad \ldots, \quad z^*P^{n-1}q = 0,$$

where $z \in \mathbf{R}^n$, $z \neq 0$. Since matrix P satisfies its own characteristic equation we may affirm that

$$z^*P^iq = 0 \qquad (i = 0, 1, 2, \ldots).$$

Hence by of (1.2.13)

$$z^*(pI_n - P)^{-1}q = 0$$

for all $p \in \Gamma$ where Γ is a region of convergence of the matrix series from the right side of (1.2.13). Thus

$$z^*\delta(p)(pI_n - P)^{-1}q = 0 \qquad (p \in \Gamma)$$

or

$$z^*Q(p) = 0 \qquad (p \in \Gamma)$$

So Assertion 3 is false.

By a) and b) we have demonstrated that Assertions 2 and 3 are equivalent.

Thus Theorem 1.2.1 is proved. □

Definition 1.2.2 *The linear block (1.2.1), (1.2.2) (resp. the pair (P, r)) is called* observable *if for any $t_1 < t_2$ and any triples $\{x'(t), \xi'(t), \sigma'(t)\}$, $\{x''(t), \xi''(t), \sigma''(t)\}$ defined on $[t_1, t_2]$ and satisfying (1.2.1), (1.2.2) the relations $\xi' \equiv \xi''(t)$, $\sigma'(t) \equiv \sigma''(t)$ $(t \in [t_1, t_2])$ imply the equality $x'(t) \equiv x''(t)$ $(t \in [t_1, t_2])$.*

The two notions, controllability and observability, are connected closely with each other by means of Kalman duality theorem.

Theorem 1.2.2 (Kalman duality principle) *The pair (P, r) is observable if and only if the pair (P^*, r) is controllable.*

Proof. Let the triples $\{x'(t), \xi'(t), \sigma'(t)\}$ and $\{x''(t), \xi''(t), \sigma''(t)\}$ satisfy (1.2.1), (1.2.2) on interval $[t_1, t_2]$. Let $\xi'(t) = \xi''(t)$ and $\sigma'(t) = \sigma''(t)$ for $t \in [t_1, t_2]$. Let

$$x(t) = x'(t) - x''(t), \quad \xi(t) = \xi'(t) - \xi''(t), \quad \sigma(t) = \sigma'(t) - \sigma''(t).$$

Then

$$\frac{dx}{dt} = Px, \quad \sigma(t) = r^*x(t) = 0, \quad t \in [t_1, t_2]. \tag{1.2.14}$$

Thus

$$r^*e^{P(t-t_1)}x(t_1) = 0, \qquad t \in [t_1, t_2]$$

or

$$x^*(t_1)e^{P^*(t-t_1)}r = 0, \qquad t \in [t_1, t_2]. \tag{1.2.15}$$

Note that $x(t) = \exp[P^*(t - t_1)]x(t_1)$. So the identity $x(t) \equiv 0$ for $t \in [t_1, t_2]$ is equivalent to $x(t_1) = 0$. Thus the observability of (P, r) is equivalent to the fact that (1.2.15) implies $x(t_1) = 0$.

a). Suppose that (P, r) is not observable. Then there exists a nonzero n-vector $z = x(t_1)$ such that

$$z^* e^{P^*(t-t_1)} r \equiv 0, \qquad t \in [t_1, t_2]$$

or

$$\begin{aligned} z^* r \;+\; & \frac{z^* P^* r (t-t_1)}{1!} + \frac{z^*(P^*)^2 r (t-t_1)^2}{2!} + \cdots + \\ & + \frac{z^*(P^*)^{n-1} r}{(n-1)!}(t-t_1)^{n-1} + \cdots \equiv 0, \quad t \in [t_1, t_2]. \end{aligned} \tag{1.2.16}$$

Hence all the coefficients of the series are equal to 0, which contradicts the fact that the matrix

$$\|r, P^* r, \ldots, (P^*)^{n-1} r\|$$

has the rank of n. Thus (P^*, r) is not controllable.

b). Suppose (P^*, r) is not controllable. Then there exists a nonzero $z \in \mathbb{R}^n$ such that

$$z^* r = 0, \quad \ldots, \quad z^*(P^*)^{n-1} r = 0.$$

Since every matrix satisfies its characteristic equation we have

$$z^*(P^*)^i r = 0 \qquad (i = 0, 1, 2, \ldots).$$

Hence by virtue of (1.2.16)

$$z^* e^{P^*(t-t_1)} r = 0.$$

Thus the pair (P, r) is not observable.

The theorem is proved. □

The duality principle gives the opportunity to reformulate Theorem 1.2.1 in order to receive the necessary and sufficient conditions of observability. We shall need the designations

$$\begin{aligned} \delta'(p) &= \det(pI_n - P^*), \\ Q'(p) &= \delta'(p)(pI_n - P^*)^{-1} r = Q'_1 p^{n-1} + \cdots Q'_{n-1} p + Q'_n \end{aligned}$$

where Q'_j $(j = 1, \ldots, n)$ are $n \times \ell$-matrices.

Theorem 1.2.3 *The following assertions are equivalent and each of them is equivalent to the observability of linear block (1.2.1), (1.2.2).*

1. *If S is a non-singular $n \times n$-matrix then matrices $\widehat{P} = S^{-1} P S$ and $\widehat{r} = S^* r$ could not be of the form*

$$\widehat{P} = \left\| \begin{array}{cc} P_{11} & 0 \\ P_{12} & P_{22} \end{array} \right\|, \qquad \widehat{r} = \left\| \begin{array}{c} r_1 \\ 0 \end{array} \right\|$$

where P_{11} is a $k \times k$-matrix $(0 \leqslant k < n)$ and r_1 is a $k \times \ell$-matrix.

2. *The $n\times n\ell$-matrix $\|r, P^*r, \ldots, (P^*)^{n-1}r\|$ has the rank of n.*

3. *The $n\times n\ell$-matrix $\|Q'_1, \ldots, Q'_n\|$ has the rank of n.*

We present here one more criterion of controllability of (P, q) (and dual criterion of observability of (P, r)).

Theorem 1.2.4

1. *The pair (P, q) is controllable if and only if the rank of $n\times(n+m)$-matrix $\|P - pI, q\|$ is equal to n for any complex number p.*

2. *The pair (P, r) is observable if and only if the rank of $n\times(n+\ell)$-matrix $\|P^* - pI, r\|$ is equal to n for any complex number p.*

Proof. It is sufficient to prove the first part of the theorem.

Necessity. Let (P, q) be controllable. Suppose that the hypothesis

$$\operatorname{rank}\|P - pI, q\| = n$$

is false. Then there exist a n-vector $z \neq 0$ and a complex number p_0 such that

$$P^*z = p_0 z, \qquad q^*z = 0. \tag{1.2.17}$$

Let p be an arbitrary complex number different from any eigenvalue of P. Then

$$z^*(P - pI) = z^*(p_0 - p)$$

and

$$z^*(p_0^* - p)^{-1}q = z^*(P - pI)^{-1}q.$$

It follows from (1.2.17) that

$$z^*(P - pI)^{-1}q = 0.$$

Then

$$z^*Q(p) = 0.$$

Hence it follows (as p is an arbitrary number) that

$$z^*Q_1 = 0, \ldots, z^*Q_n = 0,$$

which contradicts Assertion 3 of Theorem 1.2.1. Consequently,

$$\operatorname{rank}\|P - pI, q\| = n.$$

Sufficiency. Suppose that (P, q) is not controllable. Then according to Assertion 1 of Theorem 1.2.1 there exists a non-singular matrix S such that matrices $\widehat{P} = S^{-1}PS$ and $\widehat{q} = S^{-1}q$ are of the form

$$\widehat{P} = \underbrace{\left\| \begin{matrix} P_{11} & P_{12} \\ 0 & P_{22} \end{matrix} \right\|}_{k} \}k, \qquad \widehat{q} = \left\| \begin{matrix} q_1 \\ 0 \end{matrix} \right\| \}k .$$

Suppose p_0 is an eigenvalue of P_{22}^* and $z_0 \neq 0$ is a corresponding eigenvector. Let us construct a vector $\widehat{z} = \mathrm{col}\| \underbrace{0, \ldots, 0}_{k}, z_0 \|$. Then

$$\widehat{P}^* \widehat{z} = p_0 \widehat{z}, \qquad \widehat{q}^* \widehat{z} = 0.$$

Let now $\widehat{z} = S^* z$. We have

$$P^* z = p_0 z, \qquad q^* z = 0.$$

It means that

$$\mathrm{rank}\, \| P - p_0^* I, q \| < n,$$

which contradicts the hypothesis of the theorem. Thus we have proved that (P, q) is controllable. □

1.3 The Transfer Function of Controllable and Observable Linear Block

Consider the linear block (1.2.1), (1.2.2) with real matrices P, q, r, scalar input and scalar output, i.e. let $m = \ell = 1$. In this case the transfer function from the input ξ to the output $(-\sigma)$ has the form

$$\chi(p) = r^*(P - pI_n)^{-1} q = \frac{\alpha(p)}{\delta(p)}, \tag{1.3.1}$$

where $\delta(p) = \det(pI - P)$ and the degree of $\alpha(p)$ less than n.

Definition 1.3.1 *The transfer function (1.3.1) is called* non-degenerate *if α and δ are co-prime polynomials.*

Theorem 1.3.1 *In scalar case of $m = \ell = 1$ the linear block (1.2.1), (1.2.2) is observable and controllable if and only if its transfer function is non-degenerate.*

Proof. Let us demonstrate at first that if the transfer function (1.3.1) is non-degenerate then the linear block (1.2.1), (1.2.2) is controllable. Suppose the opposite. Then Assertion 1 of Theorem 1.2.1 is false. It means that there exists a non-singular matrix S such that

$$\widehat{P} = S^{-1} P S = \left\| \begin{matrix} P_{11} & P_{12} \\ 0 & P_{22} \end{matrix} \right\|, \qquad \widehat{q} = S^{-1} q = \left\| \begin{matrix} q_1 \\ 0 \end{matrix} \right\|,$$

where P_{11} is a $k \times k$-matrix ($k < n$) and q_1 is a k-vector. Let us introduce $\widehat{r} = S^* r = \mathrm{col}\| r_1, r_2 \|$, where r_1 is a k-vector. Let us now calculate the function $\widehat{\chi}(p) = -\widehat{r}^*(pI_n - \widehat{P})^{-1} \widehat{q}$. Consider the vector $L(p) = (pI_n - \widehat{P})^{-1} \widehat{q} = \mathrm{col}\| L_1(p), L_2(p) \|$, where L_1 is a k-vector and L_2 is $(n-k)$-vector. We have

$$\left\| \begin{matrix} q_1 \\ 0 \end{matrix} \right\| = (pI_n - \widehat{P}) L(p) = \left\| \begin{matrix} (pI_k - P_{11}) L_1(p) - P_{12} L_2(p) \\ (pI_{n-k} - P_{22}) L_2(p) \end{matrix} \right\|,$$

i.e.

$$L_2(p) = 0, \qquad (pI_k - P_{11})L_1(p) = q_1.$$

So

$$L(p) = \left\| \begin{matrix} (pI_k - P_{11})^{-1}q_1 \\ 0 \end{matrix} \right\|.$$

Thus

$$\widehat{\chi}(p) = -\left\| \begin{matrix} r_1 \\ r_2 \end{matrix} \right\|^* \left\| \begin{matrix} (pI_k - P_{11})^{-1}q_1 \\ 0 \end{matrix} \right\| = -r_1^*(pI_k - P_{11})^{-1}q_1.$$

So the denominator of $\widehat{\chi}(p)$ is the polynomial $\det(pI_k - P_{11})$ with the degree k, which is less than the degree of $\delta(p) = \det(pI_n - P) = n$ (the latter degree is equal to n). But $\chi(p)$ is coordinate free. Consequently, $\chi(p) = \widehat{\chi}(p)$. Thus, polynomials $\alpha(p)$ and $\delta(p)$ have common roots and function $\chi(p)$ is not non-degenerate. Just in the same way one can prove that if the pair (P, r) is not observable then the transfer function is not non-degenerate. These contradictions prove the controllability of (P, q) and observability (P, r).

Let us prove now that if the pair (P, q) is controllable and the pair (P, r) is observable then the transfer function $\chi(p)$ is non-degenerate. We shall demonstrate for the purpose that if the pair (P, q) is controllable and there exists a complex number p_0 such that $\delta(p_0) = \alpha(p_0) = 0$ then the pair (P, r) is not observable. It follows from Assertion 3 of Theorem 1.2.1 that $Q(p_0) = Q_1 p_0^{n-1} + \cdots + Q_n \neq 0$. Really, otherwise the n-vectors $Q_1, Q_2, \ldots, Q_n$ would have been linearly dependent. Let us consider $\alpha(p_0)$ and transform it.

$$\alpha(p_0) = -r^*(p_0 I_n - P)^{-1} q\delta(p_0) = -r^*\delta(p_0)(p_0 I_n - P)^{-1}q = -r^*Q(p_0).$$

Since $\alpha(p_0) = 0$ then $r^*Q(p_0) = 0$. Let us show now that $Q(p_0)$ is an eigenvector of P corresponding to eigenvalue p_0, that is $PQ(p_0) = p_0 Q(p_0)$. Indeed

$$(p_0 I_n - P)Q(p_0) = (p_0 I_n - P)\delta(p_0)(p_0 I_n - P)^{-1}q = \delta(p_0)q = 0.$$

So,

$$\begin{aligned} PQ(p_0) &= p_0 Q(p_0), \\ Q^*(p_0)r &= 0. \end{aligned}$$

Hence

$$\begin{array}{lcll} Q^*(p_0)P^*r & = & p_0^* Q^*(p_0)r & = 0, \\ Q^*(p_0)(P^*)^2 r & = & p_0^* Q^*(p_0)P^*r & = 0, \\ \ldots & & \ldots & \\ Q^*(p_0)(P^*)^{n-1}r & = & p_0^* Q^*(p_0)(P^*)^{n-2}r & = 0. \end{array}$$

So,

$$Q^*(p_0)r = 0, \quad Q^*(p_0)P^*r = 0, \quad \ldots, \quad Q^*(p_0)(P^*)^{n-1}r = 0,$$

which contradicts Assertion 2 of Theorem 1.2.3 about observability. In the same way it is easy to demonstrate that if $\chi(p)$ is not non-degenerate and the pair (P, r) is observable then the pair (P, q) is not controllable.

1.4 Stable Linear Blocks

Let us consider a linear stationary block

$$\begin{cases} \dot{x} = Px + q\xi \\ \sigma = r^* x \end{cases} \tag{1.4.1}$$

with real $n\times n$-matrix P, $n\times m$-matrix q and $n\times \ell$-matrix r. We give here the treatment of stable linear block following the book [Voronov 1979].

Definition 1.4.1 *Linear stationary block (1.4.1) is called stable if all eigenvalues of matrix P have negative real parts, i.e. if matrix P is Hurwitzian.*

Further we shall use the designation

$$|T| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij}^2}$$

for $n\times m$-matrix $T = \{\, t_{ij} \,\}_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$.

Theorem 1.4.1 *If $\xi \equiv 0$ and the block (1.4.1) is stable then there exist positive numbers $\varkappa_0$, M_1, M_2 such that*

$$\big|x(t)\big| \leqslant M_1 e^{-\varkappa_0 t}|x_0|, \tag{1.4.2}$$

$$\big|\sigma(t)\big| \leqslant M_2 e^{-\varkappa_0 t}|x_0|, \tag{1.4.3}$$

where $x_0 = x(0)$ is the initial state of the block.

Proof. The solution of the Cauchy problem

$$\dot{x} = Px, \qquad x(0) = x_0$$

has the form

$$x(t) = e^{Pt}x_0.$$

Then

$$\big|x(t)\big| \leqslant \big|e^{Pt}\big|\,|x_0|, \qquad \big|\sigma(t)\big| \leqslant |r|\,\big|e^{Pt}\big|\,|x_0|. \tag{1.4.4}$$

Let λ_i $(i = 1,\dots,n)$ be eigenvalues of P and

$$\varkappa = \max_{i=1,\dots,n}\{\,\operatorname{Re}\lambda_i\,\}.$$

In our case $\varkappa < 0$. According the well-known estimates [Gantmakher 1988, Demidovich 1967]

$$\big|e^{Pt}\big| \leqslant M_0(\varepsilon)e^{(\varkappa+\varepsilon)t}$$

where ε is an arbitrary positive number. So we may fix a certain value of ε and obtain

$$\big|e^{Pt}\big| \leqslant M_0 e^{-\varkappa_0 t} \qquad (\varkappa_0 > 0), \tag{1.4.5}$$

where $\varkappa_0 < -\varkappa$. Hence and from (1.4.4) we have (1.4.2) and (1.4.3). □

Theorem 1.4.2 *If the block (1.4.1) is stable and the norm of $\xi(t)$ is bounded for all $t > 0$ then the norm of $\sigma(t)$ is bounded for all $t > 0$ as well.*

Proof. Suppose that $|\xi(t)| < \beta$ for all $t > 0$. Let us use for (1.4.1) the Cauchy form

$$\sigma(t) = r^* e^{Pt} x_0 + \int_0^t r^* e^{P(t-\tau)} q \xi(\tau)\, d\tau.$$

Hence the estimate follows:

$$\left|\sigma(t)\right| \leqslant \left|e^{Pt}\right| |r| |x_0| + \int_0^t |r| \left|e^{P(t-\tau)}\right| |q| \beta \, d\tau.$$

Thus

$$\left|\sigma(t)\right| \leqslant \delta_1 + \delta_2 \int_0^t e^{-\varkappa_0 (t-\tau)}\, d\tau$$

or

$$\left|\sigma(t)\right| \leqslant \delta_3$$

where by δ_i $(i = 1, 2, 3)$ unessential constants are denoted. The theorem is proved. □

Theorem 1.4.3 *Let the block (1.4.1) be stable. Let the outputs $\sigma_1(t)$ and $\sigma_2(t)$ correspond to the same input $\xi(t)$ but to different initial states, x_{01} and x_{02} respectively. Then the following estimate is true*

$$\left|\sigma_1(t) - \sigma_2(t)\right| \leqslant M_3 e^{-\varkappa_0 t} \qquad (M_3 > 0,\ \varkappa_0 > 0). \tag{1.4.6}$$

Proof. Every initial state x_{0i} $(i = 1, 2)$ generates the state $x_i(t)$. The corresponding output $\sigma_i(t)$ $(i = 1, 2)$ may be represented in the Cauchy form whence

$$\sigma_1(t) - \sigma_2(t) = r^* e^{Pt} (x_{01} - x_{02}).$$

With the help of (1.4.5) we obtain (1.4.6). Thus we have demonstrated that different outputs of the linear block corresponding to the same input are asymptotically indistinguishable. Theorem 1.4.3 is proved. □

According to Definition 1.4.1 the linear block is stable if and only if the characteristic polynomial of matrix P is Hurwitzian.

We adduce here several theorems about the Hurwitzian property of a polynomial.

Consider a polynomial

$$Q(p) = a_0 p^n + \cdots + a_n$$

with real coefficients a_k $(k = 0, \ldots, n)$ and $a_0 > 0$. First of all we give here a necessary condition for this property.

Theorem 1.4.4 *If the polynomial $Q(p)$ is Hurwitzian then all its coefficients are positive.*

Proof. Let $p_k = -\tau_k \pm i\omega_k$ ($k = 1,\dots,r$; $\tau_k > 0$) and $p_j = -\gamma_j$ ($\gamma_j > 0$; $j = 1,\dots,s$) be roots of $Q(p)$. Then

$$\begin{aligned} Q(p) &= a_0 \prod_{k=1}^{r} (p + \tau_k - i\omega_k)^{m_k} \cdot (p + \tau_k + i\omega_k)^{m_k} \prod_{j=1}^{s} (p + \gamma_j)^{n_j} \\ &= a_0 \prod_{k=1}^{r} (p^2 + 2\tau_k p + \tau_k^2 + \omega_k^2)^{m_k} \cdot \prod_{j=1}^{s} (p + \gamma_j)^{n_j} \end{aligned}$$

where m_k ($k = 1,\dots,r$), n_j ($j = 1,\dots,s$) are multiplicities of corresponding roots. Since $a_0 > 0$ it is obvious that the theorem is proved. □

Remark 1.4.1. For the polynomial $Q(p)$ with $n = 2$ the hypothesis of the theorem is the sufficient condition to be Hurwitzian. If $n > 2$ it is not so.

Let us cite now two well-known criteria for the Hurwitzian property of $Q(p)$, an algebraic one and a geometrical one.

Let us introduce a matrix

$$A(p) = \begin{Vmatrix} a_1 & a_3 & a_5 & \dots & a_{2n-1} \\ a_0 & a_2 & a_4 & \dots & a_{2n-2} \\ 0 & a_1 & a_3 & \dots & a_{2n-3} \\ 0 & a_0 & a_2 & \dots & a_{2n-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{Vmatrix}$$

which is called Hurwitz matrix of $Q(p)$.

Theorem 1.4.5 (Routh–Hurwitz criterion) *The polynomial $Q(p)$ is Hurwitzian if and only if all main diagonal minors of its Hurwitz matrix $A(p)$ are positive:*

$$a_1 > 0, \qquad \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} > 0, \quad \dots, \quad \det A(p) > 0.$$

Beginning with [Hurwitz 1895] this criterion is exposed in any book which treats the stability of linear differential systems or the disposition of roots of polynomials on the complex plane.

Example 1.4.1. Let $Q(p) = p^3 + \alpha p^2 + \beta p + \gamma$. Then

$$A(p) = \begin{Vmatrix} \alpha & \gamma & 0 \\ 1 & \beta & 0 \\ 0 & \alpha & \gamma \end{Vmatrix}.$$

Consequently, necessary and sufficient conditions for $Q(p)$ to be Hurwitzian are $\alpha > 0$, $\alpha\beta > \gamma$, $\gamma(\alpha\beta - \gamma) > 0$ or $\alpha > 0$, $0 < \gamma < \alpha\beta$.

1.4.1 Hermite–Michailov Criterion

This geometrical criterion appeared in paper [Michailov 1938]. It has been exposed in various monographs. See for example [Demidovich 1967, Blaquiere 1966, Postnikov 1981].

Let us construct the hodograph of $Q(i\omega)$ on the plane $z = \zeta + i\nu$ ($i = \sqrt{-1}$):

$$\begin{cases} \zeta = \operatorname{Re} Q(i\omega), \\ \nu = \operatorname{Im} Q(i\omega) \end{cases} \qquad (-\infty < \omega < +\infty).$$

Here $\operatorname{Re} Q(i\omega)$ and $\operatorname{Im} Q(i\omega)$ are real and imaginary parts of the complex value $Q(i\omega)$ respectively. Let $M(\omega)$ be a point on the hodograph. It is evident that if $Q(p)$ has pure imaginary roots then the hodograph goes through the origin. Suppose that $Q(p)$ has no pure imaginary roots.

Let us determine the angle Δ of counterclockwise rotation of the vector $\overline{OM}(\omega)$ on the plane $\zeta O\nu$ provided ω changes from $-\infty$ to $+\infty$. Consider the factorization of the polynomial

$$Q(p) = a_0 \prod_{k=1}^{r} (p - \tau_k - i\omega_k) \cdot (p - \tau_k + i\omega_k) \cdot \prod_{j=1}^{s} (p - \gamma_j) \tag{1.4.7}$$

where r is the number of complex-conjugate roots and s is the number of real roots of $Q(p)$ (every root is counted according to its multiplicity). The angle Δ is the increment of the function[a] $\operatorname{Arg} Q(i\omega)$ along the hodograph when ω changes from $-\infty$ to $+\infty$. In virtue of factorization (1.4.7) the increment of $\operatorname{Arg} Q(i\omega)$ is the sum of increments of every co-factor of (1.4.7). It is not difficult to see that the increment of a co-factor $(i\omega - \gamma_i)$ is $(-\pi)$ for $\gamma_j > 0$ and π for $\gamma_j < 0$. The increment of a pair $(i\omega - \tau_k - i\omega_k)(i\omega - \tau_k + i\omega_k)$ is 2π for $\tau_k < 0$ and (-2π) for $\tau_k > 0$. As a result

$$\Delta = \pi(n - 2m) \tag{1.4.8}$$

where m is a number of roots of $Q(p)$ with positive real part.

The opposite assertion is also true. That is, if the increment of the angle of rotation Δ is represented by (1.4.8) then m is a number of roots of $Q(p)$ with positive real part. Consequently, the following assertion is true.

Theorem 1.4.6 (Hermite–Michailov criterion) *Suppose the polynomial $Q(p)$ has no pure imaginary roots. Then it is Hurwitzian if and only if*

$$\Delta = \pi n.$$

The hodograph of $Q(i\omega)$ is symmetric with respect to $O\zeta$. So it is sufficient to consider it for $\omega > 0$. It begins at $\omega = 0$ on the real positive semi-axis. According

[a] The function $\operatorname{Arg} z$ is a certain continuous branch of many-valued function $\arg z \pm 2\pi k$ ($k \in \mathbf{Z}$) where $\arg z \in (-\pi, \pi]$ and is the principal value of the argument.

to Theorem 1.4.6 the polynomial $Q(p)$ is Hurwitzian if and only if the hodograph envelopes the origin counterclockwise passing successively through n quadrants (where n is the degree of $Q(p)$), under ω changing from 0 to $+\infty$.

Fig. 1.4.1 [Yurevich 1975] shows hodographs of Hurwitzian polynomials with the degrees from 2 to 5.

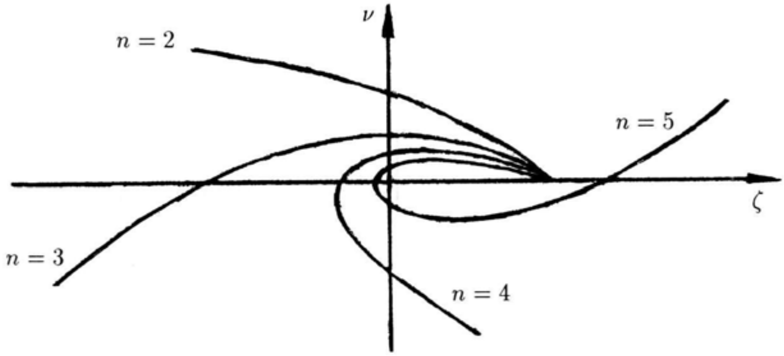


Fig. 1.4.1. Hodographs of Hurwitzian polynomials with the degrees from 2 to 5.

1.5 Stabilizability of Linear Block

Definition 1.5.1 *The linear block (1.2.1), (1.2.2) (resp. the pair (P, q)) is called* stabilizable *if there exists an $n\times m$-matrix S such that $P + qS^*$ is Hurwitzian.*

A stabilizable pair may be not controllable. For example, the pair (P, q) with Hurwitzian matrix P and zero vector q is stabilizable, but obviously it is not controllable. On the other hand if the pair is controllable it is stabilizable. This fact is stated by the following assertion.

Theorem 1.5.1 *The controllable pair (P, q) is stabilizable. More than that at the expense of matrix S one can achieve that polynomial $\delta_0(p) = \det(pI_n - P - qS^*)$ should coincide with a fixed polynomial $\Psi(p)$ of degree n. In addition, if P, q and coefficients of $\Psi(p)$ are real then the matrix S may be chosen real.*

We shall need an auxiliary assertion.

Lemma 1.5.1 (Schur lemma) *Let A, B, C and D be respectively $n\times n$-, $n\times m$-, $m\times n$- and $m\times m$-matrices. If $\det A \neq 0$ then*

$$\det\begin{Vmatrix} A & B \\ C & D \end{Vmatrix} = \det A \cdot \det(D - CA^{-1}B). \tag{1.5.1}$$

If $\det D \neq 0$ *then*

$$\det\begin{Vmatrix} A & B \\ C & D \end{Vmatrix} = \det D \cdot \det(A - BD^{-1}C). \tag{1.5.2}$$

Proof. The following equalities are always true:

a) $\left\| \begin{matrix} A & B \\ C & D \end{matrix} \right\| \cdot \left\| \begin{matrix} I_n & Q \\ 0 & I_m \end{matrix} \right\| = \left\| \begin{matrix} A & AQ+B \\ C & CQ+D \end{matrix} \right\|,$

b) $\left\| \begin{matrix} A & B \\ C & D \end{matrix} \right\| \cdot \left\| \begin{matrix} I_n & 0 \\ Q & I_m \end{matrix} \right\| = \left\| \begin{matrix} A+BQ & B \\ C+DQ & D \end{matrix} \right\|.$

Here Q is an arbitrary $n \times n$-matrix.

In the case of $\det A \neq 0$ we use the equality a) and put $Q = -A^{-1}B$. Then

$$\det \left\| \begin{matrix} A & B \\ C & D \end{matrix} \right\| = \det \left\| \begin{matrix} A & 0 \\ C & D - CA^{-1}B \end{matrix} \right\| = \det A \cdot \det(D - CA^{-1}B).$$

If $\det D \neq 0$ we use the equality b) and put $Q = -D^{-1}C$. Then

$$\det \left\| \begin{matrix} A & B \\ C & D \end{matrix} \right\| = \det \left\| \begin{matrix} A - BD^{-1}C & B \\ 0 & D \end{matrix} \right\| = \det D \cdot \det(A - BD^{-1}C).$$

The lemma is proved. □

Corollary 1.5.1 *If b and c are $n \times m$-matrices then*

$$\det(I_m + c^* b) = \det(I_n + bc^*). \tag{1.5.3}$$

If $m = 1$ then

$$\det(I_n + bc^*) = 1 + c^* b. \tag{1.5.4}$$

Proof. Formula (1.5.3) follows from Schur lemma if $A = I_n$, $D = I_m$, $B = -b$, $C = c^*$. □

Proof of Theorem 1.5.1. We confine ourselves to the case $m = 1$. The full proof can be found in [Gelig *et al.* 1978]. Let

$$\Psi(p) = p^n + \Psi_1 p^{n-1} + \cdots + \Psi_n.$$

We are going to find a vector s such that $\delta_0(p) = \Psi(p)$. For the purpose we express $\delta_0(p)$ in terms of $\delta(p) = \det(pI_n - P)$. Let

$$\delta(p) = p^n + b_1 p^{n-1} + \cdots + b_n.$$

Remind that

$$Q(p) = \delta(p)(pI_n - P)^{-1} q = Q_1 p^{n-1} + \cdots + Q_n.$$

We have in virtue of Corollary 1.5.1

$$\begin{aligned} \delta_0(p) &= \det(pI_n - P - qs^*) = \det(pI_n - P)\det\big(I_n - (pI_n - P)^{-1} qs^*\big) \\ &= \delta(p)\big(1 - s^*(pI_n - P)^{-1} q\big) \\ &= \delta(p)\left[1 - s^* \frac{Q(p)}{\delta(p)}\right] = \delta(p) - s^* Q(p). \end{aligned}$$

Let us demonstrate now that the equation

$$\delta(p) - s^*Q(p) = \Psi(p) \tag{1.5.5}$$

is solvable for any $\Psi(p)$. Let us equate to each other the coefficients of the polynomials in the left and in the right sides of (1.5.5):

$$\begin{aligned} p^n &: \ 1 = 1, \\ p^{n-1} &: \ b_1 - s^*Q_1 = \Psi_1, \\ p^{n-2} &: \ b_2 - s^*Q_2 = \Psi_2, \\ \dots &\quad \dots \\ p^0 &: \ b_n - s^*Q_n = \Psi_n. \end{aligned}$$

Then

$$s^*Q_i = b_i - \Psi_i \qquad (i = 1, 2, \dots, n). \tag{1.5.6}$$

Since (P, q) is controllable the n-vectors Q_i $(i = 1, \dots, n)$ are linearly independent. Consequently $\det \|Q_1, \dots, Q_n\| \neq 0$ and system (1.5.6) is satisfied by the unique n-vector s. In case P, q, Ψ_i $(i = 1, \dots, n)$ are real the n-vectors Q_i $(i = 1, \dots, n)$ are real as well and consequently n-vector s is also real. So to complete the proof it is sufficient to choose $\Psi(p)$ Hurwitzian. □

1.6 Stability of Linear Feedback System. Nyquist Criterion

Previous Sections 1.2–1.5 were devoted to linear part of system (1.1.2), (1.1.3). We are interested now in the whole system. Let us begin with the linear case, i.e. with the case when (1.1.3) has the form

$$\xi = \mu\sigma, \tag{1.6.1}$$

where μ is a real $m \times \ell$-matrix.

As a result the system under consideration may be described as follows

$$\dot{x} = (P + q\mu r^*)x. \tag{1.6.2}$$

Suppose we know the disposition of eigenvalues of P on the complex plane $\{ p = \xi + i\zeta \}$. Then we may formulate the problem about the disposition of eigenvalues of $P + q\mu r^*$ on the complex plane. Another problem is to choose matrix μ in such a way that prescribed disposition of eigenvalues of $P + q\mu r^*$ is guaranteed.

Let us dwell upon the scalar case (q and r are n-vectors and μ is a number). Consider the polynomial

$$M(p) = \det(P + q\mu r^* - pI_n).$$

Let $P_p = P - pI_n$. Then

$$M(p) = \det\{ P_p(I_n + \mu P_p^{-1} q r^*)\} = \det P_p \cdot \det(I_n + \mu P_p^{-1} q r^*).$$

It follows from Corollary 1.5.1 that

$$\det(I_n + \mu P_p^{-1} q r^*) = 1 + \mu r^* P_p^{-1} q.$$

Thus

$$\det(P + \mu q r^* - p I_n) = \mu \det P_p \cdot (\mu^{-1} + r^* P_p^{-1} q). \tag{1.6.3}$$

Let us investigate both sides of (1.6.3) relying on the geometric approach described in Section 1.4 in connection with Hermite–Michailov criterion. Suppose $\det(P+\mu q r^* - pI_n)$ has m_1 roots with positive real parts and $\det P_p$ has m_2 roots with positive real parts. (We assume that both polynomials have no pure imaginary roots). It follows from (1.4.8) that the rotation angle of the hodograph of $\det(P + \mu q r^* - pI_n)$ is

$$\Delta_1 = \pi(n - 2m_1)$$

and the rotation angle of the hodograph of P_p is

$$\Delta_2 = \pi(n - 2m_2)$$

(provided that ω changes from $-\infty$ to $+\infty$). The hodograph of the function

$$\mu^{-1} + r^*(P - i\omega I_n)^{-1} q$$

is the closed curve which starts (for $\omega = -\infty$) and finishes (for $\omega = +\infty$) at the same point $(\mu^{-1}, 0)$. Thus the increment of the rotation angle is $2\pi m_0$ where m_0 is an integer. So according to (1.6.3) we have

$$\pi(n - 2m_1) = \pi(n - 2m_2) + 2\pi m_0.$$

Hence

$$m_1 = m_2 - m_0.$$

It follows that with the help of m_0 we may "handle" the stability of (1.6.2).

The choice of m_0 is connected with the choice of μ^{-1}. Indeed, as the transfer function of the linear block (1.1.2) has the form

$$\chi(p) = r^*(P - pI_n)^{-1} q$$

one may represent m_0 as follows

$$m_0 = \frac{1}{2\pi} \operatorname{Arg}(\mu^{-1} + \chi(i\omega))\Big|_{\omega=-\infty}^{\omega=+\infty}.$$

Consequently, m_0 is a number of counterclockwise rotations of the function $\chi(i\omega)$ around $(-\mu^{-1}, 0)$ on the complex plane (while ω changes from $-\infty$ to $+\infty$). Thus with the help of the hodograph of $\chi(i\omega)$ on the complex plane we may choose the value of μ^{-1} in such a way that the stability of (1.6.2) is guaranteed. We have proved the following assertion.

Theorem 1.6.1 (Nyquist criterion [Nyquist 1932]) *Suppose matrix P has no pure imaginary eigenvalues and has m_p eigenvalues with positive real parts. Then the linear system (1.6.2) with the matrix $(P + \mu q r^*)$ is stable if and only if the hodograph of frequency response $\chi(i\omega) = r^*(P - i\omega I_n)^{-1} q$ makes m_p counterclockwise rotations around $(-\mu^{-1}, 0)$ when ω changes from $-\infty$ to $+\infty$.*

The theorem is also exposed in almost all books devoted to automatic control. See for example [Oldenburrg 1955, Louden 1954, Roitenberg 1978, Chezari 1964]

Example 1.6.1. Suppose P has two eigenvalues with positive real parts. The hodograph of $\chi(i\omega)$ is shown in Fig. 1.6.1. It is clear that if $\mu^{-1} \in (B, A)$ then system (1.6.2) is stable.

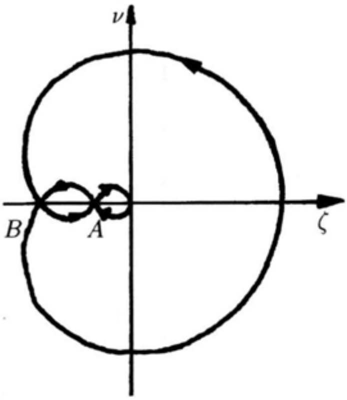


Fig. 1.6.1. The hodograph of $\chi(i\omega)$.

1.7 Lyapunov Stability. Direct Lyapunov Method

In this section we proceed to stability investigation of essentially nonlinear feedback systems. The foundation of this investigation is the Lyapunov stability theory. That is why first of all we adduce here basic definitions and theorems of this theory. The Lyapunov stability theory was started by the monograph [Lyapunov 1892]. It was then exposed and developed in numerous books [Bellman 1953, Malkin 1966, Malkin 1949, Chetaev 1990, Andronov *et al.* 1965, Barbashin 1967b, Barbashin 1970, Halanay 1966, La Salle & Lefschetz 1961, Lefschetz 1965, Letov 1962, Zubov 1957, Yoshizawa 1960, Hahn 1967, Bhatia & Szegö 1967, Coppel 1965, Kuntsevich & Lychak 1977, Lakshmikanthan 1978], [Matrosov *et al.* 1980, Voronov & Matrosov 1987, Michel & Miller 1977, Persidskij 1976, Siljak 1978, Rouche *et al.* 1977]. Rather full list of Lyapunov type theorems and bibliography devoted to Lyapunov stability theory may be found in [Rouche *et al.* 1977].

In this section we consider the ordinary differential equation

$$\dot{x} = f(t, x) \tag{1.7.1}$$

where $f : \mathbf{R}_+ \times \mathbb{R}^n \to \mathbf{R}^n$ is continuous and locally Lipschitz continuous in the second argument. For $t_0 \geqslant 0$ and $x_0 \in \mathbf{R}^n$ we denote by $x(t, t_0, x_0)$ the solution of (1.7.1) having $x(t_0, t_0, x_0) = x_0$. (Sometimes we shall use an abridged notation $x(t)$ when it does not lead to uncertainty). Note that the assumption on f guarantees that there exits a unique solution of (1.7.1) for any initial condition within the considered region.

The assumption on f guarantees also that the solutions of (1.7.1) are continuous with respect to initial conditions. This means that for a given solution $x(t, t_0, x_0)$ $(t \in [t_0, \theta))$ and arbitrary ε, $t_1 \geqslant t_0$ and $T \geqslant 0$ $(t_1 + T < \theta)$ there exists a number $\delta = \delta(\varepsilon, t_1, T)$ such that $|x(t_1, t_0, x_0) - x_1| < \delta$ and $t \in [t_1, t_1 + T]$ imply $|x(t, t_0, x_0) - x(t, t_1, x_1)| < \varepsilon$. Thus if two solutions of (1.7.1) start in the neighboring points they remain neighboring on a certain time interval.

Let us suppose now that every solution $x(t, t_0, x_0)$ of (1.7.1) may be continued on $[t_0, +\infty)$. In contrast to continuity with respect to initial conditions Lyapunov stability guarantees that if two solutions of (1.7.1) start in the neighboring points they remain neighboring on the whole semi-axis $[t_0, +\infty)$.

Definition 1.7.1 ([Lyapunov 1892]) *A solution $x(t, t_0, x_0)$ of (1.7.1) is* Lyapunov stable *if for each $t_1 \geqslant t_0$ and each $\varepsilon > 0$ there exists $\delta = \delta(t_1, \varepsilon)$ such that $|x(t_1, t_0, x_0) - x_1| < \delta$ and $t > t_1$ imply $|x(t, t_0, x_0) - x(t, t_1, x_1)| < \varepsilon$.*

Geometric illustration of Definition 1.7.1 is given in Fig. 1.7.1. It is clear that we may always choose $\delta \leqslant \varepsilon$. Thus graphs of all the solutions which are sufficiently close to $x(t, t_0, x_0)$ at $t = t_1 \geqslant t_0$ remain for all $t \geqslant t_1$ in as much as desired narrow ε-band surrounding the graph of $x(t, t_0, x_0)$.

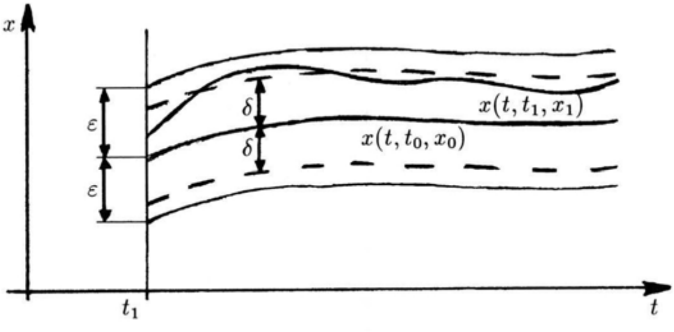


Fig. 1.7.1. Geometric illustration of Definition 1.7.1.

For autonomous systems

$$\dot{x} = f(x) \tag{1.7.2}$$

the geometric illustration is often presented in the phase space of the system. Let $n = 2$ and $f(c) = 0$ with $c = \mathrm{col}(c_1, c_2)$. Let the circle $C(r)$ with the radius r has its center in the point (c_1, c_2) of the plane $\mathbf{R}^2$. Then the solution $x(t) \equiv c$ is Lyapunov stable if any phase trajectory of (1.7.2) which begins in the circle $C(\delta)$ of sufficiently small radius $\delta > 0$ remains in the circle $C(\varepsilon)$ of arbitrary small radius ε (see Fig. 1.7.2).

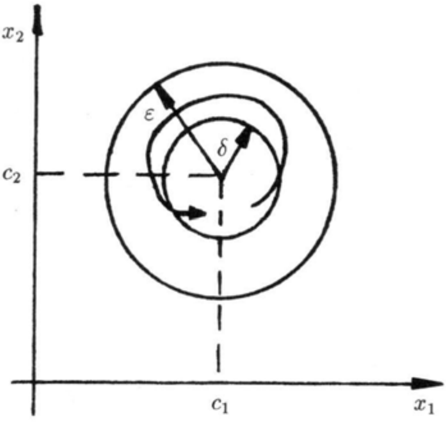


Fig. 1.7.2. Geometric illustration of Definition 1.7.1 for autonomous system (1.7.2).

Definition 1.7.2 ([Lyapunov 1892]) *A solution $x(t, t_0, x_0)$ of (1.7.1) is* asymptotically stable *if it is Lyapunov stable and if for each $t_1 > t_0$ there exists $\eta > 0$ such that $|x(t_1, t_0, x_0) - x_1| < \eta$ implies*

$$\lim_{t\to\infty} \left| x(t, t_0, x_0) - x(t, t_1, x_1) \right| = 0. \tag{1.7.3}$$

Definition 1.7.3 ([Barbashin & Krasovsky 1952]) *If a solution $x(t, t_0, x_0)$ is Lyapunov stable and if for each $t_1 \geqslant t_0$ and each $x_1 \in \mathbf{R}^n$ the relation (1.7.3) holds then $x(t, t_0, x_0)$ is said to be* globally asymptotically stable.

One of the most effective methods of global stability investigation is the direct or second method of A. M. Lyapunov. This method is an analytical realization of certain geometrical approaches of H. Poincaré [Poincaré 1881]. A central idea of H. Poincaré is the idea of a set of transversal surfaces (i.e. having no points of contact with the vector-field $f(t, x)$) which surround the solution under consideration and fill entirely a neighborhood of this solution (Fig. 1.7.3). Such surfaces may be intersected by the trajectories only in one direction.

In the Lyapunov direct method we consider continuous functions $V : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}$ which are nonincreasing along solutions of system (1.7.1). They are called Lyapunov

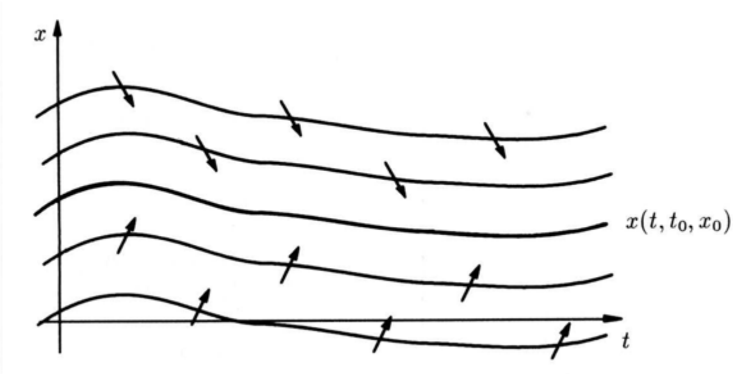


Fig. 1.7.3. Transversal surfaces surround a solution $x(t, t_0, x_0)$.

functions. Level surfaces of autonomous Lyapunov functions produce transversal surfaces for the vector-field (1.7.1).

Let us suppose that a function $V(t, x)$ is differentiable with respect to all its arguments. The function

$$\dot{V}_{(1.7.1)}(t, x) = \frac{\partial V(t, x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial V(t, x)}{\partial x_i} f_i(t, x) \tag{1.7.4}$$

where x_i and f_i $(i = 1, \dots, n)$ are components of x and f respectively, is called the derivative of V with respect to system (1.7.1).

We assume further that $f(t, 0) \equiv 0$. Note that we may always succeed in getting this relation. Indeed, suppose we are investigating the stability of the solution $\bar{x}(t)$ of (1.7.1). Let $y(t) = x(t) - \bar{x}(t)$. Since $\dot{\bar{x}}(t) = f(t, \bar{x}(t))$, we have

$$\dot{y} = \varphi(t, y),$$

where

$$\varphi(t, y) = f\big(t, y + \bar{x}(t)\big) - f\big(t, \bar{x}(t)\big).$$

Let us introduce a set $K_H = \{ x \in R^n, \ |x| < H \}$ and a class $A(K_H)$ of functions $W(x)$ which are defined and continuous for $x \in K_H$ and such that $W(0) = 0$ and $W(x) > 0$ for $x \neq 0$.

Theorem 1.7.1 ([Lyapunov 1892]) *Let $H > 0$. Suppose a continuously differentiable function $V(t, x)$ is defined for $t > 0$, $x \in K_H$ and the following assumptions are true:*

(i) $V(t, 0) = 0$, there exists $W(x) \in A(K_H)$ such that $V(t, x) \geqslant W(x)$,

(ii) the derivative of $V(t, x)$ with respect to (1.7.1) is nonpositive.

Then the trivial solution $x(t) \equiv 0$ of (1.7.1) is Lyapunov stable.

Proof. Let $0 < \varepsilon < H$. Consider the set $\{ x \mid |x| = \varepsilon \}$. Let α be the smallest value of $W(x)$ on this set. Such value exists since $\{ x \mid |x| = \varepsilon \}$ is a compact. It is obvious that $\alpha > 0$. Since $V(t,x)$ is continuous and $V(t_0, 0) = 0$ for arbitrary t_0, there exists a certain number δ such that

$$0 \leqslant V(t_0, x) < \alpha \quad \text{for } x \in K_\delta.$$

Let us consider the Cauchy problem for (1.7.1) with $x(t_0) = x_0 \in K_\delta$. Suppose that there exists a value t_1 such that $|x(t_1)| = \varepsilon$ though $|x(t)| < \varepsilon$ for $t \in [t_0, t_1)$. Then according to assumption (ii) we have

$$V\big(t_1, x(t_1)\big) \leqslant V\big(t_0, x(t_0)\big) < \alpha.$$

On the other hand assumption (i) guarantees that

$$V\big(t_1, x(t_1)\big) \geqslant W\big(x(t_1)\big) \geqslant \alpha.$$

This contradiction proves that $x(t) \equiv 0$ is Lyapunov stable. □

Theorem 1.7.2 ([Lyapunov 1892]) *Let all the assumptions of Theorem 1.7.1 hold. Suppose in addition that there exist functions $W_0 \in A(K_H)$, $W_1 \in A(K_H)$ such that*

$$V(t,x) \leqslant W_0(x), \quad \dot V_{(1.7.1)}(t,x) \leqslant -W_1(x) \quad (x \in K_H).$$

Then the trivial solution $x(t) \equiv 0$ of (1.7.1) is asymptotically stable.

Proof. Since all the assumptions of Theorem 1.7.1 are fulfilled, the trivial solution is Lyapunov stable. This gives the opportunity to determine $\alpha > 0$ such that if $|x(t_0)| < \alpha$ then $|x(t, t_0, x_0)| < H$ for all $t > t_0$. Let us introduce $L = \sup_{|x|=H} W_0(x)$. Let $\varepsilon < \alpha$ and $\ell = \inf_{\varepsilon \leqslant |x| \leqslant H} W(x)$. Determine a number η such that $\ell > \sup_{|x| \leqslant \eta} W_0(x)$ and introduce $m = \inf_{\eta \leqslant |x| \leqslant H} W_1(x)$. Let us demonstrate that any solution with $|x(t_0)| \geqslant \eta$ satisfies the inequality $|x(t_1)| < \eta$ for all $t_1 > t_0$. Indeed, $V(t, x(t)) \leqslant V(t_0, x(t_0)) - \int_{t_0}^{t} W_1(x(s))\, ds$. Let $T = (L + mt_0)/m$. If the relation $|x(t, t_0, x_0)| \geqslant \eta$ remained for all $t > t_0$ we should deduce from the latter that for $t > T$

$$V\big(t, x(t)\big) \leqslant W_0\big(x(t_0)\big) - m(t - t_0) < L - m\left(\frac{L + mt_0}{m} - t_0\right) = 0,$$

which is impossible since $V(t,x)$ is nonnegative. Thus for a certain $t_1 > t_0$ we have

$$|x(t_1, t_0, x_0)| < \eta.$$

Let us demonstrate now that for all $t \geqslant t_1$ it is true that

$$|x(t, t_0, x_0)| < \varepsilon. \tag{1.7.5}$$

Suppose that for $\bar{t} > t_1$ inequality (1.7.5) is violated. Then the chain of inequalities holds:

$$\ell \leqslant W\big(x(\bar{t}\,)\big) \leqslant V\big(\bar{t}, x(\bar{t}\,)\big) \leqslant V\big(t_1, x(t_1)\big) \leqslant W_0\big(x(t_1)\big) \leqslant \sup_{|x|\leqslant\eta} W_0(x) < \ell,$$

which contains a contradiction. Thus asymptotical stability of $x(t) \equiv 0$ is proved. □

Theorem 1.7.3 (Barbashin–Krasovsky theorem [Barbashin & Krasovsky 1952]) *Suppose there exist a continuously differentiable function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ and continuous functions $W(x)$, $W_0(x)$, $W_1(x) \in A(\mathbb{R}^n)$ such that the following conditions are true:*

(i) $W(x) \leqslant V(t,x) \leqslant W_0(x)$;

(ii) $\dot{V}_{(1.7.1)}(t,x) \leqslant -W_1(x)$;

(iii) $\lim_{|x|\to\infty} W(x) = \infty$.

Then the trivial solution $x(t) \equiv 0$ of (1.7.1) is globally asymptotically stable.

Proof. Assumptions (i) and (ii) coincide with the hypotheses of Theorem 1.7.2. Thus the trivial solution $x(t) \equiv 0$ is asymptotically stable. Let us demonstrate that any solution of (1.7.1) is bounded on $[0, +\infty)$. Really, from hypotheses (ii) and (i) we have that for $t \geqslant t_0$

$$W\big(x(t)\big) \leqslant V\big(t, x(t)\big) \leqslant V\big(t_0, x(t_0)\big).$$

Hence the boundedness of $W(x(t))$ follows. Suppose that $x(t, t_0, x_0)$ is not bounded. Then we can choose a sequence $\{\, t_k \,\}$ such that $x(t_k, t_0, x_0) \to +\infty$ for $k \to +\infty$. Then it follows from hypothesis (iii) that $W(x(t_k, t_0, x_0)) \to +\infty$ for $k \to +\infty$ which contradicts the boundedness of $W(x(t))$. Thus every solution $x(t, t_0, x_0)$ is bounded and there exists $H > 0$ such that for all $t > t_0$

$$\big|x(t, t_0, x_0)\big| < H.$$

It is evident that $W(x)$, $W_0(x)$, $W_1(x) \in A(K_H)$. Now we are within the framework of the proof of Theorem 1.7.2. Repeating literally the argument of this proof for any solution of (1.7.1) we obtain that $x(t, t_0, x_0) \to 0$ for $t \to +\infty$. The theorem is proved. □

Remark 1.7.1. According Definition 1.4.1 the linear equation

$$\dot{x} = Px \tag{1.7.6}$$

is stable if $n\times n$-matrix P is Hurwitzian, i.e. if all its eigenvalues have negative real parts. Note that this stability definition does not contradict the stability definitions of this section. More than that Definition 1.4.1 implies the global asymptotic stability of (1.7.6). Indeed, it follows from Theorem 1.4.1 that all the solutions of (1.7.6) satisfy the inequality[b]

$$\left|x(t,x_0)\right| < M_0 e^{-\varkappa_0 t}|x_0| \qquad (M_0,\ \varkappa_0 > 0).$$

Hence Lyapunov stability is evident ($\delta = \varepsilon/M_0$) as well as the limit relation

$$\lim_{t\to\infty} x(t,x_0) = 0 \qquad (x_0 \in \mathbf{R}^n).$$

1.8 Absolute Stability of Control Systems

Let us consider the nonlinear system (1.1.2), (1.1.3) in the scalar case of $m = l = 1$, i.e.

$$\begin{cases} \dot{x} = Px + q\xi, & \sigma = r^* x \\ \xi = \varphi(t,\sigma) \end{cases} \tag{1.8.1}$$

where P is real $n\times n$-matrix, q and r are real n-vectors, $\varphi : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ and is continuous and locally Lipschitz in the second argument. Let

$$\varphi(t,0) \equiv 0. \tag{1.8.2}$$

Note that system (1.8.1), (1.8.2) has a trivial solution $x \equiv 0$.

System (1.8.1) may be represented as (1.7.1) with

$$f(t,x) = Px + q\varphi(t, r^* x).$$

So all definitions and theorems of the previous section may be applied to (1.8.1). We shall be particularly interested in global asymptotic stability of (1.8.1), the global asymptotic stability of the class of systems being of the utmost interest. The latter problem is called the problem of absolute stability. Its strict formulation is as follows.

Suppose that for certain μ_1, $\mu_2 \in \mathbf{R}$ with $\mu_1 \leqslant \mu_2$ it holds

$$\mu_1 \leqslant \frac{\varphi(t,\sigma)}{\sigma} \leqslant \mu_2 \qquad \text{for all} \quad t \in \mathbf{R}_+,\ \sigma \neq 0. \tag{1.8.3}$$

In this case we say that φ *belongs to the class* $M[\mu_1,\mu_2]$. (Since φ is continuous it implies (1.8.2)). Note that in (1.8.3) it is possible to consider $\mu_1 = -\infty$ (but then $\mu_2 \neq +\infty$) or $\mu_2 = +\infty$ (then $\mu_1 \neq -\infty$). Geometrical illustration of (1.8.3) is given in Fig. 1.8.1. For every $t_0 \in \mathbf{R}_+$ the curve $\xi = \varphi(t_0,\sigma)$ lies on the plane $\{\sigma,\xi\}$ in the sector formed by the two lines $\xi = \mu_1\sigma$ and $\xi = \mu_2\sigma$.

Definition 1.8.1 ([Lur'e & Postnikov 1944]) *System (1.8.1) is said to be absolutely stable with respect to the class* $M[\mu_1,\mu_2]$ *if for any nonlinear function* φ *satisfying (1.8.3) the equilibrium* $x = 0$ *of (1.8.1) is globally asymptotically stable.*

[b] Since (1.7.1) is an autonomous system we may put $t_0 = 0$ for all solutions and omit it in designation of a solution.

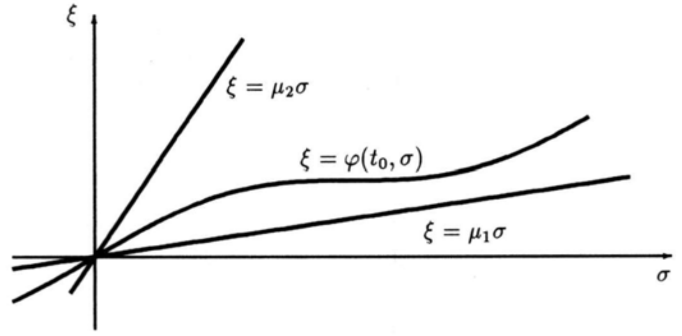


Fig. 1.8.1. Geometrical illustration of "sector condition" (1.8.3).

The notion of absolute stability for the first time appeared in the paper [Lur'e & Postnikov 1944]. Since then a lot of articles and books have appeared where the problem of absolute stability is investigated. Numerious results are exposed in monographs [Barkin 1982, Vavilov 1970, Lur'e 1951, Nelepin 1975, Mutter 1973, Aizerman & Gantmakher 1964, Voronov 1979, Gelig *et al.* 1978, Letov 1962, Lefschetz 1965, Popov 1966, Rasvan 1983, La Salle & Lefschetz 1961, Naumov 1972, Postnikov 1979, Yakubovich 1987] and surveys [Pyatnitsky 1968, Voronov 1982].

There are two parallel directions in the theory of absolute stability. One goes back to V. M. Popov [Popov 1959], the other to V. A. Yakubovich [Yakubovich 1962] and R. Kalman [Kalman 1963]. In both cases sufficient conditions of absolute stability of (1.8.1) are formulated in terms of frequency response of its linear part. But the techniques that are used in either case are different.

The Popov approach is to apply the Parseval relationship to certain functionals (of ξ and x) in order to get their upper estimates. In 1959 with the help of a priori integral estimates V. M. Popov [Popov 1959] formulated frequency-domain criterion of absolute stability. Popov criterion was systematically exposed for the first time in [Aizerman & Gantmakher 1964].

In 1962 V. A. Yakubovich [Yakubovich 1962] proved the Popov criterion by means of the method of matrix inequalities which uses the tools of Lyapunov functions. In 1963 the result was independently obtained by R. Kalman [Kalman 1963]. The method of matrix inequalities turned out to be rather fruitful. It gave the opportunity to obtain new frequency-domain criteria of stability. It gave the opportunity to study instability and oscillations as well. In sections 1.9–1.12 we present the basis of the method of matrix inequalities.

1.9 The Employment of Lyapunov Functions for Investigation of Absolute Stability

By means of Lyapunov functions the problem of absolute stability for (1.8.1) may be reduced to a certain algebraic problem. Let us use Barbashin–Krasovskij theorem (Theorem 1.7.3) and choose the Lyapunov function of the form

$$V(x) = x^* H x \qquad (x \in \mathbf{R}^n)$$

where $H = H^*$ is an unknown positive definite $n \times n$-matrix. Thus hypotheses (i) and (iii) of the theorem are fulfilled automatically. Further,

$$\begin{aligned} \dot{V}_{(1.8.1)}(x) &= \frac{\mathrm{d}V\big(x(t)\big)}{\mathrm{d}t} = 2x^* H \dot{x} = 2x^* H(Px + q\xi) \\ &= 2x^* H\big[Px + q\varphi(t, r^* x)\big] \qquad \text{for} \quad t \geqslant 0,\ x \in \mathbf{R}^n. \end{aligned}$$

So function V guarantees absolute stability with respect to $M[\mu_1, \mu_2]$ if $\dot{V}_{(1.8.1)}(x) \leqslant -W(x) \in A(\mathbb{R}^n)$ for any φ satisfying (1.8.3).

The system of inequalities (1.8.3) is equivalent to the inequality

$$(\xi - \mu_1 \sigma)(\mu_2 \sigma - \xi) \geqslant 0. \tag{1.9.1}$$

Let us introduce quadratic forms G and F by

$$G(x, \xi) = 2x^* H(Px + q\xi) \qquad x \in \mathbb{R}^n,\ \xi \in \mathbb{R}$$

and

$$F(x, \xi) = (\xi - \mu_1 r^* x)(\mu_2 r^* x - \xi) \qquad x \in \mathbf{R}^n,\ \xi \in \mathbb{R}.$$

(μ_1 and μ_2 are assumed to be finite here). With the help of these quadratic forms the absolute stability problem may now be formulated as follows:

Problem A. Find a positive definite matrix $H = H^*$ such that

$$\begin{gathered} G(x, \xi) < 0 \quad \text{for all} \quad x \in \mathbb{R}^n,\ \xi \in \mathbb{R} \\ \text{with} \quad F(x, \xi) \geqslant 0 \quad \text{and} \quad |x|^2 + \xi^2 \neq 0. \end{gathered}$$

Problems of this type often appear when it is necessary to construct a Lyapunov function. It means actually that one looks for quadratic form which is negative provided that another quadratic form is nonnegative.

Sometimes there arises a problem similar to the problem A but with a non-strict inequality.

Problem A′. Find a positive definite matrix $H = H^*$ such that

$$G(x, \xi) \leqslant 0 \quad \text{for all} \quad x \in \mathbf{R}^n,\ \xi \in \mathbb{R} \quad \text{with} \quad F(x, \xi) \geqslant 0.$$

Let us replace the problem A (A′) by another one which is more convenient for investigation. Let us define the quadratic form by

$$S(x,\xi) = G(x,\xi) + \tau F(x,\xi) \qquad (x \in \mathbf{R}^n,\ \xi \in \mathbf{R})$$

where τ is a real parameter.
Problem B. Find for a certain $\tau \geqslant 0$ a positive definite matrix $H = H^*$ such that

$$S(x,\xi) < 0 \quad \text{for all} \quad x \in \mathbf{R}^n,\ \xi \in \mathbf{R} \quad \text{with} \quad |x|^2 + \xi^2 \neq 0.$$

Problem B′. Find for a certain $\tau \geqslant 0$ a positive definite matrix $H = H^*$ such that

$$S(x,\xi) \leqslant 0 \quad \text{for all} \quad x \in \mathbf{R}^n,\ \xi \in \mathbf{R}.$$

It was proved by V. A. Yakubovich [Yakubovich 1971] that the problems A and A′ are equivalent to the problems B and B′ respectively.

As a result the absolute stability problem has led us to certain algebraic problems.
Problem C. Suppose a quadratic form $F(x,\xi)$ of two vectors $x \in \mathbf{R}^n$ and $\xi \in \mathbf{R}^m$ is given as well as an $n\times n$-real matrix P and an $n\times m$-real matrix q. Formulate the assumptions that guarantee the existence of positive definite matrix $H = H^*$ such that

$$2x^* H(Px + q\xi) + F(x,\xi) < 0 \quad \text{for all} \quad x \in \mathbf{R}^n,\ \xi \in \mathbf{R}^m \quad \text{with} \quad |x|^2 + |\xi|^2 \neq 0. \tag{1.9.2}$$

Problem C′. Suppose a quadratic form $F(x,\xi)$ of two vectors $x \in \mathbf{R}^n$ and $\xi \in \mathbf{R}^m$ is given as well as an $n\times n$-real matrix P and $n\times m$-real matrix q. Formulate the assumptions that guarantee the existence of positive definite matrix $H = H^*$ such that

$$2x^* H(Px + q\xi) + F(x,\xi) \leqslant 0 \quad \text{for all} \quad x \in \mathbf{R}^n,\ \xi \in \mathbf{R}^m. \tag{1.9.3}$$

The problems C and C′ are formulated for real P, q, x, ξ. But they may be extended to the case of complex vectors and matrices. Indeed, any real quadratic form $G(u) = u^*Qu$ ($u \in \mathbf{R}^n$) with a real $n\times n$-symmetric matrix $Q = Q^*$ may be extended to a Hermitian form G_c by

$$G_c(u + iv) := G(u) + G(v) \quad \text{for any} \quad u,\ v \in \mathbf{R}^n.$$

This procedure is called *the extension of a quadratic form to a Hermitian one*. In coordinates $(u_1, u_2, \dots, u_n)$ this extension means that any term $u_j u_k$ of G is replaced by $\operatorname{Re} u_j^* u_k$. It is obvious that for real values $u \in \mathbf{R}^n$ we have $G_c(u) = G(u)$.

For the complex case relations (1.9.2) and (1.9.3) are respectively as follows:

$$2\operatorname{Re} x^* H(Px + q\xi) + F_c(x,\xi) < 0 \quad \text{for all} \quad x \in \mathbf{C}^n,\ \xi \in \mathbf{C}^m \text{with} \quad |x|^2 + |\xi|^2 \neq 0, \tag{1.9.4}$$

$$2\operatorname{Re} x^* H(Px + q\xi) + F_c(x,\xi) \leqslant 0 \quad \text{for all } x \in \mathbf{C}^n,\ \xi \in \mathbf{C}^m. \tag{1.9.5}$$

Sometimes a problem arises to guarantee the negative definiteness of the quadratic form G provided that several quadratic forms are nonnegative, namely

$$G(x,\xi) < 0 \quad \text{for all} \quad x \in \mathbf{R}^n,\ \xi \in \mathbf{R}^m$$
$$\text{with} \quad F_i(x,\xi) \geqslant 0 \quad (i = 1,\ldots,k) \quad \text{and} \quad |x|^2 + |\xi|^2 \neq 0.$$

Here $F_i(x,\xi)$ $(i = 1,\ldots,k)$ are certain quadratic forms. This problem may be replaced by the following one: find a certain set of parameters $\tau_i \geqslant 0$ $(i = 1,\ldots,k)$ and a matrix $H = H^*$ such that

$$S(x,\xi) = G(x,\xi) + \textstyle\sum_{i=1}^{k} \tau_i F_i(x,\xi) < 0 \quad \text{for all} \quad x \in \mathbf{R}^n,\ \xi \in \mathbf{R}^m$$
$$\text{with} \quad |x|^2 + |\xi|^2 \neq 0.$$

The process of substitution of the latter problem for the former one is called S-procedure. So the replacement of the problem A (A′) by the problem B (B′) was carried out by S-procedure with $k = 1$. It has already been mentioned that for $k = 1$ the two problems are equivalent. In this case the S-procedure is said to be "lossless". We have to note that in case $k \geqslant 2$ it is not so.

1.10 Yakubovich–Kalman Frequency-Domain Theorem

The following theorem gives the necessary and sufficient conditions of existence of matrix H satisfying relations (1.9.5). It goes back to V. A. Yakubovich [Yakubovich 1962] and R. Kalman [Kalman 1963]. (See also [Lefschetz 1965, Popov 1966, Miller & Michel 1982, Knobloch & Kwakernaak 1986, Nelepin 1975, Gelig *et al.* 1978]). We expound it here in accordance with the monographs [Nelepin 1975] and [Gelig *et al.* 1978].

Let P and q be complex matrices of orders $n \times n$ and $n \times m$ respectively and

$$\mathcal{G}(x,\xi) = x^* G x + 2\,\mathrm{Re}(x^* D \xi) + \xi^* \Gamma \xi$$

be a Hermitian form of $x \in \mathbf{C}^n$ and $\xi \in \mathbf{C}^m$, the complex matrices $G = G^*$, $\Gamma = \Gamma^*$ and D having the orders $n \times n$, $m \times m$ and $n \times m$ respectively.

Theorem 1.10.1 (Yakubovich–Kalman) *Suppose that the pair (P,q) is controllable. Then there exists a matrix $H = H^*$ satisfying the inequality*

$$2\,\mathrm{Re}\big[x^* H (Px + q\xi)\big] + \mathcal{G}(x,\xi) \leqslant 0 \qquad (x \in \mathbf{C}^n,\ \xi \in \mathbf{C}^m) \tag{1.10.1}$$

if and only if

$$\mathcal{G}\big[(i\omega I_n - P)^{-1} q\xi,\ \xi\big] \leqslant 0 \tag{1.10.2}$$

for all $\xi \in \mathbf{C}^m$ and all $\omega \in \mathbf{R}$ with $\det(P - i\omega I_n) \neq 0$. *In case the matrices P and q and coefficients of $\mathcal{G}$ are real the matrix H is real as well. If the matrix P is Hurwitzian, $G \geqslant 0$ and the pair (P, D) is observable then any matrix H satisfying (1.10.1) is positive definite.*

We must note here that it is rather easy to prove that inequality (1.10.2) is necessarily fulfilled if inequality (1.10.1) is true. But it is rather laborious to prove that the validity of (1.10.2) is sufficient to the fulfillment of (1.10.1). We shall introduce in this connection a number of auxiliary definitions and designations. We shall also prove four special lemmas.

Proof of the fact that (1.10.2) necessarily follows from (1.10.1). Suppose that there exists a matrix $H = H^*$ such that (1.10.1) holds. Let us consider x and ξ which are connected by the relation

$$x = (i\omega I_n - P)^{-1} q\xi \tag{1.10.3}$$

where $\xi \in \mathbb{C}^m$, $\omega \in \mathbf{R}$ and $\det(P - i\omega I_n) \neq 0$. Then inequality (1.10.1) takes the form

$$2\,\mathrm{Re}(i\omega x^* Hx) + \mathcal{G}\big((i\omega I_n - P)^{-1} q\xi,\, \xi\big) \leqslant 0$$

The first term of the left side of the latter inequality is equal to 0 since $H = H^*$ and $i\omega x^* Hx$ is a pure imaginary number. So we have that

$$\mathcal{G}\big((i\omega I_n - P)^{-1} q\xi,\, \xi\big) \leqslant 0$$

for all $\xi \in \mathbb{C}^m$ and $\omega \in \mathbf{R}$ with $\det(P - i\omega I_n) \neq 0$. By this we have proved that (1.10.1) implies (1.10.2). □

To prove that (1.10.2) implies (1.10.1) we ought to carry out certain spade work.

i). Let us recall here a number of designations of the previous sections:

$$\begin{aligned} P_p &:= pI_n - P \qquad (p \in \mathbb{C}); \\ \delta(p) &:= \det P_p = p^n + \delta_1 p^{n-1} + \cdots + \delta_{n-1} p + \delta_n; \\ Q(p) &:= \delta(p) P_p^{-1} q = Q_1 p^{n-1} + \cdots + Q_{n-1} p + Q_n, \end{aligned}$$

where $Q_1, \ldots, Q_n$ are $n \times m$-matrices.

ii).

Definition 1.10.1 *Let $B(p) = B_0 p^n + B_1 P^{n-1} + \cdots + B_{n-1} p + B_n$ be a polynomial with matrix coefficients. The polynomial $B^\nabla(p) = B_0^*(-p)^n + B_1^*(-p)^{n-1} + \cdots + B_{n-1}^*(-p) + B_n^*$ is said to be the result of operator ∇ (nabla) applied to the polynomial $B(p)$.*

Note that

$$B^\nabla(i\omega) = \big[B(i\omega)\big]^* \qquad (\omega \in \mathbf{R}). \tag{1.10.4}$$

The properties of the operator ∇. Let $A(p)$ and $B(p)$ be polynomials. The following assertions are true.

1. $[A(p) + B(p)]^\nabla = A^\nabla(p) + B^\nabla(p)$;
2. $[A(p)B(p)]^\nabla = B^\nabla(p) A^\nabla(p)$;
3. $[\det B(p)]^\nabla = \det[B^\nabla(p)]$;

4. if $B^*(i\omega) = B(i\omega)$ then $B^\nabla(p) = B(p)$;

5. if $B(p_0) = 0$ then $B^\nabla(-\bar{p}_0) = 0$, where $\bar{p} = \tau - i\omega$ for $p = \tau + i\omega$.

Properties 1–3 follow from the very definition of operator ∇. Property 4 follows from (1.10.4) and the fact that identity of two polynomials on the imaginary axis implies their identity on the entire complex plane. The fifth property follows from the equality $B^\nabla(-\bar{p}_0) = B^*(p_0)$.

iii). Consider a Hermitian form $\mathcal{G}(x,\xi)$ of $x \in \mathbb{C}^n$ and $\xi \in \mathbb{C}^m$ with the Hermitian matrix $(-\widehat{\mathcal{G}})$, i.e.

$$\mathcal{G}(x,\xi) = -\left\| \begin{matrix} x \\ \xi \end{matrix} \right\|^* \widehat{\mathcal{G}} \left\| \begin{matrix} x \\ \xi \end{matrix} \right\|. \tag{1.10.5}$$

Let $(-\Gamma)$ be such that

$$\mathcal{G}(0,\xi) = -\xi^*\Gamma\xi. \tag{1.10.6}$$

Let us investigate the case of $m = 1$. Then Γ is a number. Let $\mathcal{G}(P_{i\omega}^{-1}q\xi,\, \xi) \leqslant 0$. Then

$$\mathcal{G}(P_{i\omega}^{-1}q\xi,\, \xi) = -\xi^*\Pi(i\omega)\xi, \tag{1.10.7}$$

where $\Pi(i\omega)$ is a nonnegative real function. Further,

$$-\mathcal{G}(P_{i\omega}^{-1}q\xi,\, \xi) = \xi^*\Pi(i\omega)\xi = \left\| \begin{matrix} P_{i\omega}^{-1}q\xi \\ \xi \end{matrix} \right\|^* \widehat{\mathcal{G}} \left\| \begin{matrix} P_{i\omega}^{-1}q\xi \\ \xi \end{matrix} \right\|.$$

Hence

$$\Pi(i\omega) = \left\| \begin{matrix} P_{i\omega}^{-1}q \\ 1 \end{matrix} \right\|^* \widehat{\mathcal{G}} \left\| \begin{matrix} P_{i\omega}^{-1}q \\ 1 \end{matrix} \right\|$$

and

$$|\delta(i\omega)|^2\Pi(i\omega) = \left\| \begin{matrix} Q(i\omega) \\ \delta(i\omega) \end{matrix} \right\|^* \widehat{\mathcal{G}} \left\| \begin{matrix} Q(i\omega) \\ \delta(i\omega) \end{matrix} \right\|.$$

Let us extend the right side of the latter equality to the entire complex plane and introduce the designation

$$\Phi(p) = \left\| \begin{matrix} Q(p) \\ \delta(p) \end{matrix} \right\|^\nabla \widehat{\mathcal{G}} \left\| \begin{matrix} Q(p) \\ \delta(p) \end{matrix} \right\|.$$

The polynomial $\Phi(p)$ has the following properties:

a) $\Phi(i\omega) > 0$;

b) $\Phi^\nabla(p) = \Phi(p)$;

c) the degree of $\Phi(p)$ is no high than $2n$ and the coefficient at p^{2n} is equal to $(-1)^n\Gamma$;

d) $\Gamma \geqslant 0$.

The property a) is evident since $\Phi(i\omega) = |\delta(i\omega)|^2\Pi(i\omega) \geqslant 0$. The property b) follows from the fact that $\widehat{\mathcal{G}}$ is a Hermitian matrix and from property 4 of the operator ∇. The property c) follows from (1.10.5) and (1.10.6). Indeed,

$$\Phi(p) = \left\| \begin{matrix} Q(p) \\ \delta(p) \end{matrix} \right\|^{\nabla} \left\| \begin{matrix} \otimes & \otimes \\ \otimes & \Gamma \end{matrix} \right\| \left\| \begin{matrix} Q(p) \\ \delta(p) \end{matrix} \right\| = (-1)^n \Gamma p^{2n} + \cdots .$$

Hence it is evident that

$$\Gamma = \lim_{\omega\to\infty} \frac{\Phi(i\omega)}{|\delta(i\omega)|^2} = \lim_{\omega\to\infty} \Pi(i\omega) \geqslant 0.$$

iiii). Now we present here four auxiliary assertions which form the base for the proof of the fact that (1.10.2) implies (1.10.1).

Lemma 1.10.1 (about solvability of Lyapunov equation) *Let P be a Hurwitzian $n\times n$-complex matrix. The equation*

$$P^*H + PH = G \tag{1.10.8}$$

has a unique Hermitian solution $H = H^*$ for any $n\times n$-complex matrix G. If $G = G^* \leqslant 0$ then $H = H^* \geqslant 0$ and $Hy_0 = 0$ implies $HPy_0 = 0$.

Proof. Let us demonstrate at first the uniqueness of the solution of (1.10.8). Suppose the opposite. Let H_1 and H_2 be two solutions of (1.10.8). Then $H = H_1 - H_2$ satisfies the equation

$$P^*H + HP = 0. \tag{1.10.9}$$

Consider now $x(t) = e^{Pt}a$ with an arbitrary complex vector a. It follows from (1.10.9) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(x^*(t)Hx(t)\big) = x^*(t)[P^*H + HP]x(t) = 0.$$

Hence $x^*(t)Hx(t) = a^*Ha$. Since $x(t) \to 0$ as $t \to +\infty$ we have that $a^*Ha = 0$. Consequently for two arbitrary vectors a_1 and a_2 it holds $0 = (a_1+a_2)^*H(a_1+a_2) = 2\,\mathrm{Re}\, a_2^*Ha_1$. But if we substitute now ia_1 for a_1 we shall obtain $\mathrm{Im}(a_2^*Ha_1) = 0$. Thus $a_2^*Ha_1 = 0$ for any a_1, a_2. On the other hand any element h_{jk} of matrix H may be represented as $e_j^*He_k$ where $e_j = \mathrm{col}(0,\ldots,0,1,0,\ldots,0)$ with the 1 in j-th place. Thus $H = 0$ and the solution of (1.10.8) is unique. It is easy to demonstrate that the solution is as follows

$$H = -\int_0^\infty e^{P^*t}Ge^{Pt}\,\mathrm{d}t \tag{1.10.10}$$

(it is evident that the improper integral converges). Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{P^*t}Ge^{Pt}\right) = P^*e^{P^*t}Ge^{Pt} + e^{P^*t}Ge^{Pt}P.$$

Thus,

$$P^*H + HP = -\int_0^\infty \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{P^*t}Ge^{Pt}\right)\right] \mathrm{d}t = G.$$

It is obvious from (1.10.10) that $G = G^* \leqslant 0$ implies $H = H^* \geqslant 0$. Let $G = G^* \leqslant 0$ and $Hy_0 = 0$ for a certain y_0. Then it follows from (1.10.10) that

$$\int_0^\infty y^*(t)Gy(t)\,\mathrm{d}t = 0 \qquad \text{with} \qquad y(t) = e^{Pt}y_0.$$

But the integrand is nonnegative here and consequently $Gy(t) \equiv 0$. By means of differentiating the latter identity we have

$$Ge^{Pt}Py_0 = 0.$$

Hence and from (1.10.10) it follows that $HPy_0 = 0$. The lemma is proved. □

Lemma 1.10.2 (R. Kalman) *Let $A(p)$ be a polynomial of degree $2k$, ($k \in \mathbf{Z}$), with the leading coefficient Γ satisfying the following relations*

$$A(p) = A^\nabla(p), \quad A(i\omega) \geqslant 0 \qquad \text{for all } \omega.$$

Then there exists a polynomial $X(p)$ such that

$$A(p) = X^\nabla(p) \cdot X(p).$$

Proof. Let p_j be a root of $A(p)$. Then from the fifth property of the operator ∇ (see page 34) and from the equality $A(p) = A^\nabla(p)$ it follows that $(-\overline{p}_j)$ is a root of $A(p)$ as well. Thus the roots of $A(p)$ are disposed symmetrically with respect to imaginary axis (with regard for their multiplicity). As to pure imaginary roots any one of them has even multiplicity since $A(i\omega) \geqslant 0$. So we may divide the set of all roots of $A(p)$ into two subsets in such a way that their elements are disposed on the complex plane symmetrically with respect to imaginary axis, those roots that are disposed on the imaginary axis being divided in halves between the two sets (with regard for their multiplicity). These sets are:

$$S_1 = \{\, p_1, p_2, \ldots, p_k \,\}, \qquad S_2 = \{\, -\overline{p}_1, -\overline{p}_2, \ldots, -\overline{p}_k \,\}.$$

Then

$$A(p) = \Gamma \prod_{j=1}^{k}(p - p_j) \prod_{r=1}^{k}(p + \overline{p}_r).$$

Let us introduce a polynomial

$$Y(p) = \prod_{j=1}^{k}(p - p_j).$$

Then

$$Y^\nabla(p) = \prod_{j=1}^{k}(-p - \overline{p}_j) = (-1)^k \prod_{j=1}^{k}(p + \overline{p}_j)$$

or

$$\prod_{j=1}^{k}(p+\bar{p}_j) = (-1)^k Y^{\nabla}(p).$$

So

$$A(p) = (-1)^k \Gamma Y(p) Y^{\nabla}(p).$$

For $p = i\omega$ we have

$$A(i\omega) = (-1)^k \Gamma Y(i\omega) Y^*(i\omega) = (-1)^k \Gamma \left|Y(i\omega)\right|^2.$$

The hypothesis $A(i\omega) \geqslant 0$ implies that $(-1)^k \Gamma \geqslant 0$. Consequently, we can put

$$X(p) = \sqrt{(-1)^k \Gamma}\, Y^{\nabla}(p).$$

Then $A(p) = X^{\nabla}(p) X(p)$ and the lemma is proved. □

Lemma 1.10.3 (Schvartz symmetry principle) *Let $\varphi(p)$ be a rational function such that* $\operatorname{Re}\varphi(i\omega) = 0$. *Then poles of $\varphi(p)$ are symmetric with respect to imaginary axis.*

Proof. Let $\varphi(p) = M(p)/N(p)$ where M and N are polynomials. Then

$$\operatorname{Re}\varphi(i\omega) = \frac{M(i\omega)\overline{N}(i\omega) + \overline{M}(i\omega)N(i\omega)}{2\left|N(i\omega)\right|^2}.$$

Since $\operatorname{Re}\varphi(i\omega) = 0$ it follows that

$$M(i\omega)\overline{N}(i\omega) + \overline{M}(i\omega)N(i\omega) = 0.$$

Expanding the latter identity to all the complex values we have

$$M(p)N^{\nabla}(p) + M^{\nabla}(p)N(p) = 0.$$

Then

$$\varphi(p) = \frac{M(p)}{N(p)} = -\frac{M^{\nabla}(p)}{N^{\nabla}(p)}.$$

Note that the poles of the function $\varphi(p)$ are the zeroes of polynomials $N(p)$ and $N^{\nabla}(p)$. Suppose p_0 is a pole of $\varphi(p)$, i.e. $N(p_0) = 0$. Then according to property 5 of the operator ∇ we conclude that $N^{\nabla}(-\bar{p}_0) = 0$. Thus $(-\bar{p}_0)$ is a pole of $\varphi(p)$ as well. So the poles of $\varphi(p)$ are symmetric with respect to the imaginary axis and the lemma is proved. □

Lemma 1.10.4 *Let A be a $n\times n$-matrix and b be a $n\times m$-matrix, the pair (A, b) being controllable. Let the polynomial $\delta(p) = \det A_p$ has no zeroes which are symmetric with respect to imaginary axis. Let $F(x,\xi)$ be an Hermitian form with $x \in \mathbf{C}^n$, $\xi \in \mathbf{C}^m$. Suppose that $F(x,0) = 0$ and $F(A_{i\omega}^{-1}b\xi,\xi) \equiv 0$ for all $\omega \in \mathbf{R}$ such that* $\det(A - i\omega I) \neq 0$. *Then $F(x,\xi) \equiv 0$ for all $x \in \mathbf{C}^n$, $\xi \in \mathbf{C}^m$.*

Proof. If $\omega \to +\infty$ we obtain from the identity $F(A_{i\omega}^{-1}b\xi, \xi) \equiv 0$ the identity $F(0,\xi) \equiv 0$. Taking now into account that $F(x,0) \equiv 0$ we may affirm that the Hermitian form $F(x,\xi)$ has no terms with $|x|^2$ and $|\xi|^2$. In other words the form $F(x,\xi)$ may be written as

$$F(x,\xi) = x^* f_1 \xi + \xi^* f_2 x,$$

where f_2 and f_1 are $m \times n$- and $n \times m$-matrices respectively. Since $F(x,\xi)$ is a Hermitian form it is true that

$$F(x,\xi) = \operatorname{Re} F(x,\xi) = \operatorname{Re}(x^* f_1 \xi + \xi^* f_2 x) = \operatorname{Re}(\xi^* f x),$$

where $f = f_1^* + f_2$. Our aim is to demonstrate that f is a zero matrix. Let us hold $\xi = \xi_0$ fixed and consider the function $\varphi(p) = \xi_0^* f A_p^{-1} b \xi_0$. According to the hypotheses of this lemma we have that $\operatorname{Re}\varphi(i\omega) \equiv 0$ for any permissible ω. Then from Lemma 1.10.3 it follows that the poles of $\varphi(p)$ are symmetric with respect to imaginary axis. On the other hand the poles of $\varphi(p)$ coincide with eigenvalues of A. But this matrix has no eigenvalues which are symmetric with respect to imaginary axis. Thus $\varphi(p)$ has no poles. Let us also take into account that $\varphi(p)$ is bounded for all complex p. Then according to well-known Liouville theorem $\varphi(p) = \text{const}$. But $\varphi(p) \to 0$ as $|p| \to \infty$. So $\varphi(p) \equiv 0$. We have proved that for any fixed ξ_0 it is true that $\xi_0^* f A_p^{-1} b \xi_0 \equiv 0$. It implies that $\delta(p) f A_p^{-1} b \equiv 0$. Since the pair (A,b) is controllable it follows from this identity and Assertion 3 of Theorem 1.2.1 that f is a zero matrix. By this the lemma is proved. □

Proof of the fact that (1.10.2) implies (1.10.1). The assertion we are going to prove is as a rule replaced by another assertion: if (1.10.2) is true then there exist an $n \times n$-matrix $H = H^*$, $n \times m$-matrix h and $m \times m$-matrix $\varkappa$ such that for all $x \in \mathbb{C}^n$, $\xi \in \mathbb{C}^m$ the following equality is true:

$$2\operatorname{Re}\left[x^* H(Px + q\xi)\right] + \mathcal{G}(x,\xi) = -|h^* x - \varkappa\xi|^2. \qquad (1.10.11)$$

It is clear that if (1.10.11) holds for all $x \in \mathbb{C}^n$, $\xi \in \mathbb{C}^m$ then (1.10.1) holds with the same matrix H.

We shall demonstrate the new assertion for $m = 1$ (for the full proof with $m \geqslant 1$ see [Gelig *et al.* 1978]).

Let (1.10.2) holds. Suppose that $\xi = g^* x$, where g is an n-vector. Then (1.10.11) takes the form

$$2\operatorname{Re} x^* H(P + qg^*)x = -\mathcal{G}(x, g^* x) - |h^* x - \varkappa g^* x|^2.$$

Let $P_1 = P + qg^*$. The right side of the latter equality is a quadratic form with matrix L: $x^* L x$. So we have received the Lyapunov equation

$$HP_1 + P_1^* H = L. \qquad (1.10.12)$$

The pair (P, q) is controllable. Then according to Theorem 1.5.1 we may choose vector g in such a way that matrix P_1 is a Hurwitzian one. Then according to Lemma 1.10.1 equation (1.10.12) has a unique Hermitian solution $H = H^*$.

Let $x = P_{i\omega}^{-1} q\xi$. Then (1.10.11) takes the form

$$\mathcal{G}(P_{i\omega}^{-1} q\xi, \xi) = -\left|h^* P_{i\omega}^{-1} q\xi + \varkappa\xi\right|^2.$$

(Indeed, in this case $x^* H(Px + q\xi) = i\omega|\xi|^2 q^*(P_{i\omega}^{-1})^* H(P_{i\omega}^{-1})q = i\omega x^* Hx$). Let us come back to iii) and use the equality (1.10.7):

$$\mathcal{G}(P_{i\omega}^{-1} q\xi, \xi) = -\xi^* \Pi(i\omega)\xi.$$

Then we have that

$$\Pi(i\omega) = \left(h^* P_{i\omega}^{-1} q + \varkappa\right)^* \left(h^* P_{i\omega}^{-1} q + \varkappa\right)$$

or

$$\Phi(i\omega) = \left|\delta(i\omega)\right|^2 \Pi(i\omega) = \left[h^* Q(i\omega) + \varkappa\delta(i\omega)\right]^* \left[h^* Q(i\omega) + \varkappa\delta(i\omega)\right]. \tag{1.10.13}$$

Let us extend formula (1.10.13) to all the complex plane. We have

$$\Phi(p) = \left[h^* Q(p) + \varkappa\delta(p)\right]^\nabla \left[h^* Q(p) + \varkappa\delta(p)\right]. \tag{1.10.14}$$

It was shown in iii) that $\Phi(p)$ satisfies all the hypotheses of Lemma 1.10.2 and its degree is no high than $2n$. So there exists a polynomial $\varphi(p)$ such that

$$\Phi(p) = \varphi^\nabla(p)\varphi(p).$$

The degree of $\varphi(p)$ is no high than n. Let $\varphi(p) = \varphi_0 p^n + \varphi_1 p^{n-1} + \cdots + \varphi_n$. Then

$$\varphi^\nabla(p)\varphi(p) = (-1)^n |\varphi_0|^2 p^{2n} + \cdots = (-1)^n \Gamma p^{2n} + \cdots.$$

Since $\Gamma \geqslant 0$ (according to the property d) of the polynomial $\Phi(p)$, see p. 34) the coefficient φ_0 may be received from the equality

$$|\varphi_0|^2 = \Gamma.$$

Now $\varphi(p)$ is known and we shall determine h and $\varkappa$ in such a way that

$$\varphi(p) = h^* Q(p) + \varkappa\delta(p). \tag{1.10.15}$$

The degree of $Q(p)$ is equal to $n - 1$ and the degree of $\delta(p)$ is equal to n. So by equating the coefficients at the same degrees of the two polynomials we have

$$\begin{gathered} \varkappa = \varphi_0 = \pm\sqrt{\Gamma}, \\ h^* Q_k = -\varkappa\delta_k + \varphi_k \quad (k = 1, \ldots, n). \end{gathered} \tag{1.10.16}$$

The vectors Q_k ($k = 1, \dots, n$) are linear independent. Thus h is determined uniquely from (1.10.16).

Finally, we may determine $\varkappa$ and h in such a way that the pair $(x = P_{i\omega}^{-1} q\xi, \xi)$ satisfies (1.10.11). Then whenever $\varkappa$ and h are found we may by means of (1.10.12) determine H in such a way that (1.10.11) is satisfied for the pair $(x, g^* x)$ with such n-vector g that matrix $P + qg^*$ is Hurwitzian.

Now we must prove that if $\varkappa$ is equal to $\sqrt{\Gamma}$ or $-\sqrt{\Gamma}$, n-vector h satisfies system (1.10.16) and $n \times n$-Hermitian matrix H satisfies (1.10.12), with n-vector g being chosen in such a way that matrix $P + qg^*$ is Hurwitzian, then equality (1.10.11) is fulfilled for all $x \in \mathbb{C}^n$, $\xi \in \mathbb{C}^m$.

Let us use Lemma 1.10.4. Consider a quadratic form

$$G(x, \xi) = 2 \operatorname{Re} x^* H(Px + q\xi) + \mathcal{G}(x, \xi) + |h^* x + \varkappa \xi|^2.$$

Let us bring about the change of variables $\xi = \eta + g^* x$. We shall obtain a quadratic form

$$G_1(x, \eta) = G(x, \eta + g^* x).$$

The form $G_1(x, \eta)$ satisfies all the hypotheses of Lemma 1.10.4. Indeed,

$$\begin{aligned} G_1(x, 0) &= G(x, g^* x) = 2 \operatorname{Re} x^* H(P_1 x) + \mathcal{G}(x, g^* x) + |h^* x + \varkappa g^* x|^2 \\ &= 2 \operatorname{Re} x^* H(P_1 x) - x^* L x. \end{aligned}$$

In virtue of (1.10.12) we have that

$$G_1(x, 0) = 0.$$

The equality $Px + q\xi = i\omega x$ is equal to the equality $P_1 x + q\eta = i\omega x$. That is why

$$G_1(P_{1_{i\omega}}^{-1} q\eta, \eta) = G(P_{i\omega}^{-1} q\xi, \xi) = 0$$

for any ω such that $\delta(i\omega) \neq 0$ and $\det(P_1 - i\omega I) \neq 0$. Since matrix P_1 is Hurwitzian it has no eigenvalues which are symmetric with respect to the imaginary axis. According to Lemma 1.10.4 we have that $G_1(x, \eta) = 0$ for any $x \in \mathbb{C}^n$, $\eta \in \mathbb{C}^m$. Consequently $G(x, \xi) \equiv 0$ for any $x \in \mathbb{C}^n$, $\xi \in \mathbb{C}^m$.

Suppose now that matrices P, q, G and D are real. Then coefficients of polynomials $Q(p)$ and $\delta(p)$ are real. From the proof of Lemma 1.10.2 it follows that coefficients of polynomial $\varphi(p)$ are real as well. Thus the solution h of system (1.10.16) is also real. With the help of real $\varkappa$ and h we determine real matrix L, i.e. $L = \overline{L}$. Then $\overline{H} P_1 + P_1^* \overline{H} = L$ and $\overline{H}$ is the solution of (1.10.12). As this solution is unique we have $H = \overline{H}$, i.e. matrix H is real. Thus (1.10.2) implies (1.10.1).

<u>Proof of the last statement of the theorem.</u> Let $H = H^*$ be an arbitrary matrix satisfying (1.10.1), let P be Hurwitzian and $G \geqslant 0$. We shall use the denotation

$$G_0 := HP + P^* H. \tag{1.10.17}$$

From (1.10.1) we have for $\xi = 0$ that $G_0 \leqslant -G$. As $G \geqslant 0$ then $G_0 \leqslant 0$ and according to Lemma 1.10.1 $H \geqslant 0$. We have only left to prove that $H > 0$. Let $Hx_0 = 0$. Then from Lemma 1.10.1 we have that $HPx_0 = 0$ which implies according to (1.10.17) that $x_0^* G_0 x_0 = 0$. Since $-x_0^* G x_0 \geqslant x_0^* G_0 x_0$ it follows that $x_0 G x_0 = 0$. So if $x = x_0$ then

$$\mathcal{G}(x_0, \xi) = 2\,\mathrm{Re}(x_0^* D\xi) + \xi^* \Gamma \xi \leqslant 0 \qquad (\xi \in \mathbb{C}^m) \tag{1.10.18}$$

whence

$$x_0^* D = 0. \tag{1.10.19}$$

Indeed, suppose the opposite. Let $\xi_0^* = x_0^* D \neq 0$ and ν be an arbitrary small positive number. Then for a certain ν

$$\mathcal{G}(x_0, \nu\xi_0) = \nu\big(|\xi_0|^2 + \nu\xi_0^* \Gamma \xi_0\big) > 0$$

which contradicts with (1.10.18). Thus (1.10.19) is true. Let us now substitute x_0 for Px_0, then for $P^2 x_0$ and so on. We get

$$x_0^* D = 0, \quad x_0^* P^* D = 0, \quad x_0^* (P^*)^2 D = 0, \quad \ldots . \tag{1.10.20}$$

The pair (P, D) is observable. Thus according to item 2 of Theorem 1.2.3 it follows from (1.10.20) that $x_0 = 0$. So Theorem 1.10.1 is proved completely. □

1.11 Theorem about Strict Frequency-Domain Inequality

Let P and q be complex matrices of order $n \times n$ and $n \times m$ respectively. Let $\mathcal{G}$ be a Hermitian form of complex variables $x \in \mathbb{C}^n$, $\xi \in \mathbb{C}^m$.

Theorem 1.11.1 *Suppose the pair (P, q) is controllable and matrix P has no pure imaginary eigenvalues. Then there exists a matrix $H = H^*$ (which is real in case P, q and coefficients of $\mathcal{G}$ are real) satisfying the inequality*

$$\begin{gathered} 2\,\mathrm{Re}\, x^* H(Px + q\xi) + \mathcal{G}(x, \xi) < 0 \\ (x \in \mathbb{C}^n,\ \xi \in \mathbb{C}^m,\ |x| + |\xi| \neq 0) \end{gathered} \tag{1.11.1}$$

if and only if

$$\mathcal{G}\big((i\omega I_n - P)^{-1} q\xi, \xi\big) < 0 \qquad (\omega \in [-\infty, +\infty],\ \xi \in \mathbb{C}^m,\ |\xi| \neq 0)^{c} \tag{1.11.2}$$

Proof. The fact that (1.11.1) implies (1.11.2) may be proved just in the same way as it has been proved that (1.10.1) implies (1.10.2). Indeed, (1.11.1) means that

$$2\,\mathrm{Re}\, x^* H(Px + q\xi) + \mathcal{G}(x, \xi) \leqslant -\varepsilon\big(|x|^2 + |\xi|^2\big)$$

[c] The relation $F(\omega) < 0$, for all $\omega \in [-\infty, +\infty]$ we regard as the limit inequality $\begin{cases} F(\omega) < 0 & \omega \in (-\infty, +\infty), \\ \lim_{\omega \to \pm\infty} F(\omega) < 0 \end{cases}$

where ε is sufficiently small. Let $x = (i\omega I_n - P)^{-1}q\xi$. Then

$$\mathcal{G}\big((i\omega I_n - P)^{-1}q\xi, \xi\big) \leqslant -\varepsilon|\xi|^2$$

which is equivalent to (1.11.2).

Now we shall prove that (1.11.2) implies (1.11.1). We shall confine ourselves to the case of $m = 1$. Let us go back to item iii) of Section 1.10 and use the designations introduced there:

$$\mathcal{G}(P_{i\omega}^{-1}q\xi, \xi) = -\xi^*\Pi(i\omega)\xi.$$

Since $\det(P - i\omega I_n) \neq 0$ we may affirm that $\Pi(i\omega)$ is a continuous function of ω. According to (1.11.2) $\Pi(i\omega) > 0$ for $\omega \in [-\infty, +\infty]$ and consequently there exists a positive number Π_0 such that for all $\omega \in \mathbb{R}$ the inequality $\Pi(i\omega) \geqslant \Pi_0$ is true, i.e.

$$\mathcal{G}(P_{i\omega}^{-1}q\xi) \leqslant -\xi^*\Pi_0\xi. \tag{1.11.3}$$

Consider the Hermitian form

$$\mathcal{G}_1(x,\xi) = \mathcal{G}(x,\xi) + \varepsilon\big(|x|^2 + |\xi|^2\big)$$

with a positive number ε. For $x = P_{i\omega}^{-1}q\xi$ we have

$$\begin{aligned} \mathcal{G}_1(P_{i\omega}^{-1}q\xi, \xi) &= \mathcal{G}(P_{i\omega}^{-1}q\xi, \xi) + \varepsilon\big(|P_{i\omega}^{-1}q\xi|^2 + |\xi|^2\big) \\ &\leqslant -\xi^*\Pi_0\xi + \varepsilon\big(|P_{i\omega}^{-1}q|^2 + 1\big)|\xi|^2. \end{aligned} \tag{1.11.4}$$

The function $|P_{i\omega}^{-1}q|$ is continuous and tends to 0 as $\omega \to \infty$. Consequently this function is bounded by a certain number β. Thus one can always choose the number ε so small that

$$\varepsilon(\beta + 1)|\xi|^2 \leqslant \xi^*\Pi_0\xi.$$

Hence and from (1.11.4) it follows that

$$\mathcal{G}_1(P_{i\omega}^{-1}q\xi, \xi) \leqslant 0. \tag{1.11.5}$$

According to Theorem 1.10.1 there exists a matrix $H = H^*$ such that for all $x \in \mathbb{C}^n$ and $\xi \in \mathbb{C}^1$ the inequality

$$2\,\mathrm{Re}\, x^*H(Px + q\xi) + \mathcal{G}_1(x,\xi) \leqslant 0$$

is true. Hence it follows that

$$2\,\mathrm{Re}\, x^*H(Px + q\xi) + \mathcal{G}(x,\xi) \leqslant -\varepsilon\big(|x|^2 + |\xi|^2\big) < 0$$
$$(x \in \mathbb{C}^n,\ \xi \in \mathbb{C}^1,\ |x| + |\xi| \neq 0).$$

The theorem is proved. □

Remark 1.11.1. Theorem 1.11.1 remains valid if the hypothesis about controllability of the pair (P, q) is substituted by the hypothesis of its stabilizability. The proof of this fact is rather laborious. It may be borrowed from [Yakubovich 1973b, Gelig *et al.* 1978].

1.12 Lyapunov Matrix Inequalities. Necessary and Sufficient Conditions of Solvability

On proving Theorem 1.10.1 we came across the problem of solvability of matrix equation

$$P^*H + HP = G \tag{1.12.1}$$

with regard to the matrix $H = H^*$. (Matrices P and G were supposed to be known).

Let us now state the problem about the solvability of inequalities

$$P^*H + HP \leqslant 0 \tag{1.12.2}$$

and

$$P^*H + HP < 0 \tag{1.12.3}$$

(with respect to H). They are called Lyapunov matrix inequalities, the latter being strict and the former being non-strict one.

Theorem 1.12.1 (about solvability of the non-strict Lyapunov matrix inequality [Gelig *et al.* 1978]) *Suppose P, q and r are matrices of order $n\times n$, $n\times m$ and $n\times m$ respectively and the pair (P,q) is controllable. Then there exists a matrix $H = H^*$ (which is real in case P, q and r are real) such that*

$$P^*H + HP \leqslant 0, \qquad Hq + r = 0 \tag{1.12.4}$$

if and only if

$$\operatorname{Re} r^*(P - i\omega I_n)^{-1}q \geqslant 0 \tag{1.12.5}$$

for all $\omega \in \mathbb{R}$ with $\det(P - i\omega I_n) \neq 0$. *If P is Hurwitzian and the pair (P,r) is observable then any matrix H satisfying (1.12.4) is positive definite.*

Proof. This theorem follows immediately from Theorem 1.10.1. Really, let us consider the following Hermitian form of x and ξ

$$\mathcal{G}(x,\xi) = 2\operatorname{Re}(\xi^* r^* x) \qquad (x \in \mathbb{C}^n,\ \xi \in \mathbb{C}^m).$$

Inequality (1.10.1) takes then the form

$$x^*(HP + P^*H)x + 2\operatorname{Re} x^*Hq\xi + 2\operatorname{Re}(x^*r\xi) \leqslant 0 \quad (x \in \mathbb{C}^n,\ \xi \in \mathbb{C}^m)$$

or

$$x^*(HP + P^*H)x + 2\operatorname{Re}\big[x^*(Hq + r)\xi\big] \leqslant 0 \qquad (x \in \mathbb{C}^n,\ \xi \in \mathbb{C}^m). \tag{1.12.6}$$

Suppose that (1.12.4) are true. Then (1.12.6) is true. Then according to Theorem 1.10.1 relationship (1.12.5) is true as well. It also follows from Theorem 1.10.1 that if P is Hurwitzian and (P,r) is observable then matrix H is positive definite.

Suppose now that (1.12.5) is true for all ω with $\det(P - i\omega I_n) \neq 0$. Then according to Theorem 1.10.1 inequality (1.12.6) is also true. Let us demonstrate that the latter implies $(Hq + r) = 0$. For the purpose we suppose the opposite and put

$$\xi = \alpha(Hq + r)^* x \qquad (\alpha > 0).$$

Then

$$x^*(Hq + r)\xi = \alpha x^*(Hq + r)(Hq + r)^* x > 0 \qquad (x \neq 0).$$

So for any fixed $x \neq 0$ one can always find a certain $\alpha > 0$ such that (1.12.6) is violated. Since $Hq + r = 0$ it follows from (1.12.6) that

$$HP + P^* H \leqslant 0.$$

Thus Theorem 1.12.1 is proved. □

Theorem 1.12.2 (about solvability of the strict Lyapunov matrix inequality [Gelig *et al.* 1978]) *Suppose that the pair* (P, q) *is stabilizable,* $\operatorname{rank} q = m$ *and* $\det(P - i\omega I_n) \neq 0$ $(\omega \in \mathbf{R})$. *Then there exists a matrix* $H = H^*$ *(which is real if* P, q *and* r *are real) satisfying the relations*

$$HP + P^*H < 0 \qquad \text{and} \qquad Hq + r = 0 \tag{1.12.7}$$

if and only if

$$\operatorname{Re} r^*(P - i\omega I_n)^{-1}q > 0 \quad \text{for all} \quad \omega \in \mathbf{R} \tag{1.12.8}$$

and

$$\lim_{\omega \to +\infty} \omega^2 \operatorname{Re} r^*(P - i\omega I_n)^{-1}q > 0. \tag{1.12.9}$$

Proof. We shall prove the theorem under the assumption that the pair (P, q) is controllable. Its full proof may be found in [Gelig *et al.* 1978]. Let us use Theorem 1.11.1 at first. Consider again

$$\mathcal{G}(x, \xi) = 2\operatorname{Re}(\xi^* r^* x) \qquad (x \in \mathbf{C}^n,\ \xi \in \mathbf{C}^m).$$

Inequality (1.11.1) takes then the form

$$\begin{gathered} x^*(HP + P^*H)x + 2\operatorname{Re}\big[x^*(Hq + r)\xi\big] < 0 \\ (x \in \mathbf{C}^n,\ \xi \in \mathbf{C}^m,\ |x| + |\xi| \neq 0). \end{gathered} \tag{1.12.10}$$

Suppose (1.12.7) are true. Then (1.12.10) is also true. Then according to Theorem 1.11.1

$$\operatorname{Re} r^*(P - i\omega I_n)^{-1}q > 0 \qquad (\omega \in [-\infty, +\infty]) \tag{1.12.11}$$

which implies (1.12.8) and (1.12.9).

Suppose now that (1.12.8) and (1.12.9) are true. Let us consider the Hermitian form

$$\mathcal{G}_2(x, \xi) = \mathcal{G}(x, \xi) + \varepsilon |x|^2 \qquad (x \in \mathbf{C}^n,\ \xi \in \mathbf{C}^m)$$

where a positive number ε will be chosen later on. Then

$$\mathcal{G}_2\big((i\omega I_n - P)^{-1}q\xi, \xi\big) = \xi^*\big[2\operatorname{Re} r^*(i\omega I_n - P)^{-1}q + \varepsilon|(P - i\omega I_n)^{-1}q|^2\big]\xi. \tag{1.12.12}$$

The number ε we may always choose in such a way that

$$\mathcal{G}_2\big((i\omega I_n - P)^{-1}q\xi, \xi\big) \leqslant 0 \qquad (\xi \in \mathbf{C}^m,\ \omega \in \mathbf{R}).$$

Indeed, let us fix a certain $\Omega > 0$. For this Ω, in virtue of (1.12.8) we may determine $\varepsilon > 0$ such that the matrix of right part of (1.12.12) is negative definite for $\omega \in [-\Omega, \Omega]$. Note that the function $|(i\omega I_n - P)^{-1}q|^2$ is bounded by k/ω^2 for $|\omega| \geqslant \Omega$ where k is a positive number. On the other hand we conclude from (1.12.9) that for $|\omega| > \Omega$

$$\operatorname{Re} r^*(i\omega I_n - P)^{-1}q < -\frac{L}{\omega^2}$$

where L is a positive number. It is clear now that ε may be diminished in such a way that the matrix of right part of (1.12.12) will be negative definite for all $\omega \in \mathbf{R}$. Then according to Theorem 1.10.1 there exists a matrix $H = H^*$ such that

$$x^*(HP + P^*H)x + \varepsilon x^*x + 2\,\mathrm{Re}\big[x^*(Hq + r)\xi\big] \leqslant 0 \qquad (x \in \mathbb{C}^n,\ \xi \in \mathbb{C}^m), \tag{1.12.13}$$

whence it follows that

$$Hq + r = 0. \tag{1.12.14}$$

The proof of this fact is borrowed from the proof of Theorem 1.12.1. In virtue of (1.12.14)

$$x^*(HP + P^*H)x \leqslant -\varepsilon x^*x \qquad (x \in \mathbb{C}^m)$$

which means that

$$HP + P^*H < 0.$$

The theorem is proved. □

1.13 Circle Criterion

We may now return to the problem which was regarded in Sections 1.8, 1.9, i.e. to the problem of absolute stability of the control system

$$\begin{cases} \dot{x} = Px + q\xi, \quad \sigma = r^*x \\ \xi = \varphi(t, \sigma) \end{cases} \qquad (x \in \mathbf{R}^n) \tag{1.13.1}$$

with nonlinear function $\varphi(t, \sigma)$ satisfying the relations

$$\varphi(t, 0) \equiv 0; \quad \mu_1 \leqslant \frac{\varphi(t, \sigma)}{\sigma} \leqslant \mu_2 \quad (t \in \mathbf{R}_+,\ \sigma \neq 0). \tag{1.13.2}$$

As we have shown in Section 1.9 an important component of this problem is the problem of existence of matrix $H = H^* > 0$ satisfying the inequality

$$2\,\mathrm{Re}\big[x^*H(Px + q\xi)\big] + F_c(x, \xi) < 0 \qquad (x \in \mathbb{C}^n,\ \xi \in \mathbb{C}^1 \text{ with } |x|^2 + |\xi|^2 \neq 0) \tag{1.13.3}$$

where

$$F_c(x, \xi) = \mathrm{Re}\big[(\xi - \mu_1\sigma)^*(\mu_2\sigma - \xi)\big] \qquad (x \in \mathbb{C}^n,\ \xi \in \mathbb{C}^1,\ \sigma = r^*x). \tag{1.13.4}$$

The latter problem may be solved with the help of Theorem 1.11.1. So suppose that the matrix P has no pure imaginary eigenvalues and let

$$x = P_{i\omega}^{-1}q\xi \qquad (P_{i\omega} := i\omega I_n - P).$$

Then $\sigma = r^* P_{i\omega}^{-1} q\xi$. In terms of transfer function of the linear part of (1.13.1) we have

$$\sigma = -\chi(i\omega)\xi.$$

(We remind that $\chi(p) = r^*(P - i\omega I_n)q$). Thus

$$F_c(P_{i\omega}^{-1} q\xi, \xi) = -\operatorname{Re}\{[\mu_1\chi(i\omega) + 1]^*[\mu_2\chi(i\omega) + 1]\}|\xi|^2.$$

So according to Theorem 1.11.1 necessary and sufficient conditions of existence of matrix $H = H^*$ satisfying (1.13.3) take the form of frequency-domain inequality

$$\operatorname{Re}\{[\mu_1\chi(i\omega) + 1]^*[\mu_2\chi(i\omega) + 1]\} > 0 \quad \text{for all } \omega \in \mathbf{R}^1. \tag{1.13.5}$$

Suppose (1.13.5) is fulfilled and a certain matrix $H = H^*$ satisfies the inequality (1.13.3). Let us consider the function

$$V(x) = x^* H x.$$

It is shown in Section 1.9 that the validity of (1.13.3) is equivalent to the fact that the derivative of $V(x)$ with respect to system (1.13.1), (1.13.2) is bounded from above by a negative definite quadratic form of $x \in \mathbf{R}^n$. To satisfy Theorem 1.7.3 we have only left to prove that matrix H is positive definite. Consider the linear system

$$\begin{cases} \dot{x} = Px + q\xi, & \sigma = r^* x \\ \xi = \mu_0\sigma, & \mu_0 \in [\mu_1, \mu_2]. \end{cases} \tag{1.13.6}$$

Suppose that for a certain μ_0 the system (1.13.6) is asymptotically stable. Then its matrix $P_0 = P + q\mu_0 r^*$ is a Hurwitzian one. On the other hand inequality (1.13.3) for system (1.13.6) takes the form

$$2\operatorname{Re}(x^* H P_0 x) + (\mu_0 - \mu_1)(\mu_2 - \mu_0)|\sigma|^2 < 0 \quad (|x| \neq 0),$$

whence

$$2\operatorname{Re}(x^* H P_0 x) < 0$$

or

$$x^*(HP_0 + P_0^* H)x < 0. \tag{1.13.7}$$

Since P_0 is Hurwitzian it follows from Lemma 1.10.1 (see formula (1.10.10)) that H may be represented in the form

$$H = -\int_0^\infty e^{P_0^* t} G e^{P_0 t}\,dt$$

where $G = HP_0 + P_0^* H$. Consequently $G < 0$. It is evident then that $H > 0$.

As a result we have proved the following assertion.

Theorem 1.13.1 (Circle criterion) *Suppose that the pair (P, q) is controllable and the following hypotheses are valid:*

(i) the matrix P has no eigenvalues with zero real parts;

(ii) for a certain $\mu_0 \in [\mu_1, \mu_2]$ linear system (1.13.6) is asymptotically stable;

(iii) frequency-domain inequality (1.13.5) is fulfilled.

Then system (1.13.1) is absolutely stable with respect to $M[\mu_1, \mu_2]$.

This theorem is cited in a lot of papers and books devoted to automatic control theory. We indicate here the most popular of them [Brockett 1970, Gelig *et al.* 1978, Voronov 1981].

Theorem 1.13.1 is called "the circle criterion" because of its geometric interpretation. Let us plot in the complex plane $\{ z = u+iv \}$ the set of dots $(\operatorname{Re}\chi(i\omega), \operatorname{Im}\chi(i\omega))$ $(\omega \in \mathbf{R})$. Condition (1.13.5) states that the hodograph of $\chi(i\omega)$ lies in the domain

$$\operatorname{Re}\big[(1+\mu_1 z)^*(1+\mu_2 z)\big] > 0. \tag{1.13.8}$$

The boundary of this domain is a circle which passes through the points $z_1 = -\mu_1^{-1}$ and $z_2 = -\mu_2^{-1}$ and has the center on the real axis. In case $\mu_1 = 0$ the circle transforms into a straight line. Note that (1.13.8) defines for $\mu_1 < 0$ and $\mu_2 > 0$ an open circular disc (Fig. 1.13.1-a). For $\mu_1 = 0$ we have a complex halfplane (Fig. 1.13.1-b). For $\mu_1 > 0$, $\mu_2 > 0$ (1.13.8) corresponds to the exterior of a circular disk (Fig. 1.13.1-c).

1.14 Popov Criterion

Let us consider the autonomous control system

$$\dot{x} = Px + q\xi, \quad \sigma = r^* x \quad (x \in \mathbf{R}^n), \tag{1.14.1}$$

$$\xi = \varphi(\sigma) \tag{1.14.2}$$

with real $n \times n$-matrix P and real n-vectors q and r. The function $\varphi(\sigma)$ is assumed to be continuous and locally Lipschitz. It also satisfies the inequalities

$$0 \leqslant \varphi(\sigma)\sigma \leqslant \mu\sigma^2 \qquad (\mu \leqslant +\infty). \tag{1.14.3}$$

Theorem 1.14.1 (Popov criterion) *Let the pair (P, q) be controllable and the matrix P be Hurwitzian. Suppose that for a certain number ν the following relationship is true:*

$$\mu^{-1} + \operatorname{Re}\big[(1+i\omega\nu)\chi(i\omega)\big] > 0 \quad (0 \leqslant \omega \leqslant \infty) \tag{1.14.4}$$

with $\mu^{-1} = 0$ if $\mu = +\infty$. Then system (1.4.1), (1.4.2) is absolutely stable with respect to $M[0, \mu]$.

Proof. Consider a Lyapunov function

$$V(x) = x^* H x + \nu \int_0^\sigma \varphi(\widetilde{\sigma})\, d\widetilde{\sigma}.$$

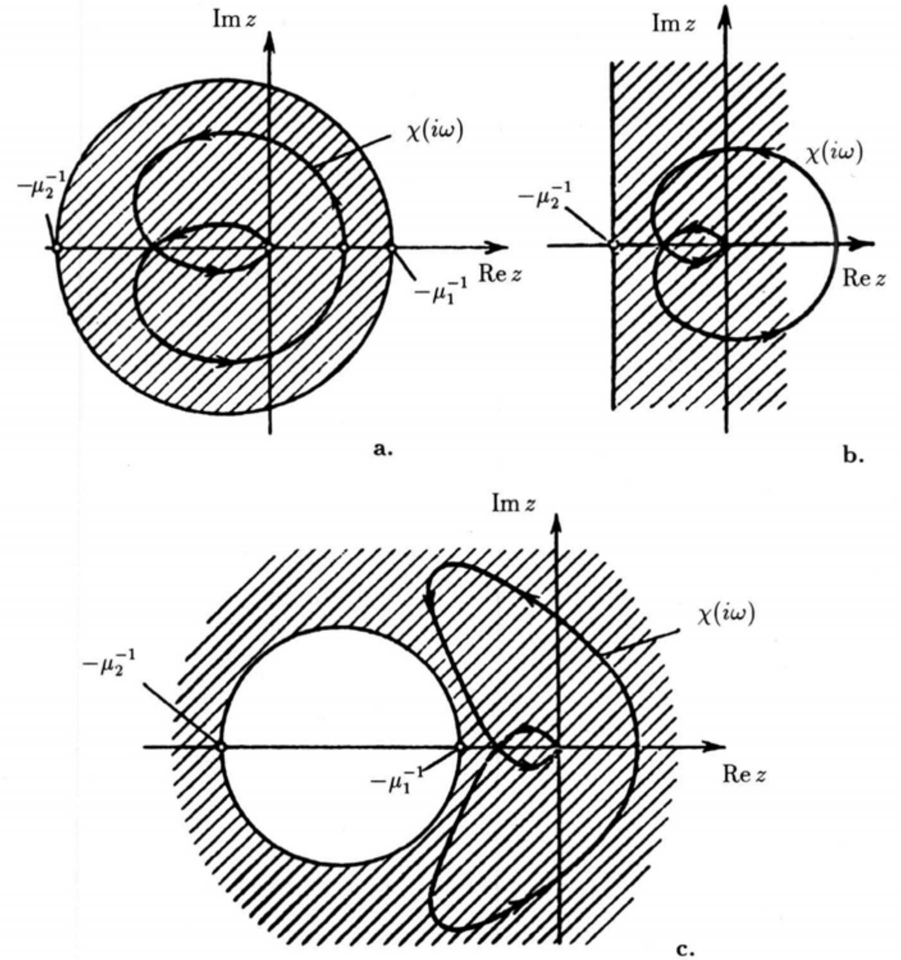


Fig. 1.13.1. Geometric interpretation of "the circle criterion".

The derivative of $V(x)$ with respect to system (1.14.1), (1.14.2) is as follows

$$\dot{V}_{(1.14.1)}(x) = 2x^*H(Px + q\xi) + \nu\xi r^*(Px + q\xi).$$

We shall demonstrate that frequency-domain inequality (1.14.4) guarantees that $\dot{V}_{(1.14.1)}$ is negative definite. For the purpose we represent inequality (1.14.3) in the form

$$\xi(\sigma - \mu^{-1}\xi) \geqslant 0 \tag{1.14.5}$$

and introduce the quadratic form

$$F(x,\xi) = \xi(\sigma - \mu^{-1}\xi) + \nu\xi r^*(Px + q\xi), \quad \sigma = r^* x. \tag{1.14.6}$$

The extension of $F(x,\xi)$ to a Hermitian form gives

$$F_c(x,\xi) = \mathrm{Re}\big[\xi^*(r^*x - \mu^{-1}\xi) + \nu\xi^* r^*(Px + q\xi)\big] \quad (x \in \mathbb{C}^n,\ \xi \in \mathbb{C}^1).$$

Let $x = P_{i\omega}^{-1} q\xi$. Then

$$r^*x = -\chi(i\omega)\xi, \qquad r^*(Px + q\xi) = i\omega r^* x$$

and

$$F_c(P_{i\omega}^{-1} q\xi, \xi) = -|\xi|^2 \,\mathrm{Re}\big[\chi(i\omega) + \mu^{-1} + \nu i\omega\chi(i\omega)\big].$$

Hence in virtue of (1.14.4) we receive that

$$F_c\big((i\omega I_n - P)^{-1} q\xi, \xi\big) < 0 \qquad (\xi \in \mathbb{C}^1,\ \xi \neq 0). \tag{1.14.7}$$

According to Theorem 1.11.1 it follows from (1.14.7) that

$$2\,\mathrm{Re}\, x^* H(Px + q\xi) + \mathrm{Re}\big[\nu\xi^* r^*(Px + q\xi)\big] + \mathrm{Re}\big[\xi^*(r^*x - \mu^{-1}\xi)\big] < 0$$
$$(x \in \mathbb{C}^n,\ \xi \in \mathbb{C}^1,\ |x| + |\xi| \neq 0)$$

whence it follows that

$$\dot V_{(1.14.1)}(x) < 0 \qquad (x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^1,\ |x| + |\xi| \neq 0). \tag{1.14.8}$$

Let us demonstrate now that $V(x) > 0$ for $x \neq 0$. For the purpose we put in (1.14.8) $\xi = 0$. Then we have

$$2x^* HPx < 0, \qquad x \neq 0, \quad x \in \mathbb{R}^n. \tag{1.14.9}$$

Since P is Hurwitzian it follows from Lemma 1.10.1 (more exact, from (1.10.10)) that

$$H = -\int_0^\infty e^{P^*t}(HP + P^*H)e^{Pt}\,\mathrm{d}t$$

and consequently in virtue of (1.14.9) $H < 0$. On the other hand $\varphi(\sigma)\sigma \geqslant 0$ and consequently $\int_0^\sigma \varphi(\sigma)\,\mathrm{d}\sigma \geqslant 0$ for any σ. So if $\nu \geqslant 0$ then $V(x) > 0$ for $x \neq 0$.

The case of $\nu < 0$ may be reduced to the case of $\nu > 0$. Indeed, let us introduce the function $\varphi_1(\sigma) = \mu\sigma - \varphi(\sigma)$. Then it follows from (1.14.3) that

$$0 \leqslant \varphi_1(\sigma)\sigma \leqslant \mu\sigma^2.$$

Let $\xi_1 = \varphi_1(\sigma)$ and $\chi_1(i\omega)$ be the transfer function of (1.14.1) from ξ_1 to $(-\sigma)$. Then on the one hand $\widetilde{\sigma} = -\chi_1(i\omega)\widetilde{\xi}_1$ and on the other hand

$$\widetilde{\sigma} = -\chi(i\omega)\widetilde{\xi} = -\chi(i\omega)(\mu\widetilde{\sigma} - \widetilde{\xi}_1).$$

So

$$\chi_1(i\omega) = -\frac{\chi(i\omega)}{1 + \mu\chi(i\omega)}.$$

It is not difficult to verify that

$$\mathrm{Re}\big[(1 - i\omega\nu)\chi_1(i\omega)\big] + \mu^{-1} = \frac{\mathrm{Re}\big[(1 + i\omega\nu)\chi(i\omega)\big] + \mu^{-1}}{\big|1 + \mu\chi(i\omega)\big|^2}.$$

Really,

$$\big[(1 - i\omega\nu)\chi_1(i\omega)\big] + \mu^{-1} = \frac{i\omega\nu\chi(i\omega) + \mu^{-1}}{1 + \mu\chi(i\omega)} = \frac{\big[i\omega\nu\chi(i\omega) + \mu^{-1}\big]\big[1 + \mu\chi^*(i\omega)\big]}{\big|1 + \mu\chi(i\omega)\big|^2}$$

$$= \big|1 + \mu\chi(i\omega)\big|^{-2}\big[i\omega\nu\chi(i\omega) + \mu^{-1} + i\mu\omega\nu|\chi(i\omega)|^2 + \chi^*(i\omega)\big].$$

Thus inequality (1.14.4) must be substituted by the inequality

$$\mu^{-1} + \mathrm{Re}\big[(1 - i\omega\nu)\chi_1(i\omega)\big] > 0.$$

So $V(x) > 0$ for $x \neq 0$ whatever sign of ν might be. Since all the hypotheses of Theorem 1.7.3 are fulfilled Theorem 1.14.1 is proved. □

Theorem 1.14.1 appeared in [Popov 1959]. It may be found in various published works on automatic control. See for example [Popov 1961, Gelig *et al.* 1978, Lefschetz 1965].

As Popov criterion admits various values of ν one must choose ν in such a way that the frequency-domain inequality (1.14.4) is fulfilled for maximum possible value of μ.

1.14.1 Geometrical Interpretation of Popov Criterion

Theorem 1.14.1 admits rather simple geometrical illustration. Let us use the notations

$$X(\omega) := \mathrm{Re}\,\chi(i\omega); \qquad Y(\omega) := \omega\,\mathrm{Im}\,\chi(i\omega)$$

and introduce the function

$$\widehat{\chi}(i\omega) := X(\omega) + iY(\omega)$$

which is called *the modified frequency response* of the linear part of (1.14.1). Let us plot in the plane $\zeta O\nu$ the curve

$$\begin{cases} \zeta = X(\omega), \\ \nu = Y(\omega) \end{cases} \qquad (\omega \in \mathbf{R}_+).$$

Definition 1.14.1 ([Aizerman & Gantmakher 1964]) *Consider on the plane $\zeta O\nu$ the straight line with the following properties:*

(i) the modified frequency response of (1.14.1) lies entirely to the right of the line;

(ii) the abscissa of its point of intersection with the ζ-axis is non-positive (it is denoted by ζ_p);

(iii) the value of ζ_p is the maximal possible with respect to all analogous values computed for other straight lines with properties (i), (ii).

This line is called Popov straight line.

Various dispositions of Popov straight line are shown in Figure 1.14.1.

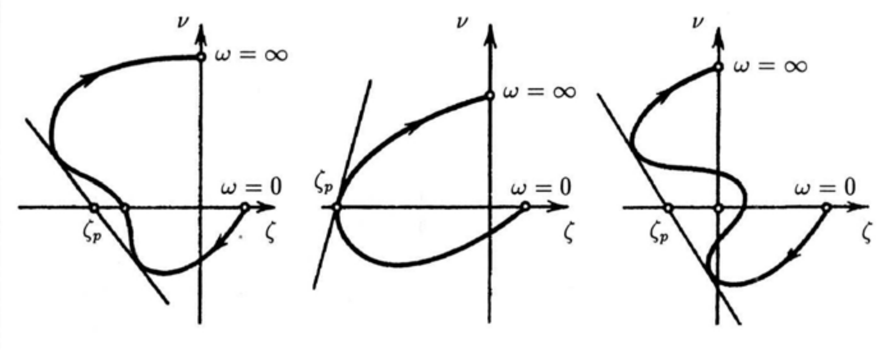


Fig. 1.14.1. Various dispositions of Popov straight line.

Since

$$\mathrm{Re}\{(1 + i\omega\nu)\chi(i\omega)\} = \mathrm{Re}\,\chi(i\omega) - \omega\nu\,\mathrm{Im}\,\chi(i\omega) = X(\omega) - \nu Y(\omega)$$

Popov inequality (1.14.4) may be rewritten in the form

$$X(\omega) - \nu Y(\omega) + \mu^{-1} > 0 \qquad (0 \leqslant \omega < +\infty).$$

Suppose that Popov straight line exists for system (1.14.1). Let $\mu_p = -\zeta_p^{-1}$ if $\zeta_p \neq 0$ and $\mu_p = \infty$ if $\zeta_p = 0$. Then according to Popov criterion system (1.14.1), (1.14.2) is absolutely stable for a class of nonlinear functions satisfying the condition

$$0 \leqslant \frac{\varphi(\sigma)}{\sigma} \leqslant \mu < \mu_p \quad (\sigma \neq 0) \qquad \text{if} \quad \mu_p \neq \infty$$

and

$$\frac{\varphi(\sigma)}{\sigma} \geqslant 0 \quad (\sigma \neq 0) \qquad \text{if} \quad \mu_p = \infty.$$

Popov straight line gives the opportunity to reveal the maximum possible value of μ for which the frequency-domain inequality (1.14.4) still holds. Thus the most wide class of nonlinearities in the frame of which absolute stability takes place is established.

We have supposed in Theorem 1.14.1 that matrix P of system (1.14.1) is a Hurwitzian one. The result can be extended on the situation when matrix P has pure imaginary eigenvalues. The cases when matrix P has pure imaginary eigenvalues are called critical cases.

Theorem 1.14.2 ([Yakubovich 1963]) *Let the transfer function $\chi(p)$ be non-degenerate. Let it have an arbitrary number of pure imaginary poles $i\omega_h$. Let also for a certain small positive number δ linear system (1.14.1), (1.14.2) with $\varphi(\sigma) = \delta\sigma$ be asymptotically stable. Suppose the following requirements are satisfied:*

(i) there exists a number ν such that the frequency domain inequality

$$\pi(\omega) := \frac{1}{\mu} + \mathrm{Re}\big[(1 + i\omega\nu)\chi(i\omega)\big] > 0$$

is true for $\{\,\omega \mid \omega \geqslant 0,\, \omega \neq \omega_h\,\}$;

(ii) $0 < \lim\limits_{\omega\to\infty} \omega^2\pi(\omega)$;

(iii) if $p = 0$ is a pole of multiplicity two the two relations

$$\int_0^{-\infty} \varphi(\sigma)\,\mathrm{d}\sigma = \infty \qquad \text{and} \qquad \int_0^{+\infty} \varphi(\sigma)\,\mathrm{d}\sigma = \infty$$

are fulfilled.

Then system (1.14.1), (1.14.2) is absolutely stable with respect to $M[\,0,\mu\,]$.

Note that if $\pi(\infty) \neq 0$ then the requirement (ii) is fulfilled automatically.

Let us consider the case when $\chi(p)$ has only two pure imaginary poles $\pm i\omega_0$ ($\omega_0 \neq 0$). In this case

$$\chi(p) = \frac{\alpha p + \beta}{p^2 + \omega_0^2} + \chi_1(p)$$

where $\chi_1(p)$ has no pure imaginary poles. Since in this case

$$\mathrm{Re}\big[(1 + i\omega\nu)\chi(i\omega)\big] = \frac{\beta - \alpha\omega^2\nu}{\omega_0^2 - \omega^2} + (1 + i\omega\nu)\chi(i\omega)$$

it is clear that the requirement (i) may be satisfied only for $\nu = \beta/\alpha\omega_0^2$. Otherwise the frequency-domain inequality is violated for ω sufficiently close to ω_0.

We do not give the proof of Theorem 1.14.2 here. It may be found in [Yakubovich 1963].

1.15 Aizerman Conjecture. Kalman Conjecture

The Aizerman conjecture is closely linked with the absolute stability problem. Let us consider the feedback control system

$$\dot{x} = Px + q\varphi(\sigma), \qquad \sigma = r^*x, \qquad \varphi(0) = 0 \tag{1.15.1}$$

where P is an $n\times n$-matrix and q and r are n-vectors. Let us take $\varphi(\sigma) = \mu\sigma$. Then we obtain the linear system

$$\dot{x} = (P + \mu qr^*)x, \tag{1.15.2}$$

Suppose that for all $\mu \in (\alpha, \beta)$, where α and β are certain numbers, it is true that all eigenvalues of the matrix $P + \mu qr^*$ have negative real parts but both $P + \alpha qr^*$ and $P + \beta qr^*$ have pure imaginary or zero eigenvalues. Then the set on the plane $\{\sigma, \varphi\}$ bounded by the two straight lines $\varphi = \alpha\sigma$ and $\varphi = \beta\sigma$ is called Hurwitzian sector (or the sector of linear stability) of system (1.15.1). We denote it by $S(\alpha, \beta)$.

Suppose that the graph of the nonlinear function $\varphi(\sigma)$ lies in the Hurwitzian sector of system (1.15.1), i.e. the following inequalities are true:

$$\alpha\sigma^2 < \sigma\varphi(\sigma) < \beta\sigma^2, \qquad \forall\sigma \neq 0, \tag{1.15.3}$$

In 1949 M. A. Aizerman [Aizerman 1949] stated the following alternative. If system (1.15.1) is always absolutely stable with respect to its Hurwitzian sector or not? This problem generated numerous investigations.

Let us present firstly a case when the Popov criterion gives the affirmative solution of the Aizerman alternative. Consider $n = 2$ and

$$\chi(p) = \frac{ap + b}{p^2 + \omega_0^2}.$$

The characteristic polynomial for the matrix of the linear system (1.15.2) is

$$\det(P + \mu qr^* - pI) = \det(P - pI)\det\left(I + \mu(P - pI)^{-1}qr^*\right).$$

According to Corollary 1.5.1

$$\det(P + \mu qr^* - pI) = \left(1 + \mu\chi(p)\right)\det(P - pI).$$

So eigenvalues of the matrix $P + \mu qr^*$ are the zeros of the polynomial

$$p^2 + a\mu p + \omega_0^2 + \mu b = 0.$$

Suppose that the Hurwitzian sector here is $S(0, \beta)$ with $\beta > 0$. It follows from Theorem 1.4.4 then that $a\mu > 0$ and $\omega_0^2 + \mu b > 0$ for $\mu \in (0, \beta)$. On the other hand the Popov criterion in this case has the form

$$\frac{1}{\mu} + \mathrm{Re}\left\{(1 + i\omega\theta)\frac{ai\omega + b}{\omega_0^2 - \omega^2}\right\} > 0 \qquad \text{for } \omega \in \mathbb{R}, \tag{1.15.4}$$

where θ is a varied parameter from $\mathbf{R}$. The left side of (1.15.4) can be transformed to the form

$$\frac{1}{\mu}+\frac{b-a\theta\omega^2}{\omega_0^2-\omega^2}.$$

It is clear that we must take $\theta = b/a\omega_0^2$, otherwise (1.15.4) will be violated. Then inequality (1.15.4) is as follows

$$\frac{1}{\mu}+\frac{b}{\omega_0^2}>0$$

and it is true in the Hurwitzian sector. So the Hurwitzian sector here coincide with the absolute stability sector.

But in general the Aizerman alternative is not solved affirmative. The first example which demonstrated that inequalities (1.15.3) are not sufficient for global asymptotic stability of nonlinear system were obtained for the case of $n = 2$ by N. N. Krasovsky [Krasovsky 1952]. This result was generalized by V. A. Pliss [Pliss 1958] and V. A. Yakubovich [Yakubovich 1958] for the multidimensional case.

In these papers it was demonstrated that system (1.15.1), (1.15.3) with the Hurwitzian sector $S(\alpha,\beta)$ can have solutions which does not tend to 0 as $t \to +\infty$. In this connection the following problem appeared. If system (1.15.1), (1.15.3) with the Hurwitzian sector $S(\alpha,\beta)$ can have periodic solutions. The first affirmative reply for such question was given by V. A. Pliss in his book [Pliss 1964]. V. A. Pliss considered the case of $n = 3$ under the assumption that the matrix P has a pair of pure imaginary eigenvalues. Further in Section 3.1 of this monograph the result of V. A. Pliss will be generalized for the case of arbitrary n.

Since condition (1.15.3) proved to be insufficient for (1.15.1) to be absolutely stable in $S(\alpha,\beta)$, another conjecture about absolute stability of (1.15.1) in its Hurwitzian sector, with more restrictive conditions of $\varphi(\sigma)$, was bought forward [Kalman 1957]. It is called Kalman conjecture. Suppose that the function $\varphi(\sigma)$ from (1.15.1) is continuously differentiable. Let $S(\alpha,\beta)$ be the Hurwitzian sector of (1.15.1). The Kalman conjecture supposes that the trivial solution of (1.15.1) must be globally asymptotically stable if $\varphi'(\sigma)\in(\alpha,\beta)$.

We must note first of all that the additional restrictions on $\varphi(\sigma)$ gave the opportunity to "strengthen" the Popov criterion (see [Yakubovich 1965, Brockett & Willems 1965]). We present here one of possible assertions of frequency-domain stability criteria.

Theorem 1.15.1 *Let the pair (P,q) be controllable and the matrix P be Hurwitzian. Let $\varphi(\sigma)$ be continuously differentiable on $\mathbf{R}$ and the following inequalities be true*

$$0\leqslant\varphi(\sigma)\sigma\leqslant\mu\sigma^2\qquad(\mu<+\infty),$$

$$-\infty<\mu_1\leqslant\frac{d\varphi}{d\sigma}\leqslant\mu_2<+\infty,$$

where $\mu_1 \leqslant 0$ and $\mu_2 \geqslant \mu$. Suppose there exist a number ν and a nonnegative number τ such that for all $\omega \in \mathbf{R}$ the inequality

$$\frac{1}{\mu} + \operatorname{Re}\left\{(1 + i\omega\nu)\chi(i\omega) + \tau\omega^2(\mu_1\chi(i\omega) + 1)^*(\mu_2\chi(i\omega) + 1)\right\} \geqslant 0 \qquad (1.15.5)$$

holds. Then system (1.15.1) is absolutely stable with respect to $M[0, \mu]$.

We will not adduce the proof of the theorem here, since its validity can be deduced from Theorem 1.19.1, formulated further in Section 1.19.

Let us now assume that $n = 3$ for (1.15.1), it has the finite Hurwitzian sector $S(0, \beta)$, $(\beta > 0)$ and the matrix $P + \beta q r^*$ has a zero eigenvalue. So the matrix P has two pure imaginary eigenvalues, and

$$\chi(p) = \frac{ap + b}{p^2 + \omega_0^2} + \frac{\varkappa}{p + \gamma} \qquad (\gamma > 0,\ \varkappa \neq 0,\ b \neq 0). \qquad (1.15.6)$$

We shall demonstrate that criterion (1.15.5) gives the affirmative solution for the Kalman conjecture in this case. Let $0 \leqslant \mathrm{d}\varphi/\mathrm{d}\sigma \leqslant \mu$. Then $\mu_1 = 0$ and $\mu_2 = \mu$. It follows that $0 \leqslant \varphi(\sigma)/\sigma \leqslant \mu$. Then (1.15.5) takes the form

$$\frac{1}{\mu} + \tau\omega^2 + \operatorname{Re}\left\{(1 + i\omega\nu + \tau\mu\omega^2)\chi(i\omega)\right\} > 0.$$

Let us substitute the varied parameter τ for τ_0/μ and work with the parameters ν and τ_0. Finally we have the frequency-domain inequality

$$\frac{1}{\mu} + \frac{\tau_0\omega^2}{\mu} + \operatorname{Re}\left\{(1 + i\omega\nu + \tau_0\omega^2)\chi(i\omega)\right\} > 0. \qquad (1.15.7)$$

Note that

$$\chi(i\omega) = \frac{ai\omega + b}{\omega_0^2 - \omega^2} + \frac{\varkappa(\gamma - i\omega)}{\omega^2 + \gamma^2}$$

and

$$\operatorname{Re}\left\{(1 + \nu i\omega + \tau_0\omega^2)\chi(i\omega)\right\} = \frac{b + b\tau_0\omega^2 - a\omega^2\nu}{\omega_0^2 - \omega^2} + \frac{\omega^2(\varkappa\gamma\tau_0 + \varkappa\nu) + \varkappa\gamma}{\omega^2 + \gamma^2}.$$

It is clear that we ought to choose

$$\nu = \frac{b(1 + \tau_0\omega_0^2)}{a\omega_0^2}$$

lest the term

$$\frac{b + b\tau_0\omega^2 - a\omega^2\nu}{\omega_0^2 - \omega}$$

should change its sign in the neighborhood of $\omega = \omega_0$. So (1.15.7) takes the form

$$\frac{\tau_0}{\mu}\omega_0^4 + \omega^2\left(\frac{\tau_0\gamma}{\mu} + \frac{1}{\mu} + \frac{b}{\omega_0^2} + \varkappa(\gamma\tau_0 + \nu)\right) + \gamma^2\left(\frac{1}{\mu} + \frac{b}{\omega_0^2} + \frac{\varkappa}{\gamma}\right) > 0. \qquad (1.15.8)$$

Let us demonstrate that (1.15.8) is true for all $\omega \in \mathbf{R}$ if $\mu \in (0, \beta)$.

For the purpose let us consider the characteristic polynomial of $P + \mu q r^*$. It is as follows

$$\begin{aligned}\Omega(p) &:= \det(P + \mu q r^* - pI) = \big(1 + \mu\chi(p)\big)\det(P - pI)\\ &= (p^2 + \omega_0^2)(p + \gamma) + \mu\big[\varkappa(p^2 + \omega_0^2 + (ap + b)(p + \gamma))\big] =\\ &= p^3 + p^2\big(\gamma + \mu(\varkappa + a)\big) + p\big(\omega_0^2 + \mu(\gamma a + b)\big) + \omega_0^2\gamma + \mu(\varkappa\omega_0^2 + \gamma b).\end{aligned}$$

Since the matrix $P + \mu q r^*$ is Hurwitzian all the coefficients of its characteristic polynomial are positive (Theorem 1.4.1). So we have

$$\frac{1}{\mu} + \frac{\varkappa + a}{\gamma} > 0, \qquad (1.15.9)$$

$$\frac{1}{\mu} + \frac{\gamma a + b}{\omega_0^2} > 0, \qquad (1.15.10)$$

and

$$\frac{1}{\mu} + \frac{b}{\omega_0^2} + \frac{\varkappa}{\gamma} > 0. \qquad (1.15.11)$$

It is clear now that it is sufficient to prove that there always exists $\tau_0 > 0$ such that the coefficient at ω^2 is nonnegative. The latter can be represented as follows

$$\frac{\tau_0\gamma}{\mu} + \frac{1}{\mu} + \frac{b}{\omega_0^2} + \varkappa(\gamma\tau_0 + \nu) = \tau_0\left(\frac{\gamma^2}{\mu} + \gamma\varkappa + \frac{\varkappa b}{a}\right) + \frac{1}{\mu} + \frac{b}{\omega_0^2} + \frac{\varkappa b}{a\omega_0^2}.$$

It is evident that for $\varkappa > 0$, $b > 0$ the latter expression is positive. So we have to consider all other cases.

Since (1.15.11) is true we have

$$\tau_0\left(\frac{\gamma^2}{\mu} + \gamma\varkappa + \frac{\varkappa b}{a}\right) + \frac{1}{\mu} + \frac{b}{\omega_0^2} + \frac{\varkappa b}{a\omega_0^2} > \tau_0\left(\frac{\varkappa b}{a} - \frac{\gamma^2 b}{\omega_0^2}\right) + \varkappa\left(\frac{b}{a\omega_0^2} - \frac{1}{\gamma}\right).$$

Let us show that for the cases $\varkappa < 0$, $\beta < 0$ and $\varkappa\beta < 0$ either

$$\varkappa\left(\frac{b}{a\omega_0^2} - \frac{1}{\gamma}\right) > 0 \qquad (1.15.12)$$

or

$$\frac{\varkappa b}{a} - \frac{\gamma^2 b}{\omega_0^2} > 0. \qquad (1.15.13)$$

Then we always can find τ_0 with the required property.

(i). If $\varkappa < 0$, $b < 0$ then (1.15.12) is true.

(ii). Let $\varkappa > 0$, $b < 0$. Then we compare (1.15.11) and (1.15.10) and use the fact that the upper boundary of the Hurwitzian sector is determined by the equality

$$\frac{1}{\mu} + \frac{\varkappa}{\gamma} + \frac{b}{\omega_0^2} = 0.$$

(Indeed, for this very value of μ the characteristic equation $\Omega(p) = 0$ has a zero root). We obtain

$$\frac{\gamma a}{\omega_0^2} > \frac{\varkappa}{\gamma}.$$

Hence and from $\beta < 0$ (1.15.13) follows.

(iii). Let $\varkappa < 0$, $\beta > 0$. Then we compare (1.15.11) and (1.15.9). We obtain $a/\gamma > b/\omega_0^2$. Hence and from $\varkappa < 0$ (1.15.12) follows.

If $\mu = \beta$ then $\mu^{-1} + b/\omega_0^2 + \varkappa/\gamma = 0$ and other coefficients of $\Omega(p)$ are positive.

Thus if $\mu \in (0, \beta)$ and $0 \leqslant \mathrm{d}\varphi/\mathrm{d}\sigma \leqslant \mu$ the trivial solution of system (1.15.1) is globally asymptotically stable. So if $\chi(p)$ is given by (1.15.6), system (1.15.1) has a finite Hurwitzian sector $S(0, \beta)$ with a zero eigenvalue of $P + \beta q r^*$ then the Kalman conjecture has the affirmative solution.

In [Barabanov 1988] it is proved that the Kalman conjecture is true for $n = 3$. But for $n \geqslant 4$ the examples are known when the conjecture fails [Barabanov 1988].