Chapter 1

Basic Concepts used for Description of Motion and Deformation of a Continuum

1 Lagrangian and Eulerian Description of Motion

Within the scope of classical mechanics, all particles of a continuous medium (or, more briefly, continuum) can be distinguished each from other, i.e., can be individualized. The individual particles are associated with triples of numbers (ξ_1, ξ_2, ξ_3). Such a triple is called the *Lagrangian coordinates* of the corresponding individual particle. The Lagrangian coordinates are used to indicate this particle, i.e., to serve as its "name" (in the same manner as integers serve as "names" of particles in the case of their discrete configuration). The coordinates of the point occupied by a particle at the initial instant are ordinarily used as the Lagrangian coordinates of this particle. The motion of a continuum and processes taking place in it are described by fields of physical quantities (velocity, pressure, temperature, etc.). If these quantities are considered as functions of the Lagrangian coordinates (ξ_1, ξ_2, ξ_3) and time t, the description is referred to as *Lagrangian* or *material*. From this viewpoint, events are considered to take place in the individual particles. The function $\boldsymbol{r}(\xi_1, \xi_2, \xi_3, t)$, or *motion law*, is used as a basic kinematical characteristic in the Lagrangian description. The vector $\boldsymbol{r}(\xi_1, \xi_2, \xi_3, t)$ is the position (relative to a chosen frame of reference) of the particle (ξ_1, ξ_2, ξ_3) at the instant t. If a coordinate system is chosen in the three-dimensional Euclidean space of vectors $\boldsymbol{r}$ (i.e., a one-to-one correspondence is set between the vectors and triples of numbers $\boldsymbol{r} \leftrightarrow (x_1, x_2, x_3)$), then the motion law is represented also by the functions $x_i = f_i(\xi_1, \xi_2, \xi_3, t)$, $i = 1, 2, 3$. The velocity and acceleration of the particles of a continuum are defined by the formulae

$$v(\xi, t) = \frac{\partial r(\xi, t)}{\partial t} , \qquad a(\xi, t) = \frac{\partial v(\xi, t)}{\partial t} ,$$

where $\xi = (\xi_1, \xi_2, \xi_3)$. In general, the rate of change in some quantity A in an individual particle of a continuum is called the *substantive* or *total derivative* of this quantity with respect to time. In the Lagrangian description this is simply the partial derivative $\partial A(\xi, t)/\partial t$. Physical quantities characterizing motion of a continuum and processes taking place can be considered as functions of the spatial coordinates (x_1, x_2, x_3) and time t. From this viewpoint, events are considered to take place at the points of the space. Such viewpoint is referred to as *Eulerian* or *spatial*. The velocity field $v(x_1, x_2, x_3, t)$ is used as a basic kinematical characteristic in the Eulerian description; spatial coordinates are called Eulerian. The vector $v(x_1, x_2, x_3, t)$ is the velocity of a particle occupying the point (x_1, x_2, x_3) at the instant t. In the Eulerian description the substantive derivative of some quantity A is denoted with $dA(x, t)/dt$, where $x = (x_1, x_2, x_3)$, and computed by the formula

$$\frac{dA(x,t)}{dt} = \frac{\partial A(x,t)}{\partial t} + v_1(x,t)\,\frac{\partial A(x,t)}{\partial x_1} + v_2(x,t)\,\frac{\partial A(x,t)}{\partial x_2} + v_3(x,t)\,\frac{\partial A(x,t)}{\partial x_3} .$$

In particular, the acceleration $a(x, t)$ in Eulerian description is determined by the formula

$$a(x,t) = \frac{\partial v(x,t)}{\partial t} + v_1(x,t)\,\frac{\partial v(x,t)}{\partial x_1} + v_2(x,t)\,\frac{\partial v(x,t)}{\partial x_2} + v_3(x,t)\,\frac{\partial v(x,t)}{\partial x_3} ,$$

where $v_i(x, t)$ are velocity components.

Lagrangian and Eulerian viewpoints are equivalent: if processes are described from one of them, the description from the other is obtained with a simple procedure. *To change over the description from Lagrangian to Eulerian*, the functions representing the motion law $x_i = f_i(\xi_1, \xi_2, \xi_3, t)$, $i = 1, 2, 3$, should be resolved with respect to the Lagrangian coordinates, i.e., the inverse functions $\xi_\alpha = g_\alpha(x_1, x_2, x_3, t)$, $\alpha = 1, 2, 3$, should be found. Then, for any quantity, the Lagrangian description of which $A(\xi_1, \xi_2, \xi_3, t)$ is known, the Eulerian description is obtained as the composite function $A(g_1(x, t), g_2(x, t), g_3(x, t), t)$. *To change over the description from Eulerian to Lagrangian*, the solution of the ordinary differential equations

$$\frac{dx_i}{dt} = v_i(x_1, x_1, x_3, t) , \quad i = 1, 2, 3 ,$$

subjected to the initial conditions

$$x_1|_{t=0} = \xi_1 , \qquad x_2|_{t=0} = \xi_2 , \qquad x_3|_{t=0} = \xi_3$$

should be found. This solution $x_i = f_i(\xi_1, \xi_2, \xi_3, t)$, $i = 1, 2, 3$, when found for all values of the parameters (ξ_1, ξ_2, ξ_3), gives the motion law in Lagrangian description, and (ξ_1, ξ_2, ξ_3) are the Lagrangian coordinates of the particles. Then, for any quantity $B(x_1, x_2, x_3, t)$, the Eulerian description of which is known, the Lagrangian description is obtained as the composite function $B(f_1(\xi, t), f_2(\xi, t), f_3(\xi, t), t)$. A motion of a continuum is well, although incompletely, represented by the *particle paths* and *streamlines*. The locus of the positions of a particle (ξ_1, ξ_2, ξ_3) at all instants is called the *path* of this particle. A curve, determined for a given instant t_0, whose tangent at any point x is in the direction of the velocity vector $\boldsymbol{v}(x, t_0)$, is called a *streamline*. The equation of a streamline at an instant t_0 has the following form

$$\frac{dx_1}{v_1(x, t_0)} = \frac{dx_2}{v_2(x, t_0)} = \frac{dx_3}{v_3(x, t_0)} .$$

In general, streamlines depend upon the instant t_0 for which they are found. When a motion is steady, the streamlines do not depend upon the instant t_0 and coincide with the particle paths. A motion is referred to as *steady* if the velocity field in the Eulerian description does not depend upon the time t. In the problems of this section, (x_1, x_2, x_3) and (ξ_1, ξ_2, ξ_3) denote respectively spatial Cartesian and Lagrangian coordinates.

— PROBLEMS —

⊙ **1.1** Introduce a spatial coordinate system and Lagrangian coordinates of the particles in the following cases: a) a rigid body executes a translatory motion in a fixed direction at constant speed v; b) a rigid body rotates about a fixed axis at constant angular speed ω.

⊙ **1.2** For translatory motions of a rigid body, find the general form of the velocity field in the Lagrangian description and that of the motion law.

⊙ **1.3** The motion law of a medium is represented by the functions

$$x_1 = \xi_1 + a\,t\,\xi_2\,, \qquad x_2 = \xi_2 + b\,t\,\xi_1\,, \qquad x_3 = \xi_3 \qquad (a, b = \text{const}).$$

Verify that the numbers (ξ_1, ξ_2, ξ_3) for an individual particle have the meaning of coordinates x_1, x_2, x_3 of the point occupied by this particle at the instant $t = 0$. Find velocity and acceleration fields in the Lagrangian description. What particle occupies the point (x_1^0, x_2^0, x_3^0) at the instant $t = 0$?

⊙ **1.4** The motion of a medium is represented by the functions

$$x_1 = \xi_1\left(1 + \frac{t}{\tau}\right), \quad x_2 = \xi_2\left(1 + 2\,\frac{t}{\tau}\right), \quad x_3 = \xi_3\left(1 + \frac{t^2}{\tau^2}\right)$$

$$(\tau = \text{const}).$$

a) Find the velocity and acceleration fields in the Lagrangian description. b) A particle occupied the point (a, b, c) at the instant $t = \tau$. Find its position at the instant $t = 3\tau$.

⊙ **1.5** Consider the functions $\xi_\alpha = g_\alpha(x_1, x_2, x_3, t)$, $\alpha = 1, 2, 3$, inverse, at instant t, to the functions $x_i = f_i(\xi_1, \xi_2, \xi_3, t)$, $i = 1, 2, 3$, representing a motion law. What is the physical meaning of their values? Find the values of the substantive derivatives $d\xi_\alpha/dt$.

⊙ **1.6** Find velocity and acceleration fields in the Lagrangian and Eulerian descriptions if the motion is a) a three-axis extension:

$$x_1 = a(t)\,\xi_1\,, \qquad x_2 = b(t)\,\xi_2\,, \qquad x_3 = c(t)\,\xi_3\,;$$

b) a simple shear:

$$x_1 = \xi_1 + b(t)\,\xi_2\,, \qquad x_2 = \xi_2\,, \qquad x_3 = \xi_3\,;$$

c) a uniform deformation and simultaneous rotation of a body with a fixed point:

$$x_i = A_{i1}(t)\,\xi_1 + A_{i2}(t)\,\xi_2 + A_{i3}(t)\,\xi_3\,, \qquad (det\|A_{ij}\| \neq 0).$$

⊙ **1.7** Introduce Lagrangian coordinates and find the motion law of a medium if the motion is described by the velocity field

$$v_1 = \frac{x_1}{t+\tau}\,, \qquad v_2 = \frac{2tx_2}{t^2+\tau^2}\,, \qquad v_3 = \frac{3t^2x_3}{t^3+\tau^2} \qquad (\tau = \text{const} > 0).$$

⊖ **1.8** Introduce Lagrangian coordinates and find the motion law of a medium, the streamlines, and the particle paths, if the motion is described by the velocity field a) $v_1 = \dfrac{Q(t)}{2\pi}\dfrac{x_1}{\rho^2}$, $v_2 = \dfrac{Q(t)}{2\pi}\dfrac{x_2}{\rho^2}$, $v_3 = 0$ $(\rho = \sqrt{x_1^2 + x_2^2})$, $Q(t) > 0$; b) $v_i = \dfrac{Q(t)}{4\pi}\dfrac{x_i}{R^3}$, $i = 1, 2, 3$ $(R = \sqrt{x_1^2 + x_2^2 + x_3^3})$, $Q(t) > 0$; c) $v_1 = -Ax_1$, $v_2 = Bx_2$, $v_3 = 0$, $A = \text{const} > 0$, $B = \text{const} > 0$.

⊙ **1.9** Introduce Lagrangian coordinates and find the motion law of a medium if the motion is described by the velocity field

$$v_1 = -A(t)x_1\,, \qquad v_2 = B(t)x_2\,, \qquad v_3 = 0\,,$$

$$A(t) > 0\,, \qquad B(t) > 0\,.$$

Find the streamlines and compare them with the streamlines for the special case $A, B = \text{const}$ (Problem 1.8). Give an example of the functions $A(t)$, $B(t)$ at which the streamlines and particle paths do not coincide.

⊙ **1.10** a) Can the functions representing a motion law be found from the known particle paths? b) Can a velocity field at a given instant be found from the streamlines known for this instant?

⊙ **1.11** Find and sketch the streamlines and particle paths if motions of a medium are described by the velocity fields a) $v_1 = -\omega x_2$, $v_2 = \omega x_1$, $v_3 = u$ $(\omega, u = \text{const})$; b) $v_1 = -Ax_2$, $v_2 = Bx_1$, $v_3 = 0$ $(A = \text{const} > 0$, $B = \text{const} > 0)$; c) $v_1 = -V \sin \omega t$, $v_2 = V \cos \omega t$, $v_3 = 0$ $(\omega, V = \text{const})$.

⊙ **1.12** Can the particles of a medium move with nonzero acceleration if a) the velocities of all the particles are identical? b) the velocity at any point of the space does not vary depending on the time?

⊙ **1.13** The density of every individual particle of an incompressible medium is constant. Can the density at any point of the space depend upon the time?

⊙ **1.14** While a medium moves with the velocity field

$$v_1 = -\omega x_2 , \qquad v_2 = \omega x_1 , \qquad v_3 = 0 \qquad (\omega = \text{const}),$$

the temperature field

$$T = T_0 e^{\frac{-t}{\tau} - (\frac{x_1}{a})^2 - (\frac{x_2}{b})^2 - (\frac{x_3}{c})^2} \qquad (T_0, \tau, a, b, c = \text{const}).$$

is created in the space (with the help of appropriately distributed sources). Find the rate of change in temperature in an individual particle at an instant t_0 if this particle is situated at the point of the space with the coordinates $x_1 = a$, $x_2 = b$, $x_3 = c$ at this instant.

⊙ **1.15** A medium moves with the velocity field

$$v_1 = kx_1 , \qquad v_2 = -kx_2 , \qquad v_3 = 0 \qquad (k = \text{const})$$

and the density field

$$\rho = \rho_0 + Ax_2 e^{kt} \qquad (\rho_0, A = \text{const}).$$

Find the rate of change in density in each of the particles of the medium.

⊖ **1.16** The position of an individual particle (ξ_1, ξ_2, ξ_3) at any instant t is given by the relationships

$$x_i = f_i(\xi_1 + Ut, \xi_2, \xi_3) , \qquad i = 1, 2, 3 , \qquad (U = \text{const}).$$

Show that a) the motion is steady, b) the parametric equations of the streamlines have the form $x_i = f_i(\tau, \xi_2^0, \xi_3^0)$, $i = 1, 2, 3$, where τ is a parameter, and the pair of numbers ξ_2^0, ξ_3^0 specifies a streamline.

⊖ **1.17** A medium moves so that the paths of all particles lie on rays originating from a point O, and the speed v and density ρ depend only upon the instant t and distance x to the point O. The mass contained, at the instant $t = 0$, within the sphere with the center O passing through some particle is often used as one of Lagrangian coordinates ξ of this particle to describe such a motion (referred to as spherically symmetric). Show that the expression

$$\xi = \int_0^x 4\pi R^2 \rho(R,t)dR$$

is valid for the Lagrangian coordinate ξ of a particle at a distance x from the point O at an instant t. Show that the speed and density of the medium depends only upon ξ and t in the Lagrangian description. Find the equation for these functions $\tilde{v}(\xi,t)$, $\tilde{\rho}(\xi,t)$ (including also the function $x(\xi,t)$) with transforming the equation

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} + 2\frac{\rho v}{x} = 0$$

expressing the mass conservation law in the Eulerian description.

⊙ **1.18** The motion of a medium is represented by the functions

$$x_1 = \xi_1, \qquad x_2 = \xi_2\left(1 + \frac{t}{\tau}\right), \qquad x_3 = \xi_3\,\frac{1}{1+\frac{t}{\tau}} \qquad (\tau = \text{const}).$$

a) Find the velocity and acceleration fields. b) Find the velocity, at the instant $t = 2\tau$, of the particle situated at the point with the coordinates (a, a, a) at the instant $t = \tau$.

⊙ **1.19** Find the velocity and acceleration fields in Lagrangian and Eulerian descriptions if the motion of the medium is represented by the functions

$$x_1 = \xi_1 + c(t)\xi_2\,, \qquad x_2 = \xi_2 + c(t)\xi_3\,, \qquad x_3 = \xi_3.$$

⊙ **1.20** A medium moves with the velocity field

$$v_1(x,t) = at\,, \qquad v_2(x,t) = -u\frac{x_2}{x_1}\,, \qquad v_3(x,t) = 0 \qquad (a, u = \text{const})$$

and the temperature field

$$T = T_0\left(1 + \frac{t^2}{\tau^2}\right) \qquad (T_0, \tau = \text{const}).$$

Find the rate of change in temperature at the instant $t = \tau$ in the individual particle situated at the point with the coordinates

$$x_1 = u^2/a\,, \qquad x_2 = 2u^2/a\,, \qquad x_3 = 3u^2/a\,.$$

2 Tensors and their Cartesian Components

Expressions with indices. Quantities of the same type are often denoted by one letter equipped with some number of indices. The Kronecker symbols δ_{ij} ($\delta_{ij} = 1$ at $i = j$, $\delta_{ij} = 0$ at $i \neq j$) is an example of a set of such quantities. Further, in absence of special notes, different indices are implied to take independently each of the values 1, 2, 3.

To shorten the notation, the following *summation convention* is adopted: If in a one-term expression composed of letters with indices some index appears twice, this expression denotes the sum of the corresponding terms taken for every value of this index. For example, $a_{ij}\, b_{jkl}$ denotes the sum $a_{i1}\, b_{1kl} + a_{i2}\, b_{2kl} + a_{i3}\, b_{3kl}$. There may be several such pairs of indices in a term; every pair denotes independent summation.

Tensors. For a pair of vectors $\boldsymbol{a}$, $\boldsymbol{b}$, their tensor product $\boldsymbol{ab}$ is introduced. Tensor products can be added and multiplied by a number. A tensor product is linear with respect to each of the cofactors:

$$(\alpha_1 \boldsymbol{a}_1 + \alpha_2 \boldsymbol{a}_2)(\beta_1 \boldsymbol{b}_1 + \beta_2 \boldsymbol{b}_2) = \alpha_1\beta_1 \boldsymbol{a}_1\boldsymbol{b}_1 + \alpha_1\beta_2 \boldsymbol{a}_1\boldsymbol{b}_2 + \alpha_2\beta_1 \boldsymbol{a}_2\boldsymbol{b}_1 + \alpha_2\beta_2 \boldsymbol{a}_2\boldsymbol{b}_2 \ .$$

All possible linear combinations of tensor products form a linear space, elements of which are called tensors of second rank. The set of the tensor products $\boldsymbol{e}_i\boldsymbol{e}_j$ can be used as a basis in this space ($\boldsymbol{e}_i$ is a basis of the original vector space). In particular, if $\boldsymbol{e}_i$ is an orthonormal basis of the Euclidian space, a tensor of second rank $\mathbf{t}$ can be written in the form $\mathbf{t} = t_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$. The numerical coefficients t_{ij} are called the components of the tensor $\mathbf{t}$ in this basis.

Let $\boldsymbol{e}_i$ and $\boldsymbol{e}'_j$ be two orthonormal bases, such that $\boldsymbol{e}_i = A_{ij}\boldsymbol{e}'_j$. Then t_{ij} and t'_{kl} are the components of a tensor of second rank $\mathbf{t}$ in these bases if and only if

$$t_{ij} = A_{ik}A_{jl}t'_{kl} \ .$$

This formula is referred to as the tensor transformation rule (in orthonormal bases). Only orthonormal bases will be used further in this section.

Tensors of third, fourth, etc., ranks and their components are introduced with the help of the tensor products $\boldsymbol{abc}$, $\boldsymbol{abcd}$, etc. (similar to tensors of second rank). The tensor products $\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k$, $\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l$, etc., can be used as bases in the spaces of tensors of third, fourth, etc., ranks. A vector is a tensor of first rank. A number (independent of a chosen basis) is called a scalar and is a tensor of zeroth rank.

Multiplication by a number α is defined for any tensor $\mathbf{t}$, the components of the product $\alpha\mathbf{t}$ being $(\alpha\mathbf{t})_{ij...m} = \alpha t_{ij...m}$ where $t_{ij...m}$ are the components of the tensor $\mathbf{t}$. Addition of two tensors $\mathbf{a}$ and $\mathbf{b}$ of the same rank is defined, the components of the sum $\mathbf{a} + \mathbf{b}$ being the sum of the components $(\mathbf{a} + \mathbf{b})_{ij...m} = a_{ij...m} + b_{ij...m}$. Besides of these operations, multiplication of any two tensors is determined. For example, the tensor product **AB** of the tensors $\mathbf{A} = A_{ij}\mathbf{e}_i\mathbf{e}_j$ and $\mathbf{B} = B_{klm}\mathbf{e}_k\mathbf{e}_l\mathbf{e}_m$ is the tensor of rank 5

$$\mathbf{AB} = A_{ij}B_{klm}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l\mathbf{e}_m .$$

For any tensor of rank not less than 2, the *contraction* with respect to a chosen pair of indices is defined, the result also being a tensor. Components of the contracted tensor are obtained from summation of the components of the original tensor with equal values of indices in the chosen pair; the summation is performed for every specified set of values of the rest indices. For example, the contraction of the tensor $\mathbf{Q} = Q_{ijkl}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l$ with respect to the first and the third indices is the tensor

$$\mathbf{q} = q_{jl}\mathbf{e}_j\mathbf{e}_l, \quad q_{jl} = Q_{ijil} .$$

A contraction is often performed in a tensor product, e.g., the tensor with the components $C_{il} = A_{ij}B_{jl}$ is the result of the contraction in the tensor product with the components $A_{ij}B_{kl}$. Contractions can be performed simultaneously with respect to every of chosen pairs of indices, e.g., the tensor with the components $p_{ij} = A_{ijkl}\varepsilon_{kl}$ is the result of the two contractions in the tensor product with the components $A_{ijkl}\varepsilon_{mn}$.

For any second rank symmetric tensor $\mathbf{t}$ ($t_{ij} = t_{ji}$), there exists an orthonormal basis $\mathbf{e}_i^*$ in which the components $t_{12}^* = t_{21}^*$, $t_{13}^* = t_{31}^*$, $t_{23}^* = t_{32}^*$ vanish. The straight lines, along which the vectors of this basis are directed, are called the *principal axes* of the tensor $\mathbf{t}$. The components t_{11}^*, t_{22}^*, t_{33}^* in this basis are called the *principal components* or *eigenvalues* of the tensor $\mathbf{t}$.

The eigenvalues and eigenvectors of a tensor $\mathbf{t}$ are defined as the numbers λ and vectors with components v_i satisfying the following system of equations

$$t_{ij}v_j = \lambda v_i .$$

Therefore the eigenvalues can also be obtained from the criterion for existence of a nonzero solution v_i of this system $\det \|t_{ij} - \lambda\delta_{ij}\| = 0$, i.e.,

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

where

$$I_1 = t_{ii} , \quad I_2 = \frac{1}{2}(t_{ii}t_{jj} - t_{ij}t_{ij}) , \quad I_3 = \det \|t_{ij}\| .$$

The numbers I_1, I_2, I_3 are independent of the orthonormal basis in which the components t_{ij} are considered in these formulae (see Problem 2.15). They are called the *invariants* of the tensor $\mathbf{t}$. Any number-valued function of t_{ij} possessing the same property is also called an invariant of the tensor $\mathbf{t}$. Any invariant of the tensor $\mathbf{t}$ can be expressed in terms of the invariants I_1, I_2, I_3.

— PROBLEMS —

Expressions with indices

⊙ **2.1** Write out in detail the following expressions using only numerical values of the indices (not their letter notations)

a) t_{ii};

b) $p_{ij}u_j$, u_jp_{ij}, $p_{ij}u_i$, u_ip_{ij};

c) $q_{ij}a_ib_j$, $q_{ij}b_ja_i$, $b_jq_{ij}a_i$, $a_iq_{ij}b_j$, $a_ib_jq_{ij}$, $b_ja_iq_{ij}$, $q_{ij}a_jb_i$;

d) $a_{ij}b_{ij}$, $a_{ji}b_{ji}$, $a_{ij}b_{ji}$, $b_{ij}a_{ji}$.

Indicate equal expressions.

⊙ **2.2** Compute

a) sums δ_{ii}, $\delta_{ij}\delta_{ji}$, $\delta_{ij}\delta_{jk}\delta_{ki}$;

b) the same sums if all the indices run the values $1, 2, \ldots, n$.

⊙ **2.3** Write the formula for computation of the substantive derivative in Eulerian description dA/dt in the short form using the summation convention.

Tensors

⊙ **2.4** Let t_{ij} be the components of a tensor in an orthonormal basis $\boldsymbol{e}_i$.

a) Show that the numbers $\tau_{ij} = t_{ji}$ (for example, $\tau_{12} = t_{21}$) are the components of a certain tensor.

b) If u_i and v_j are the components of vectors, is $\tau_{ij}u_iv_j = t_{ij}u_iv_j$? Is $t_{ij}u_iu_j = \tau_{ij}u_iu_j$?

⊙ **2.5** a) In some orthonormal basis, the components of a tensor of second rank satisfy the relationship $t_{ij} = t_{ji}$. Show that a similar relationship is valid for its components in any orthonormal basis. In this case, the tensor is referred to as *symmetric*.

b) Components of a tensor of second rank in some orthonormal basis satisfy the relationship $t_{ij} = -t_{ji}$. Show that a similar relationship is valid for its components in any orthonormal basis. In this case, the tensor is referred to as *antisymmetric*.

⊙ **2.6** Consider the sums $a_{ijk} + b_{ijk}$ of the components of tensors **a** and **b** in every orthonormal basis and show that they are the components of a tensor.

⊙ **2.7** Consider the products $B_{ijkl}\varepsilon_{mn}$ of the components of tensors **B** and $\boldsymbol{\varepsilon}$ build in every orthonormal basis and show that they are the components of a tensor. Show that their sums $B_{ijkl}\varepsilon_{kl}$ are the components of a tensor as well.

⊙ **2.8** Let s_{ij} and a_{kl} be the components of symmetric and antisymmetric tensors. Show that $s_{ij}a_{ij} = 0$ (the "complete contraction" of a symmetric and an antisymmetric tensors equals zero).

⊙ **2.9** Show that any tensor of second rank can be represented as the sum of a symmetric tensor and an antisymmetric one. Is this representation unique?

⊖ **2.10** Show that for a symmetric tensor **s** the contraction $s_{ij}u_iv_j$ can be expressed in terms of contractions of the form $s_{ij}w_iw_j$ (u_i, v_i, w_i are the components of vectors). In other words, the values of a symmetric bilinear form can be expressed in terms of the values of the corresponding quadratic form.

⊙ **2.11** Show that if a tensor of second rank **t** satisfies the relation $t_{ij}v_iv_j = 0$ for any vector $\boldsymbol{v}$, it is antisymmetric.

⊙ **2.12** The tensors $\mathbf{t}^{(s)}$ and $\mathbf{t}^{(d)}$ with the components

$$t_{ij}^{(s)} = \frac{1}{3}t_{kk}\delta_{ij}\ , \quad t_{ij}^{(d)} = t_{ij} - \frac{1}{3}t_{kk}\delta_{ij}$$

are called respectively the *spherical component* and the *deviator* of a symmetric tensor **t**.

Find

a) the deviator of a spherical component $(\mathbf{t}^{(s)})^{(d)}$;

b) the spherical component of a deviator $(\mathbf{t}^{(d)})^{(s)}$.

⊖ **2.13** Find the general form of a tensor of second rank **t** if its component t_{12} is known to equal zero in any orthonormal basis.

⊙ **2.14** Find the principal components and principal axes of the tensor with the following matrix of components in an orthogonal basis $\boldsymbol{e}_i$

$$\text{a)} \begin{pmatrix} 1 & -\sqrt{3} & 0 \\ -\sqrt{3} & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}; \quad \text{b)} \begin{pmatrix} 1 & -\sqrt{3} & 0 \\ -\sqrt{3} & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

⊙ **2.15** Show that the following functions of the components t_{ij} of a second rank symmetric tensor **t** are its invariants

a) $J_1 = t_{ii}$, $J_2 = t_{ij}t_{ij}$, $J_3 = t_{ij}t_{jk}t_{ki}$;

b) $I_1 = t_{ii}$, $I_2 = \frac{1}{2}(t_{ii}t_{jj} - t_{ij}t_{ij})$, $I_3 = \det \|t_{ij}\|$.

⊙ **2.16** Are the principal components of a second rank symmetric tensor its invariants?

⊙ **2.17** Express the invariants I_1, I_2, I_3, J_1, J_2, J_3 (see Problem 2.15) in terms of the principal components of the tensor $\mathbf{t}$.

3 Curvilinear Coordinate Systems

Solving some problems calls for use of curvilinear coordinate systems although there exists a Cartesian coordinate system in a Euclidean space. Normally, this depends upon a symmetry of the problem. For example, to study an axially symmetric body subjected to an axially symmetric load is more convenient in the cylindrical coordinate system.

Coordinate systems, local bases. A coordinate system sets a correspondence $\boldsymbol{r} \leftrightarrow (x^1, x^2, x^3)$ between vectors and triples of numbers; such a triple (x^1, x^2, x^3) is also called a point with coordinates (x^1, x^2, x^3). Numbers x^1, x^2 and x^3 are the components of a vector $\boldsymbol{r}$ only in special rectilinear coordinate systems. Note that we use upper indices for coordinates in this section. If a coordinate system is given in the space, then lines along which two coordinates are constant (coordinate lines), can be drawn through each point. A coordinate system fixes a local basis of the Euclidean space $\boldsymbol{e}_i = \partial \boldsymbol{r}/\partial x^i$ corresponding to a point (x^1, x^2, x^3). In general, this basis is not orthonormal. The basis $\boldsymbol{e}^k$ satisfying the relationships $\boldsymbol{e}^k \cdot \boldsymbol{e}_i = \delta_i^k$ ($\boldsymbol{a} \cdot \boldsymbol{b}$ is the scalar product of vectors $\boldsymbol{a}$, $\boldsymbol{b}$) is referred to as dual to $\boldsymbol{e}^k$. The dual basis exists, is unique, and can be found by a standard procedure (Problem 3.1). If g_{ij} is a set of the scalar products $g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$, and g^{ij} is a set of the elements of the matrix $\|g^{ij}\|$ reciprocal to the matrix $\|g_{ij}\|$, the following relationships are valid

$$\boldsymbol{e}^i = g^{ik}\boldsymbol{e}_k, \qquad \boldsymbol{e}_j = g_{jk}\boldsymbol{e}^k, \qquad g^{ij} = \boldsymbol{e}^i \cdot \boldsymbol{e}^j \ .$$

Vector and tensor fields. A vector-valued function on a domain (surface, curve) is called a vector field on the domain (surface, curve) or, often, simply a vector. If a coordinate system x^i is chosen, a vector field is represented with use of local bases $\boldsymbol{e}_i$ or the dual bases $\boldsymbol{e}^j$ in the form

$$\boldsymbol{v} = v^i\boldsymbol{e}_i = v_j\boldsymbol{e}^j\,, \qquad \text{with } v^i = g^{ik}v_k, \quad v_j = g_{jk}v^k. \tag{3.1}$$

The quantities v^i and v_j are called respectively the *contravariant* and *covariant* components of the vector field $\boldsymbol{v}$ in the coordinate system x^i. Thus, we use upper indices for contravariant components and lower ones for covariant components. In the case of an orthogonal curvilinear coordinate system *physical* components are also often used (see Problems 3.17, 3.19). If, besides coordinate system x^i, another coordinate system x'^k is used, its basis, dual basis and the components of a vector field $\boldsymbol{v}$ in it (denoted with primes) are bound with the bases e_i, e^j and the components of the vector field v in the coordinate system x^i by the relationships

$$\begin{aligned} e'_i &= \frac{\partial x^k}{\partial x'^i} e_k, \quad v'_i = \frac{\partial x^k}{\partial x'^i} v_k \quad \text{- the covariant transformation rule,} \\ e'^i &= \frac{\partial x'^i}{\partial x^k} e^k, \quad v'^i = \frac{\partial x'^i}{\partial x^k} v^k \quad \text{- the contravariant transformation rule.} \end{aligned} \tag{3.2}$$

The following relationships are also valid

$$e_i = \frac{\partial x'^k}{\partial x^i} e'_k \,, \qquad v_i = \frac{\partial x'^k}{\partial x^i} v'_k \,,$$

$$e^i = \frac{\partial x^i}{\partial x'^k} e'^k \,, \qquad v^i = \frac{\partial x^i}{\partial x'^k} v'^k \,,$$

$$\frac{\partial x^i}{\partial x'^k} \frac{\partial x'^k}{\partial x^j} = \delta^i_j \,, \qquad \frac{\partial x'^i}{\partial x^k} \frac{\partial x^k}{\partial x'^j} = \delta^i_j \,.$$

A tensor-valued function on a domain (surface, curve) is called a tensor field on the domain (surface, curve) or, often, simply a tensor. If a coordinate system x^i is chosen, a tensor field, e.g. of second-rank, is represented in the following forms

$$\mathbf{t} = t^{ij}\, e_i e_j = t_{k\cdot}^{\cdot j}\, e^k e_j = t^{i}_{\cdot l}\, e_i e^l = t_{kl}\, e^k e^l \,.$$

The numerical coefficients t^{ij}, t_{kl} and $t_{k\cdot}^{\cdot j}$, $t^{i}_{\cdot l}$ are called respectively the contravariant, covariant and mixed components of the tensor in the coordinate system x^i. The dots in the notations $t_{k\cdot}^{\cdot j}$, $t^{i}_{\cdot l}$ are used to indicate the sequence of the indices. For example, in the expression $A^{i}_{\cdot k}$, the index i is the first, and the index k is the second. The components of a tensor in a coordinate system x'^k are bound with its components in a coordinate system x^i by the so-called *tensor transformation rule*: the covariant transformation rule is used for every lower index, and the contravariant one is used for every upper index, e.g.,

$$t'^{ij\cdot}_{\cdot\cdot k} = \frac{\partial x'^i}{\partial x^p} \frac{\partial x'^j}{\partial x^q} \frac{\partial x^r}{\partial x'^k}\, t^{pq\cdot}_{\cdot\cdot r} \,.$$

For any two coordinate systems x^i and x'^k, $t^{pq\cdot}_{\cdot\cdot r}$ and $t'^{ij\cdot}_{\cdot\cdot k}$ are the components of a tensor $\mathbf{t}$ in these coordinate systems if and only if they are bound by the tensor transformation rule. The tensor of third-rank is considered here as an example; the formulated statement is valid for a tensor of any rank.

Metric tensor. The quantities

$$g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j, \qquad g_{i\cdot}^{\cdot j} = g^{j}_{\cdot i} = \delta^j_i, \qquad g^{ij} = \boldsymbol{e}^i \cdot \boldsymbol{e}^j$$

are the components of some tensor of second-rank $\mathbf{g}$ referred to as *metric* (see Problem 3.4). The length squared of an arc element is determined by the relationship

$$ds^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = g_{ij}\, dx^i dx^k .$$

Contravariant, covariant and mixed components of a second-rank tensor are bound by the following relationships

$$t^{i}_{\cdot j} = g^{ik}\, t_{kj}, \qquad t_{i\cdot}^{\cdot j} = g_{ik}\, t^{kj}, \qquad t^{ij} = g^{ik} g^{jl}\, t_{kl} .$$

Similar formulae are valid also for a tensor of any rank, e.g.,

$$t^{ijq}_{\cdot\cdot\cdot l} = g^{qk}\, t^{ij\cdot\cdot}_{\cdot\cdot kl}$$

an index is raised or lowered as if it is the only one, cf. Formulae 3.1.

Tensor operations. The *product of a tensor* $\mathbf{t}$ *and a number* α is the tensor with components obtained by multiplying the components of a $\mathbf{t}$ by α, e.g.,

$$\alpha\mathbf{t} = \alpha(t^{ij}\boldsymbol{e}_i\boldsymbol{e}_j) = (\alpha t^{ij})\boldsymbol{e}_i\boldsymbol{e}_j = (\alpha t^{i}_{\cdot j})\boldsymbol{e}_i\boldsymbol{e}^j = (\alpha t_{i\cdot}^{\cdot j})\boldsymbol{e}^i\boldsymbol{e}_j = (\alpha t_{ij})\boldsymbol{e}^i\boldsymbol{e}^j$$

A *sum* is defined for any two tensors $\mathbf{a}$ and $\mathbf{b}$ of rank r, and its components are obtained by adding the components of $\mathbf{a}$ and $\mathbf{b}$ with the same location of indices. For example, the sum of $\mathbf{a} = a_{ij}\boldsymbol{e}^i\boldsymbol{e}^j$ and $\mathbf{b} = b_{kl}\boldsymbol{e}^k\boldsymbol{e}^l$ is

$$\mathbf{a} + \mathbf{b} = (a_{ij} + b_{ij})\boldsymbol{e}^i\boldsymbol{e}^j = (a_{i\cdot}^{\cdot j} + b_{i\cdot}^{\cdot j})\boldsymbol{e}^i\boldsymbol{e}_j = (a^{i}_{\cdot j} + b^{i}_{\cdot j})\boldsymbol{e}_i\boldsymbol{e}^j = (a^{ij} + b^{ij})\boldsymbol{e}_i\boldsymbol{e}_j .$$

A *tensor product* is defined for any two tensors. For example, the tensor product $\mathbf{AB}$ of $\mathbf{A} = A_{ij}\boldsymbol{e}^i\boldsymbol{e}^j$ and $\mathbf{B} = B_{klm}\boldsymbol{e}^k\boldsymbol{e}^l\boldsymbol{e}^m$ is a tensor of rank 5

$$\mathbf{AB} = A_{ij}B_{klm}\boldsymbol{e}^i\boldsymbol{e}^j\boldsymbol{e}^k\boldsymbol{e}^l\boldsymbol{e}^m .$$

It can also be expressed in terms of the components with another location of indices, e.g., $\mathbf{AB} = A^{i}_{\cdot j}B_{k\cdot m}^{\cdot l}\boldsymbol{e}_i\boldsymbol{e}^j\boldsymbol{e}^k\boldsymbol{e}_l\boldsymbol{e}^m$. For any tensor of rank not less than two, a *contraction* is defined with respect to a chosen pair of indices one of which must be upper, and the other, lower. For example, the contraction of the tensor $\mathbf{Q} = Q_{i\cdot\cdot l}^{\cdot jk\cdot}\boldsymbol{e}^i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}^l$ with respect to the first and the third indices is the tensor

$$\mathbf{q} = q^{j}_{\cdot l}\boldsymbol{e}_j\boldsymbol{e}^l , \quad q^{j}_{\cdot l} = Q_{i\cdot\cdot l}^{\cdot ji\cdot} .$$

It makes no difference which of the two chosen indices is upper and which is lower, since the equality $Q_{i\cdot\cdot l}^{\cdot ji\cdot} = Q^{ij\cdot\cdot}_{\cdot\cdot il}$ is valid (see Problem 3.25).

Covariant differentiation. The partial derivatives of a vector field $\boldsymbol{v}$ with respect to coordinates x^k can be expressed in terms of the components of a second rank tensor $\nabla \boldsymbol{v}$ according to the formulae

$$\frac{\partial \boldsymbol{v}}{\partial x^k} = (\nabla_k v^i)\boldsymbol{e}_i = (\nabla_k v_j)\boldsymbol{e}^j .$$

Here we use special notation for the components of the tensor $\nabla \boldsymbol{v}$: $(\nabla \boldsymbol{v})_{k\cdot}^{\cdot i} = \nabla_k v^i$, $(\nabla \boldsymbol{v})_{kj} = \nabla_k v_j$. The quantities $\nabla_k v^i$, $\nabla_k v_j$ are called *covariant derivatives* of, respectively, contravariant and covariant components of a vector field $\boldsymbol{v}$. They are found with the use of the formulae

$$\nabla_k v^i = \frac{\partial v^i}{\partial x^k} + \Gamma^i_{sk} v^s, \qquad \nabla_k v_j = \frac{\partial v_j}{\partial x^k} - \Gamma^s_{jk} v_s, \tag{3.3}$$

where the quantities Γ^i_{jk} are the coefficients in the decomposition of the derivatives of the basis vectors into components in this basis

$$\frac{\partial \boldsymbol{e}_i}{\partial x^k} = \Gamma^l_{ik} \boldsymbol{e}_l$$

(leading to one more formula $\partial \boldsymbol{e}^j / \partial x^k = -\Gamma^j_{lk} \boldsymbol{e}^l$). The quantities Γ^i_{jk} are called the *Christoffel symbols.* They can be expressed in terms of derivatives of the metric tensor components

$$\Gamma^j_{ik} = \frac{1}{2} g^{jl} \left(\frac{\partial g_{kl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^l} \right) .$$

The partial derivatives of any tensor field $\mathbf{t}$ with respect to coordinates x^k are expressed in terms of the components of some tensor with rank one higher than $\mathbf{t}$, namely $\nabla \mathbf{t}$. The components of $\nabla \mathbf{t}$ are called covariant derivatives of the corresponding components of the tensor $\mathbf{t}$ and found with the use of formulae similar to (3.3), e.g., if $\mathbf{t} = t^{l\cdot n}_{\cdot m \cdot} \boldsymbol{e}_l \boldsymbol{e}^m \boldsymbol{e}_n$,

$$\frac{\partial \mathbf{t}}{\partial x^k} = (\nabla_k t^{l\cdot n}_{\cdot m \cdot}) \boldsymbol{e}_l \boldsymbol{e}^m \boldsymbol{e}_n,$$

$$\nabla_k t^{l\cdot n}_{\cdot m \cdot} = \frac{\partial t^{l\cdot n}_{\cdot m \cdot}}{\partial x^k} + \Gamma^l_{sk} t^{s\cdot n}_{\cdot m \cdot} - \Gamma^s_{mk} t^{l\cdot n}_{\cdot s \cdot} + \Gamma^n_{sk} t^{l\cdot s}_{\cdot m \cdot} .$$

In particular, if T is a scalar, $\nabla_k T = \partial T / \partial x^k$ are the components of the vector grad T:

$$\text{grad } T = \nabla_k T \boldsymbol{e}^k = \frac{\partial T}{\partial x^k} \boldsymbol{e}^k .$$

— PROBLEMS —

Coordinate system

⊖ **3.1** a) Show that the dual basis e^k with respect to the given basis e_i exists and is unique, and that the following relationships are valid

$$e^i = g^{ik} e_k, \qquad e_j = g_{jk} e^k$$

where g^{ik} are the components of the matrix $\|g^{ij}\|$ reciprocal to the matrix $\|g_{ij}\|$, $g_{ij} = e_i \cdot e_j$. b) Verify the relationship $e^i \cdot e^j = g^{ik}$. c) Verify the relationships

$$u \cdot v = u^i v_i = u_i v^i, \qquad v^i = e^i \cdot v, \qquad v_i = e_i \cdot v \,.$$

d) Verify that a basis and its dual basis are expressed in terms of each other according to the formulae

$$e^1 = \frac{e_2 \times e_3}{V_*}, \quad e^2 = \frac{e_3 \times e_1}{V_*}, \quad e^3 = \frac{e_1 \times e_2}{V_*}, \quad V_* = e_1 \cdot (e_2 \times e_3)$$

$$e_1 = \frac{e^2 \times e^3}{V^*}, \quad e_2 = \frac{e^3 \times e^1}{V^*}, \quad e_3 = \frac{e^1 \times e^2}{V^*}, \quad V^* = e^1 \cdot (e^2 \times e^3) \,.$$

⊙ **3.2** a) Find the basis dual to an orthonormal one. b) Is the basis dual to an orthogonal basis of non-unit vectors orthogonal? c) A basis e_i is formed by unit vectors every two of which make the angle $\pi/3$. Find the basis dual to it. Find the contravariant components of the vector $v = ae^1 + be^2 + ce^3$.

⊙ **3.3** Prove the statement formulated in Problem 2.5 in the case when the basis e_i is arbitrary.

⊖ **3.4** For every coordinate system, the sets of numbers are considered: 1) $g_{ij} = e_i \cdot e_j$, 2) g^{ij} – the components of the matrix $\|g^{ij}\|$ reciprocal to the matrix $\|g_{ij}\|$. Show that they are respectively covariant and contravariant components of one and the same tensor (*metric tensor*). Find mixed components of this tensor.

⊙ **3.5** Show that the following sets of relationships for components of a tensor are equivalent a) $s_{ij} = s_{ji} \iff s^{kl} = s^{lk} \iff s^{m}_{\cdot n} = s_{n}^{\cdot m}$ b) $a_{ij} = -a_{ji} \iff a^{kl} = -a^{lk} \iff a^{m}_{\cdot n} = -a_{n}^{\cdot m}$ It means that any of the sets of relationships (a) or (b) can be accepted as a definition of, respectively, a symmetric or antisymmetric second rank tensor (see Problems 2.5 and 3.3).

⊙ **3.6** Are the following equalities valid for a second rank tensor $\mathbf{t}$? a) $t^{i}_{\cdot j} = t_{j}^{\cdot i}$ b) $t^{i}_{\cdot i} = t_{k}^{\cdot k}$

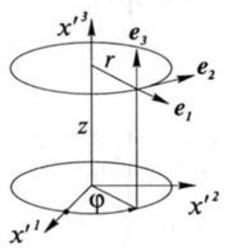

Figure 3.1

⊖ **3.7** Cylindrical coordinates $x^1 = r$, $x^2 = \varphi$, $x^3 = z$ (see Figure 3.1) are bound with Cartesian ones x'^1, x'^2, x'^3 by the relationships

$$x'^1 = r\cos\varphi, \qquad x'^2 = r\sin\varphi, \qquad x'^3 = z\ .$$

a) Find the basis of a cylindrical coordinate system at the points M_1 $(r = 5\,, \varphi = 0\,, z = 0)$ and M_2 $(r = 10\,, \varphi = \pi/6\,, z = 1)$ (express in terms of the basis $\mathbf{e'}_i$ of a Cartesian coordinate system x'^i). b) Find covariant, contravariant and mixed components of the metric tensor in a cylindrical coordinate system. c) Find the dual basis at M_1 and M_2.

⊙ **3.8** Decompose the basis vectors $\mathbf{e}'^1$, $\mathbf{e}'^2$ of plane Cartesian coordinates x'^1, x'^2 over the basis of the polar coordinate system $x^1 = r$, $x^2 = \varphi$ bound with the Cartesian coordinates by the relationships

$$x'^1 = r\cos\varphi, \qquad x'^2 = r\sin\varphi\ .$$

⊙ **3.9** A tensor of second rank $\mathbf{p}$ has the following components in a cylindrical coordinate system: $p^{11} = a$, $p^{22} = b/r^2$, the other components equal zero. Find its components in the Cartesian coordinate system (see Problem 3.7).

⊖ **3.10** Spherical coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \lambda$ (see Figure 3.2) are bound with Cartesian ones x'^1, x'^2, x'^3 by the relationships

$$x'^1 = r\sin\theta\cos\lambda, \qquad x'^2 = r\sin\theta\sin\lambda, \qquad x'^3 = r\cos\theta\ .$$

Find the basis of a spherical coordinate system (express in terms of the basis $\mathbf{e'}_i$ of a Cartesian system). Find covariant, contravariant and mixed components of the metric tensor in a spherical coordinate system.

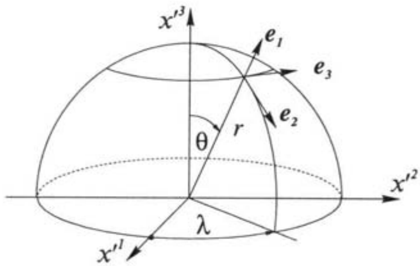

Figure 3.2

⊙ **3.11** A vector field is given in a Cartesian coordinate system x'^i:

$$\boldsymbol{v} = \frac{x'^1\boldsymbol{e}'_1 + x'^2\boldsymbol{e}'_2 + x'^3\boldsymbol{e}'_3}{\sqrt{(x'^1)^2 + (x'^2)^2 + (x'^3)^2}}.$$

Find its components in the spherical coordinate system (see Problem 3.10).

⊙ **3.12** Find the length squared of the differential element $dx^1\boldsymbol{e}_1 + dx^2\boldsymbol{e}_2 + dx^3\boldsymbol{e}_3$ in elliptic coordinates $x^1 = r$, $x^2 = \varphi$, $x^3 = z$ bound with Cartesian coordinates x, y, z by the relationships

$$x = \sqrt{r^2 + a^2}\cos\varphi, \qquad y = r\sin\varphi \qquad (r \geq 0).$$

⊙ **3.13** Find the length squared of the element $dx^1\boldsymbol{e}_1 + dx^2\boldsymbol{e}_2 + dx^3\boldsymbol{e}_3$ in ellipsoidal coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$ (with the coordinate surfaces $r = \text{const}$ in the shape of oblate ellipsoids of revolution) bound with Cartesian coordinates x, y, z by the relationships

$$x = \sqrt{r^2 + a^2}\sin\theta\cos\varphi, \quad y = \sqrt{r^2 + a^2}\sin\theta\sin\varphi, \quad z = r\cos\theta.$$

⊙ **3.14** Express in terms of the basis vectors $\boldsymbol{e}_i$ or $\boldsymbol{e}^i$ of a curvilinear coordinate system x^i a) a unit tangent vector to a coordinate line x^1 (i.e., to a curve $x^2 = \text{const}$, $x^3 = \text{const}$); b) an angle crossed at by coordinate lines x^2 and x^3 in a given point; c) a unit normal to a coordinate surface (x^2x^3) (i.e., to a surface $x^1 = \text{const}$).

⊖ **3.15** Sometimes, e.g., when studying a stream that flows about a body, it is convenient to use a special curvilinear coordinate system associated with the surface of the body. For a planar stream considered in its plane, such a coordinate system is introduced as follows. Let, in the plane of the stream, the boundary of the body be

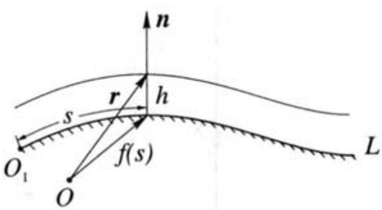

Figure 3.3

a smooth curve L given in the parametric representation $\boldsymbol{r} = \boldsymbol{f}(s) = a(s)\boldsymbol{e}_x + b(s)\boldsymbol{e}_y$ where s is the length of the arc of L, $\boldsymbol{e}_x, \boldsymbol{e}_y$ is the basis of a Cartesian coordinate system x, y on the plane. Then, in the vicinity of the curve, this coordinate system associates the point $\boldsymbol{r} = \boldsymbol{f}(s) + \boldsymbol{n}(s)h$ with the pair of numbers (s, h), where $\boldsymbol{n}(s)$ is the unit normal to L, and h is the distance from L (see Figure 3.3). For the coordinate system $x^1 = s$, $x^2 = h$, find the basis and the covariant, contravariant, and mixed components of the metric tensor.

Physical components of vectors and tensors.

⊖ **3.16** In mechanics, the length dimension is often assigned to Cartesian coordinates, and their basis vectors are regarded as dimensionless. In this case, for a spherical coordinate system (see Problem 3.10), find the dimensions of a) the coordinates, b) the vectors of the basis and dual basis, c) the covariant and contravariant components of the metric tensor, d) the covariant and contravariant components of a velocity vector.

⊙ **3.17** Components of a vector (having identical dimensions in a Cartesian coordinate system) may have different dimensions in a curvilinear coordinate system, e.g., see the preceding problem. To avoid this, so-called *physical components* v_{p_i} of a vector $\boldsymbol{v}$

$$v_{\mathrm{p}_1} = v^1|\boldsymbol{e}_1|, \qquad v_{\mathrm{p}_2} = v^2|\boldsymbol{e}_2|, \qquad v_{\mathrm{p}_3} = v^3|\boldsymbol{e}_3|$$

are introduced in orthogonal curvilinear coordinate systems ($\boldsymbol{e}_i \perp \boldsymbol{e}_j$ at $i \neq j$), and the vector is expressed in the form

$$\boldsymbol{v} = v_{\mathrm{p}1}\frac{\boldsymbol{e}_1}{|\boldsymbol{e}_1|} + v_{\mathrm{p}2}\frac{\boldsymbol{e}_2}{|\boldsymbol{e}_2|} + v_{\mathrm{p}3}\frac{\boldsymbol{e}_3}{|\boldsymbol{e}_3|}.$$

a) Show that

$$v_{\mathrm{p}_1} = v_1|\boldsymbol{e}^1|, \qquad v_{\mathrm{p}2} = v_2|\boldsymbol{e}^2|, \qquad v_{\mathrm{p}3} = v_3|\boldsymbol{e}^3|$$

and

$$\frac{\boldsymbol{e}_1}{|\boldsymbol{e}_1|} = \frac{\boldsymbol{e}^1}{|\boldsymbol{e}^1|}, \qquad \frac{\boldsymbol{e}_2}{|\boldsymbol{e}_2|} = \frac{\boldsymbol{e}^2}{|\boldsymbol{e}^2|}, \qquad \frac{\boldsymbol{e}_3}{|\boldsymbol{e}_3|} = \frac{\boldsymbol{e}^3}{|\boldsymbol{e}^3|} .$$

(This triple of vectors is called the physical basis associated with the considered orthogonal coordinate system). b) Express a scalar product $\boldsymbol{u} \cdot \boldsymbol{v}$ in terms of physical components of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. c) Similarly to physical components of a vector, introduce physical components of a second rank tensor.

⊙ **3.18** a) Express the physical basis $\boldsymbol{e}_r, \boldsymbol{e}_\varphi, \boldsymbol{e}_z$ associated with a cylindrical coordinate system (see Problems 3.7, 3.17) in terms of bases $\boldsymbol{e}_i$ of the cylindrical and $\boldsymbol{e}'_i$ of the Cartesian coordinate systems. b) A body rotates around an axis with the angular speed $\omega(t)$. Regarding the rotation axis as the coordinate line x^3 of a cylindrical coordinate system, find the physical components of the velocity and acceleration vectors of the body points in this system.

⊖ **3.19** Show that the physical basis associated with a cylindrical coordinate system x^i (see Problem 3.18a) is not the local basis of any coordinate system $y^k(x^1, x^2, x^3)$.

Tensor transformation rule.

⊙ **3.20** a) Show that the elements of the matrix $\|b_{ij}\|$ of a bilinear form are the covariant components of a tensor of second rank. (A bilinear form **b** is a scalar-valued function of two vector arguments linear with respect to each of them. For every coordinate system, a matrix $\|b_{ij}\|$ exists such that $\mathbf{b}(\boldsymbol{u}, \boldsymbol{v}) = b_{ij}u^i v^i$ for any two vectors $\boldsymbol{u}$, $\boldsymbol{v}$; this matrix is called the matrix of the bilinear form **b**.) b) A bilinear form sets into correspondence to a pair of vectors their scalar product. Find the matrix of this bilinear form.

⊖ **3.21** a) Show that the elements of the matrix $\|a^i_{\cdot j}\|$ of a linear operator are the mixed components of a tensor of second rank. (A linear operator **a** is a linear vector-valued function of a vector argument. For every coordinate system a matrix $\|a^i_{\cdot j}\|$ exists such that $\mathbf{a}\boldsymbol{v} = a^i_{\cdot j} v^j \boldsymbol{e}_i$ for any vector $\boldsymbol{v}$; the matrix $\|a^i_{\cdot j}\|$ is called the matrix of the linear operator **a**). b) A linear operator projects a vector on the plane orthogonal to the given unit vector $\boldsymbol{n} = n^i \boldsymbol{e}_i$. Find the matrix of this operator.

⊖ **3.22** Find the general form of a second-rank tensor **t** if its component t_{12} is known to equal zero in any coordinate system.

Tensor operations

⊙ **3.23** For two second rank tensor **a** and **b**, consider the sums $a^{ij} + b_{ij}$ of their components in every coordinate system. Show that they are not components of tensor unless $\mathbf{a} = 0$ or $\mathbf{b} = 0$.

⊙ **3.24** Find the contravariant components of the sum of the tensors $\mathbf{a} = \mathbf{e}_1\mathbf{e}_1$ and $\mathbf{b} = \mathbf{e}^2\mathbf{e}^2$ where $\mathbf{e}_i$ is the basis of the coordinate system x^i, $x^1 = x'_1 + x'_2$, $x^2 = x'_2$, $x^3 = x'_3$ and x'_i are Cartesian coordinates.

⊙ **3.25** Prove the following equality for the components of a tensor $\mathbf{Q} = Q_{ijkl}\mathbf{e}^i\mathbf{e}^j\mathbf{e}^k\mathbf{e}^l$

$$Q_{ij\cdot l}^{\cdot\cdot i\cdot} = Q_{\cdot jil}^{i\cdot\cdot\cdot} .$$

⊙ **3.26** Consider the components of tensors $\mathbf{B} = B_{ijkl}\mathbf{e}^i\mathbf{e}^j\mathbf{e}^k\mathbf{e}^l$ and $\boldsymbol{\varepsilon} = \varepsilon_{ij}\mathbf{e}^i\mathbf{e}^j$ in every coordinate system and prove that a) sums $B_{ijkl}\varepsilon^{kl}$ are the components of a tensor, b) the following equalities are valid

$$B^{ijkl}\varepsilon_{kl} = B^{ij\cdot l}_{\cdot\cdot k\cdot}\varepsilon^{k\cdot}_{\cdot l} = B^{ijk\cdot}_{\cdot\cdot\cdot l}\varepsilon^{\cdot l}_{k\cdot} = B^{ij\cdot\cdot}_{\cdot\cdot kl}\varepsilon^{kl}_{\cdot\cdot} .$$

⊖ **3.27** Consider the numbers t_{ii}, $t_{ij}t_{ij}$, $t_{ij}t_{jk}t_{ki}$ determined by the components of a second-rank tensor **t** in an arbitrary coordinate system. Do they depend upon this system?

⊖ **3.28** Consider the numbers $J_1 = t_{ii}$, $J_2 = t_{ij}t_{ij}$, $J_3 = t_{ij}t_{jk}t_{ki}$ determined by the components of a second-rank tensor **t** in a Cartesian coordinate system (they do not depend upon this system, see Problems 2.15, 3.27). Find the formulae expressing these numbers in terms of the mixed components of the tensor **t** in an arbitrary coordinate system.

⊖ **3.29** For a second-rank symmetric tensor **t**, the eigenvalues and eigenvectors are defined as the numbers λ and vectors with components v'_i satisfying the system of equations

$$t'_{ij}v'_j = \lambda v'_i$$

in a Cartesian coordinate system. a) Derive the equations determining the numbers λ and components v'_i in an arbitrary coordinate system in terms of (i) covariant, (ii) contravariant and (iii) mixed components of the tensor **t**. b) Express the coefficients of equation determining eigenvalues λ

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

in terms of the mixed components of tensor **t** in an arbitrary coordinate system.

Levi-Civita tensor. Calculations in curvilinear coordinate systems

⊙ **3.30** Show that the numbers $\epsilon_{ijk} = \boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k)$ are the components of a third-rank tensor (called Levi-Civita tensor).

⊖ **3.31** Verify that Levi-Civita tensor has the following properties: a) it is antisymmetric with respect to every pair of indices, i.e., the relationships

$$\epsilon_{ijk} = -\epsilon_{jik}, \qquad \epsilon_{ijk} = -\epsilon_{ikj}, \qquad \epsilon_{ijk} = -\epsilon_{kji}$$

are valid; b) only those its components do not equal zero whose values of the indices are obtained by permutation of the values $1, 2, 3$; the corresponding components equal ϵ_{123} or $-\epsilon_{123}$ if the permutation is, respectively, even or odd, i.e., the relationships

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312}, \qquad \epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -\epsilon_{123}$$

are valid; c) the component ϵ_{123} is expressed in terms of $g = \det \|g_{ij}\|$ according to the formula

$$\epsilon_{123} = \begin{cases} \sqrt{g}, & \text{if } \boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3 \text{ - is a right-handed basis,} \\ -\sqrt{g}, & \text{if } \boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3 \text{ - is a left-handed basis.} \end{cases}$$

⊖ **3.32** a) Formulate the properties of contravariant components of Levi-Civita tensor similar to those of covariant components from Problem 3.31a,b. Prove that they are valid. b) Show that the component ϵ^{123} is expressed in terms of $g = \det \|g_{ij}\|$ according to the formula

$$\epsilon^{123} = \begin{cases} 1/\sqrt{g}, & \text{if } \boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3 \text{ - is a right-handed basis,} \\ -1/\sqrt{g}, & \text{if } \boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3 \text{ - is a left-handed basis.} \end{cases}$$

⊖ **3.33** Verify the validity of the following relationships for Levi-Civita tensor

$$\begin{aligned} \epsilon^{ijk}\epsilon_{pqr} &= \delta^i_p\delta^j_q\delta^k_r + \delta^i_q\delta^j_r\delta^k_p + \delta^i_r\delta^j_p\delta^k_q - \\ &\quad -\delta^i_q\delta^j_p\delta^k_r - \delta^i_p\delta^j_r\delta^k_q - \delta^i_r\delta^j_q\delta^k_p \\ \epsilon^{ijk}\epsilon_{pqk} &= \delta^i_p\delta^j_q - \delta^i_q\delta^j_p \\ \epsilon^{ikl}\epsilon_{jkl} &= 2\delta^i_j \\ \epsilon^{ijk}\epsilon_{ijk} &= 6 \end{aligned}$$

⊖ **3.34** Show that a vector product $\boldsymbol{a} \times \boldsymbol{b}$ may be represented with the use of Levi-Civita tensor in the form

$$\boldsymbol{a} \times \boldsymbol{b} = \epsilon^{ijk} a_i b_j \boldsymbol{e}_k = \epsilon_{pqr} a^p b^q \boldsymbol{e}^r$$

in any coordinate system. (This is one of the reasons for which Levi-Civita tensor is convenient in calculations.)

⊖ **3.35** a) Show that the mixed product of vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ may be represented with the use of Levi-Civita tensor in the form

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \epsilon_{ijk}\, a^i b^j c^k.$$

in any coordinate system. b) Express components of the double vector product $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ of vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in terms of the components of these vectors.

⊙ **3.36** Show that the volume of the parallelepiped constructed on vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ equals $|\epsilon_{ijk} a^i b^j c^k|$.

⊙ **3.37** Consider a set of numbers $\tilde{\epsilon}_{ijk}$ defined for any right-handed basis: $\tilde{\epsilon}_{123} = \tilde{\epsilon}_{231} = \tilde{\epsilon}_{312} = 1$, $\tilde{\epsilon}_{213} = \tilde{\epsilon}_{132} = \tilde{\epsilon}_{321} = -1$, the other $\tilde{\epsilon}_{ijk} = 0$. They are called Levi-Civita symbols and often used in calculations (the notation $\tilde{\epsilon}^{ijk}$ is used for them as well. a) Does a tensor exist, the components of which are $\tilde{\epsilon}^{ijk}$ in every basis? b) Express the determinant of a given matrix and the components of the matrix reciprocal to it in terms of the components of this matrix using Levi-Civita symbols.

⊖ **3.38** a) Derive the formula for the differential of the determinant of a matrix

$$d(\det \|a^{i}_{\cdot j}\|) = \det \|a^{i}_{\cdot j}\| b^{k}_{\cdot l} da^{l}_{\cdot k}$$

where $\|b^{k}_{\cdot l}\|$ is the matrix reciprocal to the matrix $\|a^{i}_{\cdot j}\|$. b) Derive the formula for the differential of the determinant $g = \det \|g_{ij}\|$

$$dg = g\, g^{ij} dg_{ij}\ .$$

⊖ **3.39** a) Derive the formula for the differentials of the elements of the matrix $\|b^{i}_{\cdot j}\|$ reciprocal to a matrix $\|a^{i}_{\cdot j}\|$

$$db^{k}_{\cdot l} = -b^{k}_{\cdot m} b^{n}_{\cdot l} da^{m}_{\cdot n}\ .$$

b) Derive the formula for the contravariant components of the metric tensor

$$dg^{ij} = -g^{ik} g^{jl} dg_{kl}\ .$$

Covariant differentiation.

⊙ **3.40** Do the derivatives $\partial v^i/\partial x^j$, where v^i are the contravariant components of a vector, represent the components of a tensor?

⊙ **3.41** Are Christoffel symbols Γ^i_{jk} the components of a tensor?

⊙ **3.42** Is the following statement valid: if the component v^1 of a vector field vanishes in some coordinate system, then $\nabla_k v^1 = 0$?

⊙ **3.43** Find the covariant derivatives of components a) of the metric tensor, b) of Levi-Civita tensor.

⊖ **3.44** Find the Christoffel symbols a) for a cylindrical coordinate system (see Problem 3.7), b) for a polar coordinate system on a plane (see Problem 3.8).

⊖ **3.45** Find the Christoffel symbols for a spherical coordinate system (see Problem 3.10).

⊖ **3.46** Find the Christoffel symbols for the curvilinear coordinate system described in Problem 3.15.

⊖ **3.47** Derive the formula $\Gamma^j_{ji} = \partial \ln \sqrt{g}/\partial x^i$ where $g = \det \|g_{ij}\|$.

⊖ **3.48** The scalar field div $\boldsymbol{v}$ calculated by the formula

$$\text{div } \boldsymbol{v} = \frac{\partial v'^i}{\partial x'^i}$$

in a Cartesian coordinate system x'^i is called the *divergence* of the vector field $\boldsymbol{v}$. Show that the following formulae are valid in any coordinate system x^i: a) *div* $\boldsymbol{v} = \nabla_i v^i$, b) $\nabla_i v^i = \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g}\, v^i)}{\partial x^i}$.

⊙ **3.49** The vector field curl $\boldsymbol{v}$ calculated by the formula

$$\text{curl } \boldsymbol{v} = \begin{vmatrix} \boldsymbol{e}'_1 & \boldsymbol{e}'_2 & \boldsymbol{e}'_3 \\ \frac{\partial}{\partial x'^1} & \frac{\partial}{\partial x'^2} & \frac{\partial}{\partial x'^3} \\ v'^1 & v'^2 & v'^3 \end{vmatrix}$$

in a right Cartesian coordinate system x'^i with the basis $\boldsymbol{e}'_i$ is called the *curl* (*rotor*) of the vector field $\boldsymbol{v}$. Show that the formula

$$\text{curl } \boldsymbol{v} = \epsilon^{ijk} \nabla_i v_j \boldsymbol{e}_k$$

is valid for all coordinate system.

⊖ **3.50** Show that, for a symmetric tensor of second rank **b**, the formula

$$\nabla_i b^i_j = \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g}\, b^i_j)}{\partial x^i} - \frac{1}{2} b^{kl} \frac{\partial g_{kl}}{\partial x^j}$$

is valid.

⊙ **3.51** Show that the expression of $\nabla_i v_j - \nabla_j v_i$ (v_i are the components of a vector) in terms of the partial derivatives and Christoffel symbols, in fact, does not contain Christoffel symbols.

⊖ **3.52** Show that, for an antisymmetric second rank tensor ω, a) the formula $\nabla_i\omega^{ik} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}\,\omega^{ik})$ is valid, b) the expression $\epsilon^{ijk}\nabla_i\omega_{jk}$ does not contain the Christoffel symbols (similarly to the preceding problem).

⊕ **3.53** Show that a vector field $\boldsymbol{k} = k^j(x^i)\boldsymbol{e}_j(x^i)$, by choice of a certain new coordinate system $x'^l(x^i)$, can be reduced locally (in the vicinity of such a point at which $\boldsymbol{k}$ does not vanish) to the form $\boldsymbol{k} = \boldsymbol{e}'_1(x'^l)$.

⊕ **3.54** a) Consider a vector field $\boldsymbol{v}$ with the components

$$v^i = \epsilon^{ijk}\frac{\partial a}{\partial x^j}\frac{\partial b}{\partial x^k}$$

in a curvilinear coordinate system x^k (a and b are scalar fields). Show that the field $\boldsymbol{v}$ is *solenoidal*, i.e. satisfying the condition $\operatorname{div}\boldsymbol{v} = 0$. b) Consider a solenoidal vector field $\boldsymbol{v}$, $\operatorname{div}\boldsymbol{v} = 0$. Show that, in a neighborhood of a point where $\boldsymbol{v}$ does not vanish, there exist two scalar fields a and b such that the formula

$$v^i = \epsilon^{ijk}\frac{\partial a}{\partial x^j}\frac{\partial b}{\partial x^k}$$

is valid for the components of $\boldsymbol{v}$ in any coordinate system x^k. Thus, the previous formula gives locally general solution $\boldsymbol{v}$ of the equation $div\,\boldsymbol{v} = 0$.

4 Deformation. Deformation Rate. Vorticity.

Deformation. Strain tensors.

Deformation of a continuum is variation of distances between its particles. Deformation causes a response of the continuum and, in particular, results in the appearance of internal forces in it. So it is necessary to introduce quantitative deformation measures. For example, longitudinal deformation of an extended rod can be characterized by the relative elongation $(l - l_0)/l_0$ where l and l_0 are respectively the lengths of the rod in the current state and in a state relative to which deformation is measured (called the reference state). The absolute elongation $l - l_0$ is not a reasonable deformation measure: rods with lengths $l_0 = 10$ cm and $l_0 = 1$ m when extended by the same elongation $l - l_0 = 1$ mm are in different states. The choice of a reference state is rather conventional but is also often naturally grounded; e.g., as the reference state for an elastic rod, the state with zero internal forces is normally chosen. In what follows we assume that the *reference (undeformed) state*, where, by definition, strain equals zero, is realized at the initial instant $t = 0$. Let x_1, x_2, x_3 be a spatial coordinate system (Cartesian unless otherwise stipulated) with basis $\boldsymbol{e}_i$, $x_i(\xi, t)$ be the functions describing the motion, and $\xi = (\xi_1, \xi_2, \xi_3)$ be the Lagrangian coordinates equal to the spatial coordinates of the particle in its position at the initial instant, i.e., $\xi_i = x_i(\xi, 0)$. As a deformation measure the *Green strain tensor*

$$\overset{\circ}{\boldsymbol{\varepsilon}} = \overset{\circ}{\varepsilon}_{\alpha\beta} \boldsymbol{e}_\alpha \boldsymbol{e}_\beta \ , \qquad \overset{\circ}{\varepsilon}_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial x_i}{\partial \xi_\alpha}\frac{\partial x_i}{\partial \xi_\beta} - \delta_{\alpha\beta}\right)$$

and *Almansi strain tensor*

$$\boldsymbol{\varepsilon} = \varepsilon_{ij} \boldsymbol{e}_i \boldsymbol{e}_j \ , \qquad \varepsilon_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial \xi_\alpha}{\partial x_i}\frac{\partial \xi_\alpha}{\partial x_j}\right)$$

are often used; here, $\xi_\alpha(x, t)$ are the Lagrangian coordinates of the particle situated at the instant t at the point x. A *material line element* emanating from the particle ξ and corresponding to the vector $d\boldsymbol{\xi} = d\xi_\alpha \, \boldsymbol{e}_\alpha$ is the collection of particles with Lagrangian coordinates within the limits from (ξ_1, ξ_2, ξ_3) to $(\xi_1 + d\xi_1, \xi_2 + d\xi_2, \xi_3 + d\xi_3)$ occupying the infinitesimal segment. At an instant t, the position of a material line element is determined by the position $x(\xi, t)$ of its initial point and by the vector

$$d\boldsymbol{x} = dx_i \, \boldsymbol{e}_i = \frac{\partial x_i}{\partial \xi_\alpha} \, d\xi_\alpha \, \boldsymbol{e}_i$$

(see Figure 4.1). Each of the tensors $\overset{\circ}{\boldsymbol{\varepsilon}}$ and $\boldsymbol{\varepsilon}$ allows one to immediately express the

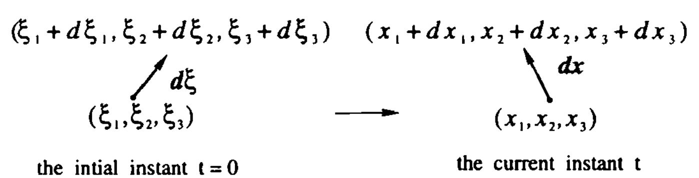

Figure 4.1

change in length squared ds^2 of a material line element (one of them, in terms of $d\boldsymbol{\xi}$, the other, in terms of $d\boldsymbol{x}$)

$$ds^2 - ds_0^2 = 2 \, \overset{\circ}{\varepsilon}_{\alpha\beta} \, d\xi_\alpha \, d\xi_\beta = 2\varepsilon_{ij} \, dx_i \, dx_j \ .$$

Here, ds_0 is the length of the material line element at instant $t = 0$. The strain tensors also allow one to find the relative elongation of any material line element, change in angle of any two material line elements (see Problems 4.3–4.4) and relative change in volume of a material solid element

$$\frac{dV - dV_0}{dV_0} = \sqrt{1 + 2\,\overset{\circ}{I}_1 + 4\,\overset{\circ}{I}_2 + 8\,\overset{\circ}{I}_3} - 1 = \frac{1}{\sqrt{1 - 2I_1 + 4I_2 - 8I_3}} - 1$$

where $\overset{\circ}{I}_i$ and I_i, $i = 1,2,3$ are the invariants of the corresponding tensors defined as $I_1 = \varepsilon_{ii}$, $I_2 = \frac{1}{2}(I_1^2 - \varepsilon_{ij}\varepsilon_{ij})$, $I_3 = \det\|\varepsilon_{ij}\|$ (see Problem 2.15). The mechanical meaning of components of Green strain tensor is illustrated by their relationship to a) the relative elongations l_1, l_2, l_3 reached at the current instant of the material line elements directed, at instant $t = 0$, along the basis vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ respectively, b) the angles $\psi_{\alpha\beta}$ ($\alpha \neq \beta$) reached at the current instant made by these elements (e.g., ψ_{23} is the angle between elements directed along $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ at the instant $t = 0$)

$$\overset{\circ}{\varepsilon}_{\alpha\beta} = \frac{1}{2}\left[(1 + l_\alpha)(1 + l_\beta)\cos\psi_{\alpha\beta} - \delta_{\alpha\beta}\right]$$

(no summation over α, β). In particular, for components with identical indices

$$\overset{\circ}{\varepsilon}_{\alpha\alpha} = \frac{1}{2}\left[(1 + l_\alpha)^2 - 1\right]$$

(no summation over α). Similarly, components of Almansi strain tensor are determined by "reverse" characteristics of deformation: with the changes in length "reached" at instant $t = 0$, relative to the length at instant t, of the material line elements directed along the basis vectors $\boldsymbol{e}_i$ and angles made by these elements (see Problem 4.17). **Components of the strain tensors can be expressed in terms of the displacement field** in the Lagrangian description

$$\boldsymbol{u}(\xi, t) = (x_\alpha(\xi, t) - \xi_\alpha)\,\boldsymbol{e}_\alpha$$

or in Eulerian description

$$\boldsymbol{w}(x, t) = (x_i - \xi_i(x, t))\,\boldsymbol{e}_i$$

(where, certainly, $\boldsymbol{u}(\xi, t) = \boldsymbol{w}(x(\xi, t), t)$). Namely, the formulae

$$\overset{\circ}{\varepsilon}_{\alpha\beta}(\xi, t) = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial \xi^\beta} + \frac{\partial u_\beta}{\partial \xi^\alpha} + \frac{\partial u_\gamma}{\partial \xi^\alpha}\frac{\partial u_\gamma}{\partial \xi^\beta}\right),$$

$$\varepsilon_{ij}(x, t) = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} - \frac{\partial w_k}{\partial x_i}\frac{\partial w_k}{\partial x_j}\right).$$

are valid. In the case when relative elongations and rotations of all the material line elements are small, i.e., when all the derivatives $\partial u_\alpha/\partial \xi_\beta \sim \delta \ll 1$ are small (or, identically, all the derivatives $\partial w_i/\partial x_j \sim \delta \ll 1$ are small), Green and Almansi strain tensor differ only by a value of order δ^2 from *linearized strain tensors* $\overset{\circ}{\boldsymbol{\varepsilon}}^{(1)}$, $\boldsymbol{\varepsilon}^{(1)}$

$$\overset{\circ}{\boldsymbol{\varepsilon}} = \overset{\circ}{\boldsymbol{\varepsilon}}^{(1)} + O(\delta^2), \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(1)} + O(\delta^2),$$

$$\overset{\circ}{\boldsymbol{\varepsilon}}{}^{(1)} = \overset{\circ}{\varepsilon}{}^{(1)}_{\alpha\beta} \boldsymbol{e}_\alpha \boldsymbol{e}_\beta \ , \quad \overset{\circ}{\varepsilon}{}^{(1)}_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial \xi_\beta} + \frac{\partial u_\beta}{\partial \xi_\alpha}\right) \ ,$$

$$\boldsymbol{\varepsilon}^{(1)} = \varepsilon^{(1)}_{ij} \boldsymbol{e}_i \boldsymbol{e}_j \ , \quad \varepsilon^{(1)}_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right) \ .$$

Green and Almansi strain tensors also differ by a value of order δ^2

$$\overset{\circ}{\boldsymbol{\varepsilon}}(\xi, t) = \boldsymbol{\varepsilon}\left(x(\xi, t), t\right) + \mathcal{O}(\delta^2) \ ,$$

$$\overset{\circ}{\boldsymbol{\varepsilon}}{}^{(1)}(\xi, t) = \boldsymbol{\varepsilon}^{(1)}\left(x(\xi, t), t\right) + \mathcal{O}(\delta^2) \ .$$

Normally one neglects the difference between the two linearized tensors and considers both as one linearized strain tensor. Depending on what is more convenient, its components are calculated by either of the two sets of formulae

$$\frac{1}{2}\left(\frac{\partial u_\alpha}{\partial \xi_\beta} + \frac{\partial u_\beta}{\partial \xi_\alpha}\right) \ , \quad \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right) \ .$$

This tensor is often called also the small strain tensor. The mechanical meanings of its components are as follows: ε_{11} is the relative elongation of a material line element directed along $\boldsymbol{e}_1$ at instant $t = 0$); ε_{22} and ε_{33} have the similar meanings; ε_{ij} when $i \neq j$ is half of the decrease in angle made by material line elements directed along $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ at the instant $t = 0$.

Transformation of a small volume of a continuum

Consider a small volume of a continuum that contains a particle ξ and consists of all material line elements emanating from ξ and corresponding to all possible infinitesimal vectors $d\boldsymbol{\xi}$. The positions of all these elements at instant t are determined by the position of the particle ξ, i.e., by the point with coordinates $x_i(\xi, t)$, and by the vectors dx (see Figure 4.1) related to $d\boldsymbol{\xi}$ by the linear transformation

$$dx_i = \frac{\partial x_i}{\partial \xi_\alpha} d\xi_\alpha$$

or, briefly,

$$d\boldsymbol{x} = \mathbf{F} d\boldsymbol{\xi} \ , \quad dx_i = F_{i\alpha} d\xi_\alpha$$

where the notation $\partial x_i / \partial \xi_\alpha = F_{i\alpha}$ is used. The linear transformation $\mathbf{F}$ is called the *distortion*, its matrix $F = \|F_{i\alpha}\|$ is called the distortion matrix or deformation gradient. The components of the strain tensors are obviously expressed in terms of the deformation gradient

$$\overset{\circ}{\varepsilon}_{\alpha\beta} = \frac{1}{2}(F_{k\alpha} F_{k\beta} - \delta_{\alpha\beta}) \ , \quad \varepsilon_{ij} = \frac{1}{2}(\delta_{ij} - H_{\gamma i} H_{\gamma j})$$

where $\|H_{\gamma j}\|$ is the matrix reciprocal to F: $F_{i\gamma}H_{\gamma j} = \delta_{ij}$. According to the theorem of polar decomposition, the matrix F can be written in the form

$$F = RU \ , \quad F_{i\alpha} = R_{i\beta}U_{\beta\alpha}$$

where $R = \|R_{i\beta}\|$ is an orthogonal matrix, and $U = \|U_{\beta\alpha}\|$ is a symmetric positive-definite matrix. The linear transformation $d\boldsymbol{\xi} \to \mathbf{U} \cdot d\boldsymbol{\xi}$ determined by the matrix U performs three extensions along the principal axes of the Green strain tensor, and the linear transformation $d\boldsymbol{\xi} \to \mathbf{R} \cdot d\boldsymbol{\xi}$ determined by the matrix R maps the principal axes of the Green strain tensor to the principal axes of the Almansi strain tensor (see Problem 4.19). So a transformation of a small volume of a continuum with center in a particle ξ can be represented as sequential performance of a) extensions along the three mutually orthogonal directions — transformation $\mathbf{U}$, b) rotation — transformation $\mathbf{R}$ (see Figure 4.2); besides that, the translation turning the point ξ into the point $x(\xi, t)$ must be performed. The transformation $\mathbf{U}$ has the meaning of pure deformation. It

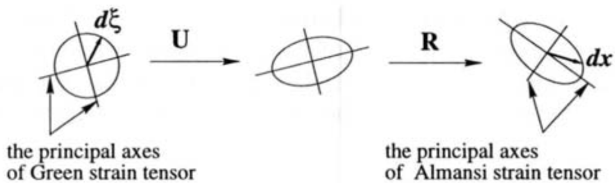

Figure 4.2

is closely connected with the Green strain tensor (see Problem 4.19).

Deformation rate, vorticity, divergence of velocity.

Many continua respond not so much to the strains, i.e., to relative elongations of material line elements, as to their rates. To describe them quantitatively, the *strain rate tensor* is introduced. Components of the strain rate tensor are expressed in terms of the velocity field $\boldsymbol{v}$ according to the formulae

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \ .$$

The mechanical meanings of these components are as follows: e_{11} is the rate of relative elongation of the material line element directed along $\boldsymbol{e}_1$ at the current instant; e_{22} and e_{33} have the similar meanings; e_{ij} at $i \neq j$ is half of the rate of decrease in angle made by the material line elements directed along $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ at the current instant.

The rate of relative change in volume of a particle is equal to the first invariant of the strain rate tensor which in turn is equal to divergence of the velocity vector

$$e_{ii} = \frac{\partial v_i}{\partial x_i} = \operatorname{div} \boldsymbol{v} \ .$$

The distribution of velocity in a small volume of a continuum is expressed in terms of the strain rate tensor and the *vorticity vector*

$$\boldsymbol{\omega} = \frac{1}{2} \operatorname{curl} \boldsymbol{v} = \frac{1}{2} \epsilon_{ijk} \frac{\partial v_k}{\partial x_j} \boldsymbol{e}_i$$

where ϵ_{ijk} are the components of Levi-Civita tensor. Namely, if $\boldsymbol{v}_0$ is the velocity of the particle situated at a point $\boldsymbol{r}_0$, the velocity of the particle situated at a point $\boldsymbol{r}_0 + \boldsymbol{\rho}$ is expressed by the Cauchy-Helmholtz formula

$$\boldsymbol{v} = \boldsymbol{v}_0 + \frac{\partial \Phi}{\partial \rho_i} \boldsymbol{e}_i + \boldsymbol{\omega} \times \boldsymbol{\rho} + O(\rho^2)$$

where $\Phi = (1/2) e_{ij} \rho_i \rho_j$. According to this formula, rotation with the angular velocity $\boldsymbol{\omega}$ equal to the vorticity vector is superimposed on the velocity connected with deformation (the second term in the formula) and the velocity of translation $\boldsymbol{v}_0$.

Use of curvilinear coordinate systems

Solution of some problems is essentially simplified if appropriate curvilinear coordinates are used instead of Cartesian ones. For example, oscillation of a spherical bubble in unbounded liquid is conveniently studied in the spherical coordinate system. Transition to a curvilinear coordinate system requires some modification of formulae valid in a Cartesian coordinate system. This is connected with the fact that the basis $\boldsymbol{e}_i = \partial \boldsymbol{r} / \partial x^i$ of a curvilinear coordinate system x^i, in general, depends upon the point x of the space. In particular, the basis $\boldsymbol{e}_i(x(\xi, t))$ at the point where a particle ξ is situated at instant t, in general, differs from the basis $\boldsymbol{e}_i(x(\xi, 0)) = \boldsymbol{e}_i(\xi)$ at the point where the particle was situated at instant $t = 0$ (here, as usual, the spatial coordinates of a particle at the initial instant are used as its Lagrangian coordinates: $\xi = x(\xi, 0)$). The displacement vector in the Lagrangian description is ordinarily decomposed in the basis $\boldsymbol{e}_i(\xi)$

$$\boldsymbol{u}(\xi, t) = u^\alpha(\xi, t)\, \boldsymbol{e}_\alpha(\xi) \ ,$$

and, in the Eulerian description, in the basis $\boldsymbol{e}_i(x)$

$$\boldsymbol{w}(x, t) = w^i(x, t)\, \boldsymbol{e}_i(x) \ ;$$

here, certainly, $\boldsymbol{w}(x(\xi,t),t) = \boldsymbol{u}(\xi,t)$. To make notation briefer, the arguments of functions are often omitted. Then, to avoid confusion, the notations $\boldsymbol{e}_\alpha(\xi) = \overset{\circ}{\boldsymbol{e}}_\alpha$ and $\boldsymbol{e}_i(x) = \boldsymbol{e}_i$ are introduced, in particular, $\boldsymbol{u} = u^\alpha \overset{\circ}{\boldsymbol{e}}_\alpha$, $\boldsymbol{w} = w^i \boldsymbol{e}_i$. Then, the Green strain tensor can be written in the form

$$\overset{\circ}{\boldsymbol{\varepsilon}} = \overset{\circ}{\varepsilon}_{\alpha\beta} \overset{\circ}{\boldsymbol{e}}{}^\alpha \overset{\circ}{\boldsymbol{e}}{}^\beta , \quad \overset{\circ}{\varepsilon}_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial x^i}{\partial \xi^\alpha}\frac{\partial x^j}{\partial \xi^\beta} g_{ij} - \overset{\circ}{g}_{\alpha\beta}\right)$$

where $\overset{\circ}{\boldsymbol{e}}{}^\mu$ is the basis dual to the basis $\overset{\circ}{\boldsymbol{e}}_\nu$ (see Section 3), $\overset{\circ}{g}_{\alpha\beta} = \overset{\circ}{\boldsymbol{e}}{}^\alpha \cdot \overset{\circ}{\boldsymbol{e}}{}^\beta$, and the components of the metric tensor $g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$ are calculated for the point $x = x(\xi, t)$. The Almansi strain tensor can be written in the form

$$\boldsymbol{\varepsilon} = \varepsilon_{ij} \boldsymbol{e}^i \boldsymbol{e}^j , \quad \varepsilon_{ij} = \frac{1}{2}\left(g_{ij} - \frac{\partial \xi^\alpha}{\partial x^i}\frac{\partial \xi^\beta}{\partial x^j} \overset{\circ}{g}_{\alpha\beta}\right)$$

where $\boldsymbol{e}^m$ is the basis dual to the basis $\boldsymbol{e}_n$, $g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$, and the components of the metric tensor $\overset{\circ}{g}_{\alpha\beta} = \overset{\circ}{\boldsymbol{e}}_\alpha \cdot \overset{\circ}{\boldsymbol{e}}_\beta$ are calculated for the point $\xi = \xi(x,t)$. The expressions of the components of the strain tensors in terms of displacement field remains the same, but the partial derivatives are replaced by the covariant ones

$$\overset{\circ}{\varepsilon}_{\alpha\beta} = \frac{1}{2}\left(\overset{\circ}{\nabla}_\alpha u_\beta + \overset{\circ}{\nabla}_\beta u_\alpha + \overset{\circ}{\nabla}_\alpha u^\gamma \overset{\circ}{\nabla}_\beta u_\gamma\right) ,$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\nabla_i w_j + \nabla_j w_i - \nabla_i w^k \nabla_j w_k\right)$$

where the covariant derivative $\overset{\circ}{\nabla}_\alpha$ is calculated with use of the Christoffel symbols $\overset{\circ}{\Gamma}{}^\kappa_{\lambda\mu}(\xi)$ determined by the usual formulae (see Section 3) in terms of the components of the metric tensor $\overset{\circ}{g}_{\beta\gamma}(\xi)$. If the relative elongations and rotations of all material line elements are small, the small strain tensor is used. It can be written in either of the two forms

$$\boldsymbol{\varepsilon}^{(1)} = \frac{1}{2}\left(\overset{\circ}{\nabla}_\alpha u_\beta + \overset{\circ}{\nabla}_\beta u_\alpha\right) \overset{\circ}{\boldsymbol{e}}{}^\alpha \overset{\circ}{\boldsymbol{e}}{}^\beta ,$$

$$\boldsymbol{\varepsilon}^{(1)} = \frac{1}{2}\left(\nabla_i w_j + \nabla_j w_i\right) \boldsymbol{e}^i \boldsymbol{e}^j .$$

Components of the strain rate tensor are expressed in terms of the velocity field by the formula

$$e_{ij} = \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right) .$$

In some cases, a special coordinate system, determined by the motion of the continuum itself, is useful and referred to as *concomitant*. It, as any coordinate system, sets correspondence between a point of the space and three numbers, namely, Lagrangian coordinates $\xi^\alpha = \xi^\alpha(x,t)$ of the particle situated at the point x at the current instant. Thus, this system is differently determined for different instants: different triples of numbers correspond to the same point of the space; rigorously speaking, they must be denoted with $(\xi^1_{(t)}, \xi^2_{(t)}, \xi^3_{(t)})$. The coordinate lines ξ^α at the instants $t_1 \neq t_2$ occupy different positions, but pass through the same particles; this is why this coordinate system is referred to as *concomitant* (with the motion of the continuum). Its basis

$$\hat{e}_\alpha = \frac{\partial \boldsymbol{r}}{\partial \xi^\alpha} = \frac{\partial \boldsymbol{r}}{\partial x^i}\frac{\partial x^i}{\partial \xi^\alpha} = \frac{\partial x^i}{\partial \xi^\alpha}\boldsymbol{e}_i \ .$$

In the concomitant coordinate system, the Almansi strain tensor can be written in the form

$$\boldsymbol{\varepsilon} = \hat{\varepsilon}_{ij}\hat{\boldsymbol{e}}^i\hat{\boldsymbol{e}}^j \ , \quad \hat{\varepsilon}_{ij} = \frac{1}{2}\left(\hat{g}_{ij} - \overset{\circ}{g}_{ij}\right)$$

where $\hat{\boldsymbol{e}}^m$ is the basis dual to the basis $\hat{\boldsymbol{e}}_n$, and $\hat{g}_{ij} = \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j$ are the components of the metric tensor. As in any coordinate system, the components $\hat{\varepsilon}_{ij}$ are expressed in terms of the displacement field by the formula

$$\hat{\varepsilon}_{ij} = \frac{1}{2}\left(\hat{\nabla}_i\hat{w}_j + \hat{\nabla}_j\hat{w}_i - \hat{\nabla}_i\hat{w}^k\hat{\nabla}_j\hat{w}_k\right) \ .$$

The components of the Almansi tensor in the basis $\hat{\boldsymbol{e}}^i$ coincide with the components of the Green tensor in the basis $\overset{\circ}{\boldsymbol{e}}{}^\alpha$, $\hat{\varepsilon}_{\alpha\beta} = \overset{\circ}{\varepsilon}_{\alpha\beta}$.

Conditions of compatibility

In some important cases, the velocity field $\boldsymbol{v}$ of a flow in the continuum is determined by one function φ and can be written in the form $\boldsymbol{v} = \operatorname{grad}\varphi$. In this case, the vector field $\boldsymbol{v}$ is referred to as *potential*, and the function φ is called its *potential*. By far not all vector fields $\boldsymbol{v}$ are potential. It is clear that, to be potential, a vector field must satisfy some conditions, since the three functions — its components — are expressed in terms of one function — potential. The necessary condition for potentiality of a field $\boldsymbol{v}$ is the relationship $\operatorname{curl}\boldsymbol{v} = 0$; it is called also the *compatibility condition* for components of a potential field $\boldsymbol{v}$. If the field $\boldsymbol{v}$ is considered in a simply connected domain, this is the sufficient condition for existence of a single-valued potential. Similarly to the relationship $\boldsymbol{v} = \operatorname{grad}\varphi$, the six components of a small strain tensor are expressed in terms of the three components of the displacement w_i and, consequently, cannot be arbitrary. They satisfy the relationships

$$\frac{\partial^2\varepsilon_{kl}}{\partial x_i\partial x_j} + \frac{\partial^2\varepsilon_{ij}}{\partial x_k\partial x_l} - \frac{\partial^2\varepsilon_{il}}{\partial x_k\partial x_j} - \frac{\partial^2\varepsilon_{kj}}{\partial x_i\partial x_l} = 0$$

(x_i is a spatial Cartesian coordinate system); these relationships also are called the *compatibility conditions for strain components*. These are the necessary conditions, and, when the tensor field of small strains $\boldsymbol{\varepsilon}$ is considered in a simply connected domain, are the sufficient conditions for the relationships

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right)$$

to be valid, $\boldsymbol{w}$ being a vector field. In other words, the compatibility conditions for strain components are the conditions of possibility, in principle, to obtain the strain as a result of a displacement. In all the problems of this section, x_i are spatial (Eulerian) coordinates, and ξ_α are Lagrangian coordinates. As Lagrangian coordinates of a particle, spatial coordinates of the point, at which the particle was situated at the initial instant (in the undeformed state), are taken. Unless otherwise stipulated, the spatial coordinate system is Cartesian.

— PROBLEMS —

Deformation. Strain tensors.

⊙ **4.1** As a result of displacement, the particles (ξ_1, ξ_2, ξ_3) of a continuum are situated at the points with the coordinates

$$x_1 = \xi_1 + a\xi_1 \ , \ x_2 = \xi_2 \ , \ x_3 = \xi_3 \quad (a = \text{const})$$

with reference to a spatial Cartesian coordinate system x_i. (This deformation is called a uniform uniaxial extension along the axis x_1.) What happened to the material line elements initially situated parallel to the coordinate axis x_1 and those orthogonal to this axis? Consider the two cases: $a > 0$ and $-1 < a < 0$?

⊖ **4.2** For a uniaxial extension (see Problem 4.1), find the displacement field in Lagrangian and Eulerian descriptions, and calculate the components of the Green and Almansi strain tensors.

⊖ **4.3** a) A material line element emanating from a particle ξ corresponds to a vector $d\boldsymbol{\xi}$. Regarding the components $\overset{\circ}{\varepsilon}_{\alpha\beta}$ of the Green strain tensor in this particle as known, find the relative elongation of the material line element by deformation. b) For a uniaxial extension (see Problem 4.1), find the relative elongations of material line elements perpendicular to the axis x_3 and making the angles $\pm\pi/4$ with the axis x_1 before deformation.

⊖ **4.4** a) Two material line elements emanating from a particle ξ correspond to vectors $d\boldsymbol{\xi}^{(1)}$ and $d\boldsymbol{\xi}^{(2)}$. Regarding the components $\overset{\circ}{\varepsilon}_{\alpha\beta}$ of the Green strain tensor in this particle as known, find the angle made by these material line elements after deformation. b) For a uniaxial extension (see Problem 4.1), find the angle made after deformation by the material line elements perpendicular to the axis x_3 and making the angles $\pm\pi/4$ with the axis x_1 before deformation.

⊙ **4.5** Find relative change in volume caused by a uniaxial extension (see Problem 4.1).

⊖ **4.6** As a result of displacement, the particles (ξ_1, ξ_2, ξ_3) of a continuum are situated at the points with the coordinates

$$x_1 = \xi_2 \ , \ x_2 = -(1+b)\xi_1 \ , \ x_3 = \xi_3 \quad (b = \text{const} > -1)$$

with reference to a spatial Cartesian coordinate system x_i. a) What happened as a result of deformation to material line elements initially situated parallel to the coordinate axes? b) Find the Green and Almansi strain tensors. c) Can the Green and Almansi tensors be regarded as coincident at $|b| \ll 1$? Compare with the answer to Problem 4.2 at $a \ll 1$.

⊕ **4.7** As a result of displacement, the particles (ξ_1, ξ_2, ξ_3) of a continuum are situated at the points with the coordinates

$$x_1 = \xi_1 + \alpha \sin(k\xi_1) \ , \ x_2 = \xi_2 \ , \ x_3 = \xi_3$$

$$(\alpha = \text{const}, \ |\alpha| < 1, \ k = \text{const})$$

with reference to a spatial Cartesian coordinate system x_i. Show that a uniaxial extension takes place in a small vicinity of each particle of the medium (see Problem 4.1). What is the relative elongation of the material line element emanating from a given point ξ and parallel to the axis x_1 before deformation? Calculate the Green deformation tensor. Indicate the particles in a small vicinity of which there is no deformation.

⊙ **4.8** As a result of displacement, the particles (ξ_1, ξ_2, ξ_3) of a continuum are situated at the points with the coordinates

$$x_i = \xi_i + a\xi_i \ , \ (i = 1, 2, 3) \ (a = \text{const} > -1)$$

with reference to a spatial Cartesian coordinate system x_i. Show that the relative elongations of all material line elements are identical. Note that such deformation is called a uniform extension or compression to what signs of a do they correspond?.

⊖ **4.9** A simple shear is deformation of a continuum described by the formulae

$$x_1 = \xi_1 + a(t)\xi_2 \ , \ x_2 = \xi_2 \ , \ x_3 = \xi_3$$

where x_i is a spatial Cartesian coordinate system, ξ_α are Lagrangian coordinates, and $a(t)$ is a function of time with $a(0) = 0$. Regarding the function $a(t)$ as given, find the Green and Almansi strain tensors. Find their principal components and principal axes. Simplify the formulae for the case $|a(t)| \ll 1$.

⊖ **4.10** Find the components of the displacement field for a simple shear (see Problem 4.9) in Lagrangian and Eulerian descriptions. Determine the components of the Green and Almansi strain tensors, expressing them in terms of the derivatives of the displacement field. Find the small strain tensor.

⊖ **4.11** For a simple shear (see Problem 4.9), find a) relative elongations of material line elements emanating from particle ξ and parallel to the axes x_1, x_2, x_3 before deformation; b) all possible material line elements whose relative elongations at the instant t equal zero.

⊙ **4.12** Find the relative change in volume of a small material element during a simple shear (see Problem 4.9) by two methods: with the use of the invariants of the Green strain tensor and those of the Almansi strain tensor.

⊖ **4.13** The small strain tensor at a point where small deformation takes place has the following matrix of components in a Cartesian coordinate system

$$\begin{pmatrix} 0.01 & 0.03 & 0 \\ 0.03 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}$$

Find the maximal and minimal relative elongations of material line elements at this point. Find the direction of the material elements whose relative elongation is a) maximal, b) minimal. Calculate the relative change in volume at this point.

⊖ **4.14** A double shear is deformation of a continuum described by the formulae

$$x_1 = \xi_1 + b(t)\xi_2 \ , \ x_2 = \xi_2 + b(t)\xi_3 \ , \ x_3 = \xi_3$$

where x_i are spatial Cartesian coordinates, ξ_α are Lagrangian coordinates, and $b(t)$ is a function of time with $b(0) = 0$. Treating the function $b(t)$ as given, find the Green and Almansi strain tensors.

⊙ **4.15** Find the components of the displacement field for a double shear (see Problem 4.14) in the Eulerian description. Find the small strain tensor.

⊖ **4.16** The positions of three material line elements in a deformed state are characterized by the vectors $d\boldsymbol{x}^{(i)} = ds\,\boldsymbol{e}_i$, $i = 1, 2, 3$ ($\boldsymbol{e}_i$ are the vectors of the orthogonal basis of a Cartesian coordinate system). Their "inverse" relative elongation

$ds_0^{(i)}/ds - 1$ (where $ds_0^{(i)}$ is the length of the corresponding element before deformation) equal l_i. The elements characterized by the vectors $d\boldsymbol{x}^{(i)}$ and $d\boldsymbol{x}^{(j)}$ in the deformed state make the angle ψ_{ij} before deformation. Prove the formula showing the mechanical meaning of the components of the Almansi strain tensor

$$\varepsilon_{ij} = \frac{1}{2}[-(1+l_i)(1+l_j)\cos\psi_{ij} + \delta_{ij}]$$

(no summation over i and j).

⊖ **4.17** Prove that the eigenvalues of a Green strain tensor $\overset{\circ}{\lambda}_\alpha$ and the eigenvalues of an Almansi strain tensor λ_i satisfy the inequalities

$$1 + 2\overset{\circ}{\lambda}_\alpha > 0\ , \quad 1 - 2\lambda_i > 0\ .$$

⊖ **4.18** Prove that for deformation with the Green strain tensor $\overset{\circ}{\boldsymbol{\varepsilon}}$ and the Almansi strain tensor $\boldsymbol{\varepsilon}$ a) the material line element directed along a principal axis of the tensor $\overset{\circ}{\boldsymbol{\varepsilon}}$ before deformation is directed along a principal axis of the tensor $\boldsymbol{\varepsilon}$ in the deformed state, b) on the contrary, the material line element directed along a principal axis of the tensor $\boldsymbol{\varepsilon}$ in the deformed state was directed along a principal axis of the tensor $\overset{\circ}{\boldsymbol{\varepsilon}}$ before deformation, c) the eigenvalues of the Green strain tensor $\overset{\circ}{\lambda}$ and of the Almansi strain tensor λ are related to each of them by

$$1 + 2\overset{\circ}{\lambda} = \frac{1}{1 - 2\lambda}\ .$$

⊖ **4.19** Prove the following polar decomposition theorem. The deformation gradient $F = \|F_{i\alpha}\|$ (and, in general, any nonsingular matrix) can be written in the form

$$F = RU\ , \quad F_{i\beta} = R_{i\alpha}U_{\alpha\beta}$$

where $R = \|R_{i\alpha}\|$ is an orthogonal matrix, and $U = \|U_{\alpha\beta}\|$ is a symmetric positive-definite matrix. The principal axes of the matrix U coincide with the principal axes of the Green strain tensor, and the corresponding eigenvalues k_α of the matrix U are expressed in terms of the eigenvalues $\overset{\circ}{\lambda}_\alpha$ of the Green tensor: $k_\alpha = \sqrt{1 + 2\overset{\circ}{\lambda}_\alpha}$. The linear transformation determined by the matrix R rotates the principal axes of the Green tensor into the principal axes of the Almansi tensor.

⊙ **4.20** For a uniaxial extension (see Problem 4.1) at $a > 0$, find the initial positions of the three mutually orthogonal material line elements which remain mutually orthogonal after deformation. Find the directions of these elements after deformation. Indicate the directions of the material line elements whose relative elongation is maximal.

⊖ **4.21** Show that the following motions cannot be reduced to each other by imposing a rigid motion (i.e., a rotation and a translation): a) a rotation around the axis x_3, b) a uniaxial extension along the axis x_1 (Problem 4.1), c) a simple shear in the plane x_1, x_2 (Problem 4.9), d) a double shear (Problem 4.14).

⊕ **4.22** Represent the transformation of a small vicinity of a particle ξ resulted from a simple shear (see Problem 4.9) in the form of the extensions along three mutually orthogonal directions, rotation and translation.

⊖ **4.23** For a uniaxial extension (Problem 4.1) with the parameter a as a function of time $a = a(t)$ such that $a(0) = 0$, a) write the formula describing transition, for an instant t, from a spatial coordinate system x_i to the concomitant coordinate system ξ^α, b) draw the coordinate lines, find the basis vectors and the components of the metric tensor for the concomitant coordinate system ξ^α for an instant t, c) find the components of the Green tensor for an instant t in the coordinate system x_i as well as the covariant, mixed and contravariant components of the Almansi strain tensor for an instant t in the concomitant coordinate system ξ^α.

⊖ **4.24** For a uniaxial extension (Problems 4.1, 4.23), choose the coordinates x_i of the space point at which a particle is situated at the instant t_* as new Lagrangian coordinates η^β of this particle. Draw the coordinate lines, find the basis vectors and the components of the metric tensor in the concomitant coordinate system η^β at the instant $t = 0$.

⊙ **4.25** For a simple shear (Problem 4.9), find, for an instant t the coordinate lines of the concomitant coordinate system passing through the space point with the coordinates $(0, 0, 0)$. Do they vary in time? Find also the basis vectors of the concomitant coordinate system and the components of the metric tensor in it.

⊖ **4.26** The axis of a cylindrical rod with circular cross section is positioned along the axis x_3 of a spatial Cartesian coordinate system x_i. The rod deforms according to the formula

$$x_1 = \xi_1 - \alpha(t)\xi_2\xi_3\ , \ x_2 = \xi_2 + \alpha(t)\xi_1\xi_3\ , \ x_3 = \xi_3$$

where ξ_α are the Lagrangian coordinates, and $\alpha(t)$ is a function of time with $\alpha(0) = 0$. Regarding the function $\alpha(t)$ as given, a) find the position at an instant t of the particles forming, at the instant $t = 0$, a cross section of the rod, the circumference bounding this cross section, its radius as well as a segment parallel to the axis of the rod and lying on its surface; b) find the displacement field in Eulerian description; c) if $|\alpha| << 1$, find the small strain tensor, the value of maximal relative elongation for the material line elements emanating from the point x and the direction of the element experiencing the maximal elongation. d) write the formulae describing the motion in the cylindrical coordinate system, taking the (cylindrical) coordinates of a particle at the instant $t = 0$ as the Lagrangian coordinates.

$\ominus$ **4.27** A tube (thick-walled circular cylinder) expands under action of internal pressure. Its deformation is described by the formulae

$$r = r_0 + f(r_0, t) \ , \ \varphi = \varphi_0 \ , \ z = z_0$$

where r, φ, z are the spatial cylindrical coordinate system (see Problem 3.7), r_0, φ_0, z_0 are the Lagrangian coordinates (the cylindrical coordinates of the initial position of the particle), and $f(r_0, 0) = 0$. Let the function $f(r_0, t)$ be given. Find a) the Green strain tensor, b) the Almansi strain tensor, c) the relative elongations of the material line elements emanating from a particle (r_0, φ_0, z_0) directed along the coordinate lines of the cylindrical coordinate system before deformation.

$\ominus$ **4.28** The components of Green and Almansi strain tensors can be of order 1 in the cases when the components of the linearized strain tensor are small or even equal zero. Verify this, using as an example the displacement field $w_1(x) = x_2$, $w_2(x) = -x_1$, $w_3 = 0$ where $x = (x_1, x_2, x_2)$ are spatial Cartesian coordinates.

$\ominus$ **4.29** The components of the linearized strain tensor can be of order 1 in the case when the components of Green and Almansi strain tensors are small or even equal zero. Verify this, using as an example the displacement field $x_i = R_{ij}\xi_j$ where x_i is a spatial Cartesian coordinate system, ξ_α are the Lagrangian coordinates, $\|R_{ij}\|$ is an orthogonal matrix.

Strain rate tensor, vorticity, divergence of velocity.

$\odot$ **4.30** Find the velocity field and components of the strain rate tensor in Eulerian description for a) a uniaxial extension (Problem 4.1), treating the parameter a as a given function of time t, b) a simple shear (Problem 4.9) and c) a double shear (Problem 4.14).

$\odot$ **4.31** Calculate the components e_{ij} of the strain rate tensor in a spatial Cartesian coordinate system x_i and the components of its deviator $e_{ij}^{(d)} = e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}$ for the flows of a medium with the velocity fields described, in this coordinate system, by the formulae a) $v_1 = Ax_1$, $v_2 = Bx_2$, $v_3 = 0$ ($A = \text{const}, B = \text{const}$); b) $v_1 = \alpha t x_1$, $v_2 = v_3 = 0$ ($\alpha = \text{const}$); c) $v_1 = \beta t x_3$, $v_2 = v_3 = 0$ ($\beta = \text{const}$).

$\odot$ **4.32** Calculate the components e_{ij} of the strain rate tensor in a spatial Cartesian coordinate system x_i for the flow of a medium with the velocity field described, in this coordinate system, by the formulae

$$v_1 = \frac{2x_2}{t} \ , \quad v_2 = \frac{2x_1}{t} \ , \quad v_3 = 0 \ .$$

Does a change in volume of material solid elements take place during this motion?

⊖ **4.33** The components of the velocity field of a medium in a spatial Cartesian coordinate system x_i at a given instant have the form

$$v_1 = kx_2 \ , \quad v_2 = v_3 = 0 \quad (k = \text{const}) \ .$$

Find, for this instant, the rate of change in angle made by material line elements emanating from the point x and situated along two straight lines making the angles $\pi/4$ with the axis x_1 and $\pi/2$ with the axis x_3.

⊖ **4.34** The covariant components of a velocity field in a spatial cylindrical coordinate system $x^1 = r$, $x^2 = \varphi$, $x^3 = z$ have the form

$$v_1 = 0 \ , \quad v_2 = k \ , \quad v_3 = 0 \quad (k = \text{const})$$

everywhere except the point $r = 0$. a) Draw the particle paths of the medium, find the value of the velocity of a particle and the physical components of the velocity. b) Calculate the components of the strain rate tensor. c) Find the vorticity vector. d) Find the principal axes of the strain rate tensor. Do they rotate in time in an individual particle? e) What is, at an instant, the angular velocity of the material line elements which are situated at this instant along the principal axes of the strain rate tensor?

⊙ **4.35** A medium experiences a uniaxial extension, i.e., its motion is described by the formulae

$$x_1 = \xi_1 + a(t)\xi_1 \ , \quad x_2 = \xi_2 \ , \quad x_3 = \xi_3$$

where x_i are spatial Cartesian coordinates, ξ_α are the Lagrangian coordinates, $a(t)$ is a function of time with $a(0) = 0$. Verify that the vorticity vector field is zero during this motion. Show that there are material line elements which rotate (simultaneously varying in length).

⊖ **4.36** Find the vorticity vector field $\boldsymbol{\omega}$ for a simple shear (Problem 4.9). Indicate the material line elements angular velocity of which equals $\boldsymbol{\omega}$ at an instant t. Find the angular velocity of the material line elements directed at the considered instant t along the axes x_1, x_2, x_3.

⊙ **4.37** The velocity distribution in a rigid body is determined by the Euler formula $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{\Omega} \times \boldsymbol{r}$ where $\boldsymbol{\Omega}(t)$ is the angular velocity, $\boldsymbol{r}$ is the radius-vector with respect to a point O, $\boldsymbol{v}_0(t)$ is the velocity of the point O. Calculate the vorticity vector for this velocity field.

⊖ **4.38** Prove that, if the strain rate tensor is identical for all particles of a medium at an instant, then the vorticity vector is also identical for all the particles at this instant.

⊖ **4.39** The strain rate tensor equals zero in all particles of a medium. Show that, in this case, the velocity field is described by the Euler formula for the velocity distribution in a rigid body $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{\Omega} \times \boldsymbol{r}$ where $\boldsymbol{r}$ is the radius-vector relative to a point O, $\boldsymbol{v}_0(t)$ is the velocity of this point, and $\boldsymbol{\Omega}(t)$ is a vector independent of $\boldsymbol{r}$ (the vector of instantaneous angular velocity).

⊖ **4.40** Consider the second rank antisymmetric tensor $\omega_{ij}e^ie^j$ determined by a vector field $\boldsymbol{v}$

$$\omega_{ij} = \frac{1}{2}\left(\nabla_i v_j - \nabla_j v_i\right)$$

a) Show that in any curvilinear coordinate system x^i, the following equality holds

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x^i} - \frac{\partial v_i}{\partial x^j}\right) ;$$

b) Show that the vorticity vector $\boldsymbol{\omega} = 1/2 \operatorname{curl} \boldsymbol{v}$ can be represented by the above-introduced tensor, since the following formulae are valid

$$\omega^\gamma = \frac{1}{2}\epsilon^{\gamma\alpha\beta}\omega_{\alpha\beta} , \quad \omega_{\alpha\beta} = \epsilon_{\gamma\alpha\beta}\omega^\gamma .$$

⊖ **4.41** For a motion of a continuum with acceleration $\boldsymbol{a}$ satisfying the condition $\operatorname{curl} \boldsymbol{a} = 0$, consider the components of vorticity vector $\hat{\omega}^\gamma$ with respect to a concomitant coordinate system. Consider also the components $\hat{\omega}_{\alpha\beta}$ of the second-rank antisymmetric tensor representing the vorticity vector as indicated in the preceding problem. Show that the following formulae are valid

$$\text{a)}\ \frac{\partial}{\partial t}\hat{\omega}_{\alpha\beta} = 0 , \quad \text{b)}\ \frac{\partial}{\partial t}\left(\hat{\omega}^\gamma\sqrt{\hat{g}}\right) = 0$$

where $\hat{g} = \det \|\hat{g}_{\alpha\beta}\|$.

⊖ **4.42** For a given in the Problem 4.41 motion of a continuum, consider at any moment a *vorticity line*, that is, a line whose tangent is a vortex vector (at every point of the line). Show that a vorticity line is frozen into the medium, that is it passes through the same particles of the medium at any moment.

Compatibility conditions

⊙ **4.43** Verify that the equality $\operatorname{curl} \boldsymbol{v} = 0$ is the necessary condition for potentiality of the vector field $\boldsymbol{v}$.

⊙ **4.44** Verify that the field velocity for a uniaxial extension (Problem 4.1) satisfies the condition for potentiality. Find the potential of this velocity field.

⊖ **4.45** Show that the compatibility condition for the small strain tensor can be written, using Levi-Civita tensor, in the form a) $\epsilon_{kip}\epsilon_{ljq}\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} = 0$ in a Cartesian coordinate system, b) $\epsilon^{kip}\epsilon^{ljq}\nabla_k\nabla_l\varepsilon_{ij} = 0$ in any coordinate system. Here, ϵ^{ijk} are the components of Levi-Civita tensor.

⊙ **4.46** Indicate the conditions which a symmetric tensor field must satisfy to be the strain rate tensor for a velocity field.

⊖ **4.47** The tensor field is given by the formulae for its components in a Cartesian coordinate system x_1, x_2, x_3 a) $\varepsilon_{11} = Ax_3^2$, $\varepsilon_{22} = Bx_1^2$, $\varepsilon_{33} = Cx_2^2$, $\varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0$, $A, B, C = \text{const}$; b) $\varepsilon_{11} = 2Ax_1x_2$, $\varepsilon_{22} = 2Bx_1x_2$, $\varepsilon_{12} = \frac{1}{2}(Ax_1^2 + Bx_2^2)$, $\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$, $A, B = \text{const}$ Is it the small strain tensor for a displacement field?

⊖ **4.48** The components f_{ij} of a second-rank tensor field are given in a Cartesian coordinate system x_k. Consider the possibility of finding a vector field $\boldsymbol{v}$ for which the relationships $\partial v_i/\partial x_j = f_{ij}$ hold. Prove that this is possible if and only if the components f_{ij} satisfy the following conditions

$$\epsilon_{lkj}\frac{\partial f_{ij}}{\partial x_k} = 0$$

(ϵ_{lkj} are the components of Levi-Civita tensor). Formulate this statement in a curvilinear coordinate system.

⊖ **4.49** The components e_{ij} and ω_j of a second rank symmetric tensor field and a vector field satisfying the condition $\operatorname{div}\boldsymbol{\omega} = 0$ are given in a Cartesian coordinate system. Consider possibility to find the velocity field $\boldsymbol{v}$ for which $\mathbf{e}$ and $\boldsymbol{\omega}$ are the strain rate and vorticity fields. Prove that this is possible if and only if the components e_{ij} and ω_j satisfy the following conditions

$$\epsilon_{lkj}\frac{\partial e_{ij}}{\partial x_k} = \frac{\partial \omega_l}{\partial x_i}$$

(ϵ_{lkj} are the components of Levi-Civita tensor). Formulate this statement in a curvilinear coordinate system.

5 Principles of Symmetry and Tensor Functions.

Let us consider algebraic properties of tensor characteristics of a medium at a given point of the three-dimensional Euclidian space with local basis e_i. Let $\mathbf{T}_1, \ldots, \mathbf{T}_N$ be a set of tensors among which there is the metric tensor $\mathbf{g}$. The *symmetry group* G of this set of tensors is the set of orthogonal transformations of the basis $e'_i = a^j{}_i e_j$ preserving the values of the components of each of these tensors. For example, if for the contravariant components of a tensor of a rank r,

$$T'^{i_1 \ldots i_r} e'_{i_1} \ldots e'_{i_r} = T^{i_1 \ldots i_r} e'_{i_1} \ldots e'_{i_r} ,$$

or

$$b^{i_1}{}_{j_1} \ldots b^{i_r}{}_{j_r} T^{j_1 \ldots j_r} = T^{i_1 \ldots i_r} \tag{5.1}$$

where $(b^i{}_j)$ is the matrix inverse to the matrix $(a^i{}_j)$, then it is said that the tensor $\mathbf{T}$ is invariant relative to the group G, determined by $(a^i{}_j)$. The symmetry group of the tensor $\mathbf{g}$ itself (isotropy) is the complete group of rotations and reflections represented by orthogonal matrices. The symmetry group of a given set of tensors containing $\mathbf{g}$ is a subgroup of the complete group of rotations and reflections or coincides with that group.

A *tensor function* $\mathbf{T} = \mathbf{F}(\mathbf{T}_1, \ldots, \mathbf{T}_N)$ is a dependence of the components of the tensor $\mathbf{T}$ upon the components of the tensors $\mathbf{T}_1, \ldots, \mathbf{T}_N$ which is invariant with respect to choice of the basis e'_i. It means that, for a set of functions of the form

$$T^{i_1 \ldots i_r} = F^{i_1 \ldots i_r}(T_1^{j_1 \ldots j_{r_1}}, \ldots, T_N^{j_1 \ldots j_{r_N}}) ,$$

the relationships

$$T'^{i_1 \ldots i_r} = F^{i_1 \ldots i_r}(T_1'^{j_1 \ldots j_{r_1}}, \ldots, T_N'^{j_1 \ldots j_{r_N}})$$

are valid for any nonsingular matrix $(a^i{}_j)$, i.e. in all coordinate systems the components of $\mathbf{T}$ are the same functions of the components of $\mathbf{T}_1$, $\mathbf{T}_2$, ..., $\mathbf{T}_N$. This condition can be written in the form

$$F^{i_1 \ldots i_r}(b^{j_1}{}_{k_1} \ldots b^{j_{r_1}}{}_{k_{r_1}} T_1^{k_1 \ldots k_{r_1}}, \ldots, b^{j_1}{}_{k_1} \ldots b^{j_{r_N}}{}_{k_{r_N}} T_N^{k_1 \ldots k_{r_N}}) = b^{i_1}{}_{k_1} \ldots b^{i_r}{}_{k_r} T_1^{k_1 \ldots k_r} \tag{5.2}$$

The equality (5.2) is a very strong restriction on the form of a function $\mathbf{F}$. By use of it one can prove, e.g., that

1. if a vector $\boldsymbol{a}$ is a function of a vector $\boldsymbol{b}$ only then

$$\boldsymbol{a} = k\, \boldsymbol{b}$$

where k is a scalar which may depend on $|\boldsymbol{b}|$ (in fact, the components of a metric tensor $\mathbf{g}$ are needed to calculate $|\boldsymbol{b}|$; so to be exact we should say that $\boldsymbol{a}$ is a function of $\boldsymbol{b}$ and $\mathbf{g}$).

2. if a second-rank tensor $\mathbf{H}$ is a function of a second-rank tensor $\mathbf{T}$ only (besides of a metric tensor $\mathbf{g}$ which stands for a unit and is used to construct scalar invariants) then

$$\mathbf{H} = k_0\mathbf{g} + k_1\mathbf{T} + k_2\mathbf{T}^2$$

where $\mathbf{T}^2 = \mathbf{T} \cdot \mathbf{T}$ and k_0, k_1, k_2 are scalar functions of invariants of the tensor $\mathbf{T}$;

3. if a symmetric second-rank tensor $\mathbf{H}$ is a function of two independent symmetric second-rank tensors $\mathbf{T}$ and $\mathbf{P}$ then

$$\mathbf{H} = k_0\mathbf{g} + k_1\mathbf{T} + k_2\mathbf{P} + k_3\mathbf{T}^2 + k_4(\mathbf{T}\cdot\mathbf{P} + \mathbf{P}\cdot\mathbf{T}) + k_5(\mathbf{T}^2\cdot\mathbf{P} + \mathbf{P}\cdot\mathbf{T}^2)$$

where k_0,...k_5 are scalar functions of invariants of the tensors $\mathbf{T}$ and $\mathbf{P}$.

It follows from the definition of a tensor function that the symmetry group of fixed values of the arguments of a tensor function is also the symmetry group of its corresponding value. This property allows the determination of the general forms of tensor functions with accuracy up to scalar coefficients without direct using of the relation (5.2). The problem of determining the general form of a tensor function $\mathbf{T} = \mathbf{F}(\mathbf{T}_1, \ldots, \mathbf{T}_N)$ (let the rank of the tensor $\mathbf{T}$ be r) can be solved in the following way: a) fix the arguments of the function $\mathbf{F}$ in the case of general situation, and, with transformation of the basis e_i, reduce them to the simplest possible form (e.g., two components of one of the vectors can be made zero, or, a second-rank symmetric tensor can be reduced to the diagonal form); b) determine the symmetry group G of the arguments; c) with the use of the relations (5.1) determine all r-rank tensors that are invariant relative to G, choose among them a set of linearly independent ones (the basis tensors), and express them in terms of the tensors $\mathbf{T}_1, \ldots, \mathbf{T}_N$; d) $\mathbf{T}$ is a linear combination of the basis tensors with coefficients that are scalar functions of invariants of tensors $\mathbf{T}_1, \ldots, \mathbf{T}_N$.

Analytic tensor functions $\mathbf{T} = \mathbf{F}(\mathbf{S})$, where $\mathbf{T}$ and $\mathbf{S}$ are second-rank tensors, are defined by the power series of the form

$$\mathbf{T} = a_0\mathbf{g} + a_1\mathbf{S} + a_2\mathbf{S}^2 + \ldots$$

which is equivalent to the series for the tensor components

$$T^i_j = F^i_j(S^k_l) = a_0\delta^i_j + a_1 S^i_j + a_2 S^i_k S^k_j + \ldots .$$

Application of the Hamilton-Cayley theorem stating that any square matrix is a root of its own characteristic polynomial, i.e.,

$$\mathbf{S}^3 - I_1\,\mathbf{S}^2 + I_2\mathbf{S} - I_3\mathbf{g} = 0$$

obviates the use of powers of the matrix (S^i_j) of orders more than two.

Let us extend the definition of a symmetry group of tensors at a point up to the definition of a symmetry group of tensor fields. Let $\mathbf{T}_1(x^i), \ldots, \mathbf{T}_N(x^i)$, be a set of tensor fields among which there is the metric tensor field **g**. The symmetry group G of this set of tensor fields is the set of transformations of Cartesian coordinate systems (x^i) preserving the form of each of these tensor fields. For example, for the contravariant components of a tensor field of a rank r regarded as functions of the variables x^i,

$$T'^{i_1 \ldots i_r}(x'^l)\, e'_{i_1} \ldots e'_{i_r} = T^{i_1 \ldots i_r}(x'^l)\, e'_{i_1} \ldots e'_{i_r} ,$$

or

$$b^{i_1}{}_{j_1} \ldots b^{i_r}{}_{j_r}\, T^{j_1 \ldots j_r}(a^l{}_i(x'^i - c^i)) = T^{i_1 \ldots i_r}(x'^i)$$

on the transformations $x'^i = b^i{}_j x^j + c^i$ where $(a^i{}_j)$, $(b^j{}_k)$ are orthogonal matrices each of which is the inverse of the other. The tensor field is said to be invariant relative to the group G. The symmetry group of the metric tensor field **g** is the complete group of motions of the Euclidian space. The tensor field invariant relative to the group of translations of the coordinate origin $x'^i = x^i + c^i$ is referred to as uniform. Let us indicate the method of construction of tensor fields invariant with respect to a given group G of the transformations $x'^i = b^i{}_j x^j$ preserving the position of the coordinate origin O. That method is based on the definition of the tensor function. Let G be a symmetry group of a set of tensors $\mathbf{T}_1, \ldots, \mathbf{T}_N$ given at a point O, and $\boldsymbol{r}$ be the radius-vector of an arbitrary point P relative to O. Compose, for the point O, the tensor function $\mathbf{T} = \mathbf{F}(\boldsymbol{r}, \mathbf{T}_1, \ldots, \mathbf{T}_N)$ whose components satisfy obviously all the necessary conditions of symmetry as functions of the components of the radius-vector $\boldsymbol{r}$. The remaining step is to translate (without variation of the components) the tensor $\mathbf{T}$ to the point P.

— PROBLEMS —

⊙ **5.1** Find the eigenvalues of the matrix of rotation by the angle φ around the axis x^3

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

⊙ **5.2** Determine the general form of a matrix of rotation around a unit vector $\boldsymbol{n}$ by an angle φ.

⊖ **5.3** Show that any orthogonal three-by-three matrix has at least one eigenvalue equal to 1 or -1. To what transformation does the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

correspond ?

⊖ **5.4** Show that an arbitrary orthogonal matrix can be represented either as a matrix of rotation around an axis (Problem 5.2) or as the product of a matrix of rotation and a matrix of reflection in the plane perpendicular to the same axis (Problem 5.3)

⊖ **5.5** Find the eigenvalues of an antisymmetric second rank tensor. Compare with the eigenvalues of the orthogonal matrix of Problem 5.1.

⊙ **5.6** Using the expansions of the functions of one variable $F(x)$ in power series over x, where 1) $F(x) = e^x$, 2) $F(x) = \ln(1+x)$, 3) $F(x) = (1-x)^{-1}$, define the corresponding analytic tensor functions $\mathbf{F}(\mathbf{S})$ replacing x by $\mathbf{S}$.

⊖ **5.7** Determine the relationship between the eigenvalues and eigenvectors of a tensor $\mathbf{S}$ and those of $\mathbf{T} = \mathbf{F}(\mathbf{S})$ if $\mathbf{F}(\mathbf{S})$ is an analytic tensor function.

⊖ **5.8** Show that $I_3(e^{\mathbf{S}}) = e^{I_1(\mathbf{S})}$ where $I_3(\mathbf{T}) = \det \|T^i{}_j\|$, $I_1(\mathbf{T}) = T^i{}_i$.

⊖ **5.9** Show that, if $\mathbf{S}$ is an antisymmetric matrix ($S_i{}^j g^{ik} = -S_i{}^k g^{ij}$), then $e^{\mathbf{S}}$ is orthogonal. Find such a matrix $\mathbf{S}$ so that the matrix $e^{\mathbf{S}}$ is the matrix of rotation of Problem 5.2.

⊙ **5.10** Show that, if a tensor $\mathbf{S}$ is symmetric then any analytic tensor function of $\mathbf{S}$ is also symmetric.

⊖ **5.11** Let all the eigenvalues λ_i $(i = 1,2,3)$ of a tensor $\mathbf{S}$ be real and different. Show that, for any analytic tensor function $\mathbf{F}(\mathbf{S})$, the formula (called the Lagrange interpolation polynomial)

$$\mathbf{F}(\mathbf{S}) = \frac{(\mathbf{S}-\lambda_2\mathbf{g})\cdot(\mathbf{S}-\lambda_3\mathbf{g})}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}F(\lambda_1)$$

$$+\frac{(\mathbf{S}-\lambda_3\mathbf{g})\cdot(\mathbf{S}-\lambda_1\mathbf{g})}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}F(\lambda_2) + \frac{(\mathbf{S}-\lambda_1\mathbf{g})\cdot(\mathbf{S}-\lambda_2\mathbf{g})}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}F(\lambda_3)$$

is valid; here, $\mathbf{g}$ is the metric tensor.

⊖ **5.12** Analyze, for the Problem 5.11, the cases of repeated eigenvalues of the tensor $\mathbf{S}$ if there are three linearly independent eigenvectors.

⊙ **5.13** Determine the symmetry group of the set of tensors ($\mathbf{g}$, $\boldsymbol{\epsilon}$) where $\mathbf{g}$ is the metric tensor, $\boldsymbol{\epsilon}$ is Levi-Civita tensor.

⊖ **5.14** Determine, for an orthonormal basis $\mathbf{e}_i$, the symmetry groups of the following sets of tensors: a) $\mathbf{g}$, $\mathbf{e}_3$; b) $\mathbf{g}$, $\boldsymbol{\epsilon}\cdot\mathbf{e}_3$; c) $\mathbf{g}$, $\mathbf{e}_3\mathbf{e}_3$ (transverse isotropy with the axes of symmetry parallel to $\mathbf{e}_3$); d) $\mathbf{g}$, $\boldsymbol{\epsilon}$, $\mathbf{e}_3$; e) $\mathbf{g}$, $\boldsymbol{\epsilon}$, $\mathbf{e}_3\mathbf{e}_3$; here, $\mathbf{g}$ is the metric tensor, $\boldsymbol{\epsilon}$ is Levi-Civita tensor.

⊖ **5.15** Determine the symmetry group of the tensors **g**, **S** where **S** is a symmetric second-rank tensor of the general form, **g** is the metric tensor (orthotropy with the planes of symmetry determined by principal axes of the tensor **S**). Consider the case of coinciding eigenvalues of the tensor **S**.

⊖ **5.16** a) Find all the second-rank tensors invariant relative to the symmetry group of the tensor **g**. b) It is known that there are three linearly independent fourth-rank tensors invariant relative to the complete group of rotations. Compose their components of the components of the tensor **g**.

⊖ **5.17** For the setting of Problem 5.14 find all the symmetric second-rank tensors invariant relative to the corresponding symmetry groups. For what groups do there exist invariant antisymmetric second-rank tensors?

⊖ **5.18** Find all the symmetric second-rank tensors which are invariant relative to the symmetry group of the tensors **g**, **S** (Problem 5.15), and compose them of the tensors **g**, **S**.

⊖ **5.19** Consider the matrix which is inverse to a nonsingular matrix of the covariant components of a second-rank tensor. Show that it is the matrix of the contravariant components of a certain new tensor, i.e., it defines a tensor function.

⊕ **5.20** It is known that there exist ten linearly independent fourth-rank tensors invariant relative to the group of transverse isotropy (Problem 5.14c). Compose their components using the components of the tensors **g** and e_3e_3.

⊙ **5.21** Determine the general form of the tensor function a) $\boldsymbol{a} = \mathbf{F}(\boldsymbol{b}, \mathbf{g})$ where $\boldsymbol{a}$, $\boldsymbol{b}$ are vectors, **g** is the metric tensor; b) $\boldsymbol{c} = \mathbf{F}(\boldsymbol{a}, \boldsymbol{b}, \mathbf{g})$ where $\boldsymbol{c}$ is also a vector.

⊖ **5.22** Prove that the tensor function of Problem 5.21a has a scalar potential $\Phi(\boldsymbol{b}, \mathbf{g})$ such that $a_i = \dfrac{\partial \Phi}{\partial b^i}$.

⊖ **5.23** Let a tensor function $\mathbf{F}(\mathbf{S}, \mathbf{g})$, where **S** is a symmetric second-rank tensor, **g** is the metric tensor regarded as constant, have the form

$$\mathbf{F} = (k_1 g_{ij} + k_2 S_{ij} + k_3 g^{kl} S_{ik} S_{jl}) e^i e^j$$

where the coefficients k_1, k_2, k_3 are the functions of the invariants of the tensor **S**. Show that the necessary conditions for existence of a scalar potential Φ of the function **F**, $F_{ij} = \dfrac{\partial \Phi}{\partial S^{ij}}$, can be represented in the form

$$\frac{1}{\alpha} \frac{\partial k_\alpha}{\partial J_\beta} - \frac{1}{\beta} \frac{\partial k_\beta}{\partial J_\alpha} = 0 \,, \quad \alpha, \beta = 1, 2, 3$$

where $J_1 = S^{i}_{\cdot i}$, $J_2 = S^{ij} S_{ij}$, $J_3 = S^{i}_{\cdot j} S^{j}_{\cdot k} S^{k}_{\cdot i}$.

⊖ **5.24** Using the interpolation polynomial of Lagrange (Problem 5.11), determine the form of the tensor functions $e^{\mathbf{S}}$, sin **S**, $\mathbf{S}^3$. Compare with the expansions in the power series. Determine the form of the scalar coefficients of the representations $a\mathbf{g} + b\mathbf{S} + c\mathbf{S}^2$.

⊖ **5.25** Calculate the tensor of moments of inertia relative to the center of mass a) for a homogeneous solid sphere, using the properties of symmetry; b) for a homogeneous ellipsoid, using the principal axes.

⊕ **5.26** Show that the tensor of moments of inertia relative to the center of mass of a homogeneous regular tetrahedron is spherical (proportional to the metric tensor **g**). Show the same for a cube and a regular octahedron. Use the properties of symmetry.

⊖ **5.27** Determine the general form of scalar, vector and second rank tensor fields invariant relative to the symmetry group of Problem 5.14d. Such fields are referred to as axially symmetrical. How do the results change for the other groups of Problem 5.14 ? Write the results in the cylindrical coordinate system.

⊖ **5.28** Determine the general form of scalar, vector and second-rank tensor fields which are invariant relative to the complete group of rotations and reflections. Such fields are referred to as spherically symmetrical. Write the results in a spherical coordinate system.