INTRODUCTION

After the fundamental work of Gardner, Green, Kruskal and Miura [GGKM] on the integrability of the *Korteweg-de Vries equation* (KdV) $u_t = 6uu_x - u_{xxx}$, a whole series of two-dimensional nonlinear KdV-like differential equations have been discovered. All these equations are integrated by the inverse scattering technique, possess infinitely many local conservation laws, parametric Bäcklund transformations, and solutions in terms of theta functions of algebraic curves as well as KdV does. The theory of these equations was named the soliton theory.

The main tool in the theory of the KdV equation is the *Lax pair* [Lax1], a representation of KdV in the form of the commutation relation $L_t = [A, L]$, where

$$L = -\frac{\partial^2}{\partial x^2} + u, \qquad A = -4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u_x.$$

The integrability and other remarkable properties of KdV result from this interpretation.

Lax pairs became rather popular. However, it was quickly realized that for many equations and for quite a few problems they must be modified. Zakharov, Shabat for the *nonlinear Schrödinger equation*, Ablowitz, Kaup, Newell and Segur for the *Sin-Gordon equation*, and Novikov for KdV suggested the so-called *zero curvature representations* which appeared to be very important for the systematic development of the theory. The new idea was to write differential equations as the relation

$$M_t - N_x = [N, M]$$

where M, N are matrix functions depending rationally on the auxiliary parameter. For example, the nonlinear Schrödinger equation (NS) $ir_t = r_{xx} + 2r^2\bar{r}$ can be written in this form for

$$M = \begin{pmatrix} -i\lambda & \bar{r} \\ r & i\lambda \end{pmatrix}, \qquad N = 2\lambda M + i\begin{pmatrix} r\bar{r} & -\bar{r}_x \\ -r_x & -r\bar{r} \end{pmatrix}.$$

The equations KdV, NS, and Sin-Gordon are still the most famous two-dimensional soliton equations, with numerous applications in both physics and mathematics. For instance, the NS equation describes the evolution of the complex envelope of ultra-short impulses in one-mode optical fiber. The stability of one-soliton impulses (which have constant velocity and form) in optical fibers is 4-5 orders greater than the stability of the "solitary waves", observed by J.Scott-Russel in a narrow canal in 1834. An impulse of length $\sim 10^{-12}$ sec can survive in an optical fiber over 100km.

An outline of the main directions. The paper [GGKM] and the work by Lax [Lax1] were the first. Then the paper [ZF] by Zakharov and Faddeev and the results by Gardner gave rise to the Hamiltonian approach in the theory of solitons and were developed algebraically in the works by Gel'fand and Dickey. Novikov [N1] and Lax [Lax2] introduced the finite zoned solutions of the KdV equation, which was a starting point of the algebraic-geometric direction.

The next achievement was a unification of all these results by means of the zero curvature representations and a generalization to the multi-component (matrix) soliton equations. Paper [ZMa1] about the N-*wave equation* and then the papers by Pohlmeyer [Poh] and Zakharov, Mikhailov [ZMi2] (in which zero curvature representations for the chiral fields were found) became important steps. The Riemann-Hilbert problem based on the theory by M. Krein was adapted to the soliton theory by Shabat. It eventually replaced the Gel'fand-Levitan-Marchenko equations (which were used to integrate KdV) in the multi-component case.

The equations of N-waves, of the *principal chiral fields* (PCF), and the *generalized Heisenberg magnet* (GHM) appeared to be the most universal soliton multi-component equations with zero curvature representations. Many others (including KdV, NS and the Sin-Gordon equation) are their reductions.

A qualitatively new type of zero curvature representations with an elliptic dependence on the spectral parameter was proposed by Borovik [Bor] and Sklyanin [Sk] for the *Landau-Lifshitz equation* of magnetism. This equation is an asymmetric variant of the equation of the Heisenberg magnet. Analogous representation of an asymmetric analogue of the PCF equation (in 2 × 2 matrices) was found in paper [Ch12]. The problem was to prove that the elliptic case is the most general and somehow list all soliton equations.

The quantum inverse scattering method originated by Faddeev and others gave birth to the *classical* r-*matrices*, which in their turn changed the classical theory a lot. Applied for the first time for the description of the Poisson brackets of integrable equations, r-matrices also appeared to be directly related to the zero curvature representations.

The problem of the classification of fundamental types of hamiltonian soliton equations was successfully formulated with the help of r-matrices and then partially solved in the paper [BD]. The zero curvature representations were subjected to the analogous algebraization in [Ch13]. It was proved that under natural assumptions the genus of the algebraic curve on which an r-matrix is defined does not exceed one. Thus the elliptic representations are indeed the most universal in the soliton theory.

Here we remark that considering more complicated coset spaces of loop groups, one can get certain zero curvature representations over an arbitrary algebraic curve. However the corresponding differential equations have essentially more complicated nature and no one has succeeded in describing and studying them so far.

Another important notion which allowed to systematize the soliton theory and involve the representation theory was the so-called *τ-function*, introduced by Hirota and Sato and then developed in the papers by Date, Jimbo, Kashiwara, Miwa and other (mainly Japanese) researchers. First elaborated for the *Kadomtsev-Petviashvili equation*, the technique of the τ-functions was transfered to many soliton equations (including matrix ones). In particular, invariant methods of obtaining formulas of determinant type for N-soliton solutions and the so called Hirota identities (closely related to the local conservation laws) were suggested. We note also the applications of τ-functions to algebraic-geometric solutions and discrete equations.

Nowadays, a large part of the mathematical theory of solitons can be included in the theory of infinite dimensional groups and their representations. What are these groups? The *loop groups* over a projective complex line and other one-dimensional complex or real curves should be mentioned first. They are closely connected with the others that are the corresponding diffeomorphism groups and the groups of matrices of infinite order.

The process of development and algebraization of the theory of solitons turned out to be and still remains fruitful not only for mathematical physics but also for modern mathematics. The theory has gone far beyond the initial range of ideas. At the present time, the quantum theory with its original methods and impressive applications plays a proponderant role. On the other hand, the possibilities of the development of the classical theory of solitons are far from being exhausted. The aim of this book is to demonstrate it.

0.1. Plan of this book.

We are trying to unify and present systematically the soliton theory on the basis of a limited number of algebraic notions. The first step in this direction was made in the work by Manin [Mani1]. Since then almost all tools of the soliton theory have radically advanced. Describing this development is another aim of this book.

Among the books written before, this one is close mostly to the books [ZMa2] and [TF2]. We would like to mention [PS] as well. Supplementing the theory presented in the monograph of Takhtajan and Faddeev, we study mainly the algebraic aspects and the matrix theory(GL_n and other Lie groups). The book can be read irrespective of whether one has read these or other books and papers, though a certain preliminary acquaintance with the soliton theory would be desirable. The readers can find proper recommendations in the introductions to each paragraph.

We give complete proofs for all statements apart from those included as exercises. The exercises which require more complicated considerations (additional research) are marked with an asterisk. In order to get familiar with the material of a chapter or a paragraph for the first time, it is recommended to read the corresponding introduction, successively looking at the theorems and corollaries (containing the main statements). Perhaps after that it is worth turning to the commentaries placed at the ends of paragraphs. Almost all paragraphs are independent units supplied with introductions and the necessary comments. Results of other paragraphs and general mathematical information are reminded.

Though a large part of this book is understandable to mathematics and physics students, it is not a textbook in a regular sense of the word. As already mentioned our main purpose is to expose modern methods of the soliton theory in the form convenient for independent study and reading special literature. We tried to compensate certain (almost inevitable) concentration, dividing the material into relatively small logically self-contained sections("solitons"). We hope that the brevity will not be an obstruction and, perhaps, make it easier to use this book as a guide in the soliton theory and its recent applications.

Issues that have been left out. In this book general mathematical theory is investigated. As for applications in theoretical physics, it is worth noting that the soliton theory plays an important role in the study of "one-dimensional" classical and quantum physical models. Multi-dimensional counterparts of the soliton techniques are also remarkably intresting. The soliton theory (especially the inverse problem technique) penetrates deeper and deeper into the numerical methods for integrable equations.

The concrete equations considered in this book are mainly selected for the following reason. The general theory of the zero curvature representations is needed for them with relatively simple modifications. Of course, there are still quite a few equations interesting from various viewpoints which are not even mentioned in this book. Some are barely discussed. The general constructions of this book can be adapted to many concrete equations. The same is true for the proofs of the basic statements, which are almost always simpler for general equations than for their reductions.

We note that the references to literature (and also the comments at the end of each paragraph) are somewhat fragmentary and, as a rule, are directly related to the questions under consideration. We did not try to give the complete list of origins. Concerning the history of the soliton theory and the relations with classical results, one can find much in the published books and survey works.

The continuation of this book [Ch11] is devoted to the invariant theory of soliton equations based on the loop groups and containes quite a few additional issues. The r-matrix technique and the general theory of τ-functions are among them. However many questions are not touched upon or hardly illustrated. Here we name some of them.

Quantum theory. Mathematical aspects of the quantum inverse problem method [F4], including the applications to the theory of Lie groups and the representation theory (see, for example, [Dr1]). Certain related questions are considered in [Ch11].

The *theory of τ functions* and the corresponding part of the representation theory of infinite dimensional groups. Recent applications of τ-functions and integrable equations to the conformal field theory, the 2D-gravity and other problems of modern mathematical physics.

Differentially-geometrical methods. We mean the results by Estabrook, Wahlquist, Lund, Regge and others (cf. comments to §1 Ch.II), their classic origins, and the continuation. The works by Griffiths on the general theory of differential forms and the results by Verdier on the chiral fields are related to this direction. We also mention an approach to the Riemann factorization problem based on the micro-differential operators (Mulase).

Besides this, there is the vast theory of *discrete equations* and *lattice models* started by Toda, Calogero, Moser, Olshanetsky, Perelomov, Kostant, Kuperschmidt and others. It is based on the group-theoretical and geometrical methods, the τ-function technique (cf., for example, [DJM] and [UT]), algebraic-geometric approach, and the hamiltonian r-matrix methods (see, [TF2]). We discuss some of them here and in [Ch11]. Of course, discrete variants of the inverse scattering problem (Ablowitz, Ladik, Flashka, MacLaughlin, Manakov and others) should not be forgotten.

In many publications multi-dimensional, integral and supersymmetric generalization of two-dimensional soliton equations are considered. The examples are the Kadomtsev-Petviashvili equation, the *duality equation* (cf. Ch.I §4.4) and some special cases of the *Einstein equation* (Belinskii, Zakharov, Neugebauer, Kramer, Nakamura, Takasaki, Wu and others). As to the integral soliton equations like the *Benny* or *Benjamin-Ono equations*, we mention the papers by Miura, Manin, Kuperschmidt, Zakharov, Case, Satsuma, Ablowitz, Kodama, Lebedev and Radul.

Concerning the algebraic theory of the soliton equations, this book and [Ch11] do not fully reflect today's situation. Say, one of the main applications in algebraic geometry will be not discussed at all, that is the proof of the so-called Novikov conjecture on the Schottky problem (characterization of the Jacobian varieties among Abelian varieties) by Mulase and Shiota. There are also deep relations with the topological field theories, and the $2D$-gravity.

Nevertheless, we hope that the present book (and its continuation [Ch11]), the

τ-function techniques from the papers by Date, Jimbo, Kashiwara, Miwa "Transformation groups of soliton equations" (see also the last chapter from [Kac]), the article [DrS1], and recent results on Virasoro and W-algebras in connection with KdV- like equations will give rather a well-balanced picture of algebraic methods of the (classical) soliton theory.

0.2. Chiral fields and Sin-Gordon equation.

One of the simplest ways to get an interesting differential equation is to impose constraints on a free particle. In the classical field theory, the analogous way is in restricting the values of free fields (vector functions on the space-time) to certain closed submanifols. Symmetries (automorphisms) of the latter induce the symmetries of corresponding equations which are called the *chiral symmetries* in physics papers. Quantities invariant relative to chiral symmetries are sometimes called simply the *invariants*. The equations of motion (the Euler-Lagrange equations) can be obtained by the method of Lagrange multipliers in both mechanics and the field theory.

By *principal chiral fields* (cf., for example, [STSF]) we mean free matrix fields with values in the manifold of invertible matrices. The chiral symmetries of such fields are generated by the action of the linear group GL_n on the left and on the right. In order to derive the equation of motion it is necessary to overcome the following difficulty: invertible matrices form an open (not closed) submanifold. For this purpose, there is a natural "algebraic-geometrical" recipe. One should consider free bi-matrix fields (g, f) (i.e., pairs of matrix valued functions g, f on the space-time), restrict the range to the closed subvariety $gf = I$, write down the corresponding equation of motion, and project it onto the first component g.

Let us denote the light cone coordinates in the two-dimensional space-time by x, t : $x = (x^0 + x^1)/2$, $t = (x^0 - x^1)/2$ (the letters ξ, η are in common use instead of x, t in physics). Choose the Lagrangian density (the density of the Lagrangian) of the free field (g, f) as

$$-\frac{1}{4}\operatorname{Sp}(g_x f_t + f_x g_t) = -\frac{1}{2}\operatorname{Sp}(g_{x^0} f_{x^0} + g_{x^1} f_{x^1}),$$

where Sp is the usual matrix trace, $g_x = \partial g(x,t)/\partial x$, $g_t = \partial g(x,t)/\partial t$ and so on. Then the Lagrangian density with the constraint $gf = \mathbf{I}$ is

$$L = -\frac{1}{4}\operatorname{Sp}(g_x f_t + f_x g_t + \Lambda(gf - \mathbf{I}))$$

for the matrix Lagrangian multiplier Λ. Calculating the variation $\delta \int L\, dx\, dt$ and setting the terms of δg, δf and $\delta\Lambda$ equal to zero, we obtain the following equations:

$$-2f_{xt} + \Lambda f = 0 = -2g_{xt} + g\lambda,$$
$$fg = \mathbf{I}.$$

Eliminating Λ and f, we come to the *principal chiral field equation* (PCF):

$$2g_{xt} = g_x g^{-1} g_t + g_t g^{-1} g_x. \tag{0.1}$$

This equation is consistent with the restriction of the values of g to an arbitrary Lie subgroup $G \subset GL_n$. To be more exact, if the initial data of the Cauchy problem of (0.1) take the values in G, then the same is true for the whole solution. In this case we call it a *G-field*. Let us suppose, for example, that $G = \{g \in GL_n,\ gg^* = \mathbf{I}\}$ where $*$ is an *anti-involution* of complex matrices, e.g., the Hermitian conjugation $\dagger$. Then the equation (0.1) is consistent with the restriction $g(x,t) \in G$. Indeed, given a field (solution) g, the function (field) $(g^*)^{-1}$ is a solution of (0.1) as well and we may apply the uniqueness theorem for differential equations. For an arbitrary group G, let us rewrite (0.1) in another form to make the consistency absolutely clear.

Set $U = g_x g^{-1}$, $V = g_t g^{-1}$. These functions are invariant with respect to the right multiplication of g by constant matrices. One has:

$$L = -\tfrac{1}{2}\,\mathrm{Sp}(g_x(g^{-1})_t) = \tfrac{1}{2}\,\mathrm{Sp}(UV).$$

It is natural to call U and V the *left currents* of g (with respect to the cone variables). Equation (0.1) results in the following *PCF system* for $U(x,t)$ and $V(x,t)$:

$$U_t + V_x = 0, \tag{0.2a}$$

$$U_t - V_x = [V, U], \tag{0.2b}$$

where the second equation is derived directly from the definition of U and V while the first is equivalent to (0.1). The field g can be uniquely recovered from the currents U, V in the connected domain of the plane $\{(x,t)\}$, if g is known at one (arbitrary) point (x', t').

If $g(x,t) \in G$, then U and V take their values in the Lie algebra $\mathfrak{g} = \mathrm{Lie}\,G$. The system (0.2) is compatible with the restriction to $\mathfrak{g}$, since it can be expressed in terms of the generators of $\mathfrak{g}$. Hence, $g(x', t') \in G \Rightarrow g(x,t) \in G$ for all x, t. In particular, $g^* g = \mathbf{I} \Rightarrow U^* + U = 0 = V^* + V$, i.e., the $*$-unitarity of g gives that the functions U and V are $*$-anti-hermitian.

The equations (0.2) can be interpreted as the compatibility condition of the system of equations depending on an auxiliary parameter λ:

$$\Phi_x - \frac{1}{1-\lambda} U\Phi = 0, \tag{0.3a}$$

$$\Phi_t - \frac{1}{1+\lambda} V\Phi = 0 \tag{0.3b}$$

for an invertible matrix valued function $\Phi(x,t;\lambda)$, or as the commutativity condition:

$$\left[\frac{\partial}{\partial x}-\frac{1}{1-\lambda}U,\ \frac{\partial}{\partial t}-\frac{1}{1+\lambda}V\right]=0. \tag{0.4}$$

Equations (0.3) and (0.4) are called the *zero curvature representation for PCF.* In order to check the equivalence of (0.4) and (0.2) it is enough to expand the commutator and set the coefficients of $\lambda(1-\lambda)^{-1}(1+\lambda)^{-1}$ and $(1-\lambda)^{-1}(1+\lambda)^{-1}$ equal to zero. Note that the field g coincides with the value of Φ at $\lambda=0$ up to a right constant factor.

Chiral fields on spheres (σ-model). Fields with values on the $n-1$ dimensional sphere, in short, S^{n-1}*-fields* are special cases of the PCF. They are defined as free fields $q(x,t)\in\mathbb{R}^n$ with values in $S^{n-1}=\{q\in\mathbb{R}^n,(q,q)=1\}$, where (q,q) is the Euclidean scalar product. Choosing the Lagrangian density of the free $\mathbb{R}^n$ field proportional to (q_x,q_t), we obtain the equation of the S^{n-1} field:

$$q_{xt}+(q_x,q_t)q=0,\qquad (q,q)=1, \tag{0.5}$$

by means of the Lagrange multiplier method. It is also called the σ-model equation or the n-field equation (cf., for example, [Poh]). This equation is invariant under constant orthogonal transformations of q.

The equation (0.5) can be embedded in (0.1) in the following way (cf.[ZMi2]). Let us impose the condition $g^2=\mathrm{I}$ on an O_n-field g. It is possible because g and g^{-1} satisfy (0.1) simultaneously. Then g can be put in the form $g=\mathrm{I}-2P$, where $P^2=P$. It follows from the orthogonality of g that the function $P(x,t)$ is an orthogonal projection on a subspace (which depends on x and t). Its dimension is a discrete invariant. When this dimension is one, we may set $Pz=(z,q)q$ for $z\in\mathbb{R}^n$ and suitable $q(x,t)\in S^{n-1}$. It is easy to see by a direct calculation that q satisfies (0.5). We will use U and V to make it absolutely clear.

Let us denote by $p\wedge q$ the skew symmetric matrix which acts on z as:

$$(p\wedge q)z=(z,p)q-(z,q)p,$$

where $p,q,z\in\mathbb{R}^n$. Then

$$\begin{aligned} U&=g_x g^{-1}=-2P_x(1-2P)=-2q_x\wedge q,\\ V&=g_t g^{-1}=2q\wedge q_t \end{aligned}$$

(we made use of the trivial relations $(q,q_x)=0=(q,q_t)$). Applying the formulas

$$\begin{aligned} (p\wedge q)&=-q\wedge p,\\ (p\wedge q)_x&=p_x\wedge q+p\wedge q_x,\\ (p\wedge q)_t&=p_t\wedge q+p\wedge q_t, \end{aligned}$$

we see that (0.2a) is equivalent to the equation

$$q_{xt} \wedge q = 0.$$

Thus the vector function q_{xt} is proportional to q. The coefficient of proportionality can be found easily and has the form exactly as in (0.5) (multiply the equation $q_{xt} = c(x,t)q$ by $(\cdot, q)$ and transform the relation $c = (q_{xt}, q)$ using the constraint $(q,q) = 1$).

From S^2-fields to Sin-Gordon equation. Solutions of (0.5) (or (0.1)) go again to solutions under the change of variables $(dx, dt) \mapsto (h\,dx, k\,dt)$, where functions $h, k > 0$ depend only on x and on t respectively. Hence, taking the conservation laws $(q_x, q_x)_t = 0 = (q_t, q_t)_x$ into account, we can normalize $q(x,t)$ by the condition $(q_x, q_x) = 1 = (q_t, q_t)$ without losing much generality. These fields q are called *S^{n-1}-fields in normalized coordinates.*

Now set $n = 3$. Then the relations

$$(q,q) = 1 = (q_x, q_x) = (q_t, q_t) \tag{0.6}$$

are sufficient to give

$$q_{xt} + (q_x, q_t)q = 0,$$

if we discard the trivial case when q_x and q_t are proportional. Really, q, q_x and q_t form a basis in $\mathbb{R}^3$. Hence the vector function q_{xt} is to be proportional to q, because it is orthogonal to q_x and q_t (it follows from (0.6) that $(q, q_x) = 0 = (q, q_t)$, $(q_{xt}, q_x) = 0 = (q_{xt}, q_t)$).

If we regard q_x and q_t as unit vector fields (at the point $q(x,t)$) on S^2, then $[q_x, q_t] = 0$. Conversely, the commutativity of unit vector fields on S^2 means that they have the form q_x, q_t for a proper q. The coordinates (x,t) are transformed by the map q into a coordinate net on S^2 with the following defining property. The opposite sides of any coordinate rectangle have the same length. Such nets are due to Tchebychef, who established their connection with the Sin-Gordon equation. Let us explain his statement.

Set $(q_x, q_t) = \cos\alpha$ (i.e., denote the *net angle* corresponding to $q(x,t)$ by α). The most direct way of getting the equation for α is to express all the derivatives of q in terms of the basis $\{q, q_x, q_t\}$. We have:

$$q_{xx} = -q - \frac{\alpha_x}{\sin\alpha} q_t + \alpha_x \cot\alpha\, q_x, \tag{0.7a}$$

$$q_{tt} = -q - \frac{\alpha_t}{\sin\alpha} q_x + \alpha_t \cot\alpha\, q_t. \tag{0.7b}$$

Thus, $(q_{xx}, q_{tt}) = 1 - \alpha_x\alpha_t \cos\alpha$. On the other hand, we obtain, using (0.5):

$$\begin{aligned}(q_{xx}, q_{tt}) &= (q_x, q_t)_{xt} - (q_{xxt}, q_t) = \\ &= 1 - \sin^2\alpha - \alpha_x\alpha_t\cos\alpha - \alpha_{xt}\sin\alpha.\end{aligned}$$

Comparing these equations, we arrive at the *Sin-Gordon equation*

$$\alpha_{xt} + \sin\alpha = 0. \tag{0.8}$$

0.3. Generalized Heisenberg magnet and VNS equation.

In the equation of the principal chiral fields (0.1) the variables x and t apper on equal footing. It is also related to system (0.2). The function V_t cannot be expressed without integration in terms of U, V and their derivatives by x. But in mechanics and the field theory the so-called evolution equations play an important role. They are solved with respect to the derivatives by the time variable t, in constrast to (0.1), (0.2). Among evolution equations,the Hamiltonian equations are especially interesting which are constructed by a Poisson bracket (symplectic structure) on the phase space and by a Hamiltonian, that is a function or a functional on this space.

As the phase space, let us take the space of matrix-valued functions $U(x) = (u_p^q(x))$ of the variable $x \in \mathbb{R}$ ($1 \leqq p, q \leqq n$ are the indices of rows and columns). Set

$$\{u_p^q(x), u_r^s(y)\} = (u_p^q(x)\delta_r^q - u_r^q(y)\delta_p^s)\delta(x-y),$$

where δ_p^q is the Kronecker symbol and δ is Dirac's delta function. Informally speaking, $u_p^q(x)$ corresponds to the matrix $I_p^q = (\delta_i^p\delta_q^j)$ (i, j are the indices of rows and columns), "located" at the point x. The Poisson brackets among $u_p^q(x)$'s are generated by the corresponding commutators of the matrices I_p^q at "coinciding" points x, y and are trivial for $x \neq y$.

The simplest interesting Hamiltonian is

$$H = \frac{1}{2}\int \mathrm{Sp}(U'(x)\, {}^tU'(x))\, dx,$$

where ^{t} is the transposition and, temporarily, the notation $U' \overset{\text{def}}{=} U_x = dU/dx$ is used. We will calculate the corresponding equation of motion

$$U_t(x) = \{H, U(x)\},$$

which defines the evolution of U with respect to t. Since

$$\{(u_p^q)'(x), u_r^s(y)\} = \delta'(x-y)(u_p^s(y)\delta_r^q - u_r^p(y)\delta_p^s),$$

one has:

$$\begin{aligned}\{H, u^s_r(y)\} &= \frac{1}{2}\sum_{p,q=1}^{n}\int dx\{(u^p_q)'(x)(u^p_q)'(x), u^s_r(y)\} = \\ &= \sum_{p,q=1}^{n}\int dx\delta'(x-y)(u^s_p(y)\delta^q_r - u^q_r(y)\delta^s_p)(u^p_q)'(x) = \\ &= -\sum_{p,q=1}^{n}\int dx\delta(x-y)(u^s_p(y)\delta^q_r - u^q_r(y)\delta^s_p)(u^p_q)''(x) = \\ &= \sum_{q=1}^{n} u^q_r(y)(u^s_q)''(y) - (u^q_r)''(y)u^s_q(y).\end{aligned}$$

Thus,

$$U_t = [U, U_{xx}]. \tag{0.9}$$

The Poisson bracket constructed above is degenerate. Its kernel (center, commutant) contains the elements of the center of the universal enveloping algebra of $\mathfrak{gl}_n$ (the Casimir operators), where $u^q_p(x)$ are substituted for I^q_p. In particular, functionals $\mathrm{Sp}(U^m)$ lie in the center of the bracket and therefore do not depend on t (i.e., are integrals of (0.9)), where $m = 1, 2, \ldots$. This allows us to fix the conjugacy class of the matrix U. Let

$$U = F_0 U_0 F_0^{-1}, \qquad U_0 = c_1 \sum_{i=1}^{p} I^i_i + c_2 \sum_{i=p+1}^{n} I^i_i, \tag{0.10}$$

where $1 \leqq p < n$, F_0 is a certain matrix (depending on x and t) and $c_1, c_2 \in \mathbb{C}$, $c \overset{\mathrm{def}}{=} c_1 - c_2 \neq 0$. The equation (0.9) with the constraint (0.10) is called the *generalized Heisenberg magnet equation* (GHM).

It is derived from (0.10), that $[U, [U, [U, X]]] = c^2[U, X]$ for an arbitrary matrix X. Since $U_x = [(F_0)_x F_0^{-1}, U]$, then $[U, [U, U_x]] = c^2 U_x$. Using the last relation we can establish the equivalence of (0.9) (under constraint (0.10)) and the equation (the *zero curvature representation*)

$$\left[\frac{\partial}{\partial x} - kU, \frac{\partial}{\partial t} - c^2k^2U - k[U, U_x]\right] = 0, \tag{0.11}$$

which holds identically for any k. This is a generalization of the analogous representation in the case $n = 2$ from [ZT, TF2].

Reduction to vector nonlinear Schrödinger equation. Supposing the constraint (0.10) is satisfied, set

$$\begin{aligned}\mathfrak{gl}_n^{\pm} &= \{X \in \mathfrak{gl}_n \,|\, [U_0, X] = \pm cX\},\\ \mathfrak{gl}_n^{0} &= \{X \in \mathfrak{gl}_n \,|\, [U_0, X] = 0\},\\ \mathfrak{gl}_n' &= \mathfrak{gl}_n^{+} \oplus \mathfrak{gl}_n^{-} = [U_0, \mathfrak{gl}_n].\end{aligned}$$

Then $[\mathfrak{gl}_n^{\pm}, \mathfrak{gl}_n^{\pm}] = 0$, $[\mathfrak{gl}_n^{\pm}, \mathfrak{gl}_n^{\mp}] \subset \mathfrak{gl}_n^{0}$. In Chapter I (Corollary 1.1) we will show that an arbitrary solution of (0.9) associated with U_0 is written in the form $U = F_0 U_0 F_0^{-1}$ for a solution $F_0(x,t)$(unique up to the right multiplication by a constant matrix) of the equation

$$F_0^{-1}(F_0)_t = [U_0, (F_0^{-1}(F_0)_x)_x] - \frac{1}{2}[F_0^{-1}(F_0)_x, [U_0, F_0^{-1}(F_0)_x]] \tag{0.12}$$

subject to the constraint $F_0^{-1}(F_0)_x(x,t) \in \mathfrak{gl}_n'$. Here we only check that $U = F_0 U_0 F_0^{-1}$ satisfies (0.9) if (0.12) and the last constraint hold true. Indeed,

$$\begin{aligned}U_{xx} &= F_0\big([F_0^{-1}(F_0)_x, [F_0^{-1}(F_0)_x, U_0]] + [(F_0^{-1}(F_0)_x)_x, U_0]\big)F_0^{-1},\\ U_t &= F_0[F_0^{-1}(F_0)_t, U_0]F_0^{-1}, \text{ and } [[F_0^{-1}(F_0)_x, [U_0, F_0^{-1}(F_0)_x]], U_0] = 0,\end{aligned}$$

which implies (0.9).

Set $F_0^{-1}(F_0)_x = R = R^{(+)} + R^{(-)}$, where $R^{(+)}(x,t) \in \mathfrak{gl}_n^{+}$, $R^{(-)}(x,t) \in \mathfrak{gl}_n^{-}$ and F_0 satisfies (0.12). Then, using (0.12), we get

$$\begin{aligned}R_t =& (F_0^{-1}(F_0)_t)_x + [R, F_0^{-1}(F_0)_t] =\\ =& [U_0, R_{xx}] - \frac{1}{2}[R_x, [U_0, R]] + \frac{1}{2}[[R, [U_0, R]], R] - \frac{1}{2}[R, [U_0, R_x]] - [[U_0, R], R].\end{aligned}$$

Taking the commutator with U_0, we find that

$$\begin{aligned}[U_0, R_t] =& cR_t^{(+)} - cR_t^{(-)} =\\ =& c^2 R_{xx}^{(+)} + c^2 R_{xx}^{(-)} + c^2([[R^{(+)}, R^{(-)}], R^{(-)}] + [[R^{(-)}, R^{(+)}], R^{(+)}]).\end{aligned}$$

Thus, we come to the system

$$\frac{1}{c} R_t^{(+)} = R_{xx}^{(+)} + [[R^{(-)}, R^{(+)}], R^{(+)}], \tag{0.13a}$$

$$-\frac{1}{c} R_t^{(-)} = R_{xx}^{(-)} + [[R^{(+)}, R^{(-)}], R^{(-)}], \tag{0.13b}$$

where the second equation is obtained from the first by the formal conjugation $R^{(\pm)} \to \varepsilon^{\pm} R^{(\mp)}$, $c \to -c$ for $\varepsilon^{\pm} \in \mathbb{C}$ such that $\varepsilon^{+}\varepsilon^{-} = 1$.

Let us take now $p = 1$ and set

$$R^{(+)} \overset{\text{def}}{=} \sum_{i=1}^{n-1} r_i^{(+)} I_1^{i+1}, \qquad R^{(-)} \overset{\text{def}}{=} \sum_{i=1}^{n-1} r_i^{(-)} I_{i+1}^{1}.$$

Then

$$[[R^{(+)}, R^{(-)}], R^{(-)}] = -2\Big(\sum_{i-1}^{n-1} r_i^{(+)} r_i^{(-)}\Big) R^{(-)},$$

$$[[R^{(-)}, R^{(+)}], R^{(+)}] = -2\Big(\sum_{i-1}^{n-1} r_i^{(+)} r_i^{(-)}\Big) R^{(+)}.$$

Let us use the vector notations :

$$\boldsymbol{r}^{(+)} = {}^t(r_i^{(+)}), \qquad \boldsymbol{r}^{(-)} = {}^t(r_i(-)), \qquad \sum_{i-1}^{n-1} r_i^{(+)} r_i^{(-)} = (\boldsymbol{r}^{(+)}, \boldsymbol{r}^{(-)}),$$

where $(r_i^{(\pm)}) \overset{\text{def}}{=} (r_1^{(\pm)}, \dots, r_{n-1}^{(\pm)})$ and ^{t} is transposition. In the vector form, (0.13) is rewritten as

(0.14a) $$\frac{1}{c}\boldsymbol{r}_t^{(+)} = \boldsymbol{r}_{xx}^{(+)} - 2(\boldsymbol{r}^{(+)}, \boldsymbol{r}^{(-)})\boldsymbol{r}^{(+)},$$

(0.14b) $$-\frac{1}{c}\boldsymbol{r}_t^{(-)} = \boldsymbol{r}_{xx}^{(-)} - 2(\boldsymbol{r}^{(+)}, \boldsymbol{r}^{(-)})\boldsymbol{r}^{(-)}.$$

To continue we fix $\omega_j = \pm 1, \omega_1 = +1$ and set $X^* = \Omega\, {}^t\overline{X}\, \Omega$, where $\overline{X}$ means the complex conjugate,

$$\Omega = \operatorname{diag}(\omega_j) = \sum_{j=1}^{n} \omega_j I_j^j, \ \widetilde{\Omega} = \sum_{j=1}^{n-1} \omega_{j+1} I_j^j.$$

Let $c_1, c_2 \in i\mathbb{R}$, $c = c_1 - c_2 = \pm i$. We will assume that the solution U is $*$-anti-hermitian. Respectively, we can take $*$-unitary F_0 and $*$-anti-hermitian R. Then $\boldsymbol{r}^{(-)} = -\widetilde{\Omega}\overline{\boldsymbol{r}}^{(+)}$ and (0.14a) and (0.14b) are equivalent. Setting $\boldsymbol{r} \overset{\text{def}}{=} \boldsymbol{r}^{(\pm)}, \sigma = \pm c^{-1}$, we arrive at

(0.15) $$\sigma \boldsymbol{r}_t = \boldsymbol{r}_{xx} + 2(\boldsymbol{r}, \widetilde{\Omega}\overline{\boldsymbol{r}})\boldsymbol{r},$$

which is called the *vector (multi-component) nonlinear Schrödinger equation* (briefly, VNS). In particular, let $n = 2$, $\sigma = i$, $\widetilde{\Omega} = \omega = \pm 1$. Then we come to the *(scalar) nonlinear Schrödinger equation* $\sigma r_t = r_{xx} + 2\omega(r, \overline{r})r$. In this case, the calculation above establishes the equivalence of NS for $\omega = 1$ and the equation of *Heisenberg magnet:*

$$\boldsymbol{s}_t = \boldsymbol{s} \times \boldsymbol{s}_{xx}, \qquad (\boldsymbol{s}, \boldsymbol{s}) = 1.$$

Here $c_1 = -c_2 = i/2$, $\boldsymbol{s} = {}^t(s_1, s_2, s_3)$, functions $s_j(x,t) \in \mathbb{R}$ are defined by means of the expansion $-2iU = \sum_{j=0}^{3} s_j \sigma_j$ with respect to the Pauli matrices $\{\sigma_j\}$, and $\times$ is the vector product.

0.4. Four key constructions.

Restricting ourselves to the principal chiral fields, let us briefly outline the main methods of the study of the two-dimensional soliton equations in this book.

A) Let the pair U and V be a solution of the system (0.2). Find invertible matrix solutions $\Phi_1(x,t)$ and $\Phi_2(x,t)$ of the system (0.3) for the two fixed values of the parameter $\lambda_1, \lambda_2 \neq \pm 1, \infty$. Take two arbitrary spaces $K_1^0, K_2^0 \subset \mathbb{C}^n$ of complementary dimensions ($\dim K_1^0 + \dim K_2^0 = n$) and construct the following spaces depending on x, t: $K_1 = \Phi_1 K_1^0$, $K_2 = \Phi_2 K_2^0$. Let $P(x,t)$ be the projector onto the space K_1 along K_2. If K_1 and K_2 are in general position, then the functions

$$\widetilde{U} = U + (\lambda_1 - \lambda_2)P_x,$$
$$\widetilde{V} = V + (\lambda_2 - \lambda_1)P_t$$

satisfy (0.2). This is the simplest example of the *Bäcklund-Darboux transformation* for the PCF. The pair $\{\widetilde{U}, \widetilde{V}\}$ constructing by constant diagonal $U = U_0, V = V_0$ (taking as an initial solution) is called the *one-soliton* pair. The corresponding g is a one-soliton PCF.

B) Let us assume that U is equivalent to a constant diagonal matrix $U_0 \overset{\text{def}}{=} \operatorname{diag}(\mu_1, \dots, \mu_n)$, where $\{\mu_j\} \subset \mathbb{C}$ (i.e., U is conjugated to U_0). Then we can construct a formal solution of (0.3) in the following form:

$$\Phi = \Big(\sum_{s=0}^{\infty} \Phi_s (1-\lambda)^s\Big) \exp\big((1-\lambda)^{-1} U_0 x\big), \qquad \Phi_0(x,t) \in GL_n.$$

As to the computation of $\{\Phi_s\}$, we have to integrate. The coefficients $\{\Phi_s\}$ are not uniquely determined. However, if all $\{\mu_j\}$ are pairwise distinct, an arbitrary solution $\widetilde{\Phi}$ has the form ΦC where C is a constant purely diagonal series of $(1-\lambda)$.

Therefore in this case the coefficients in the expansions with respect to $(1-\lambda)$ of $(\log \operatorname{diag}\Phi)_x$ and $(\log \operatorname{diag}\Phi)_t$ are polynomials of matrix elements (entries) of U, V, and their derivatives with respect to x of suitable orders. Hence the (trivial) identity

$$\frac{\partial}{\partial t}(\log \operatorname{diag}\Phi)_x = \frac{\partial}{\partial x}(\log \operatorname{diag}\Phi)_t,$$

gives an infinite series of *local conservation laws* for the system (0.2).

Under proper analytical assumptions, integrals of $(\log \operatorname{diag}\Phi_s)_x$ with respect to x from $-\infty$ to $+\infty$ exist and are *integrals* of the system (0.2) (i.e., are independent of t).

C) *Inverse scattering technique* for (0.2) is outlined as follows. Let

$$U_0 = \operatorname{diag}(\mu_1, \dots, \mu_n), \; V_0 = \operatorname{diag}(\nu_1, \dots, \nu_n) \;\text{ for constant }\; \{\mu_j, \nu_j\},$$

$S(\lambda)$ be a matrix depending only on $\lambda \in \mathbb{R} \cup \infty$, and $[S(+1), U_0] = 0 = [S(-1), V_0]$. For the sake of simplicity, we suppose that $\{\mu_j, \nu_j\}$ are purely imaginary ($\in i\mathbb{R}$). Set

$$\Xi = \exp\Big(\frac{1}{1-\lambda}U_0 x + \frac{1}{1+\lambda}V_0 t\Big),$$
$$\widetilde{S} = \Xi S \Xi^{-1}.$$

Let us assume that there exist matrix functions $\widetilde{\Phi}$, $\widetilde{\Psi}$ (depending on x, t, λ) with the following properties:

a) $\widetilde{\Phi}\widetilde{\Psi} = \widetilde{S}$;
b) $\widetilde{\Phi}$(*resp.*, $\widetilde{\Psi}$) is analytically continued to the upper (resp., lower) half plane;
c) the continuations are invertible;
d) $\widetilde{\Phi}(\lambda = \infty) = \mathbf{I}$.

If such $\widetilde{\Phi}$ and $\widetilde{\Psi}$ exist, they are uniquely determined by $\widetilde{S}$. In this case the pair $\widetilde{\Phi}, \widetilde{\Psi}$ is called a solution of the *regular Riemann-Hilbert problem* for $\widetilde{S}$.

Set $\Phi = \widetilde{\Phi}\Xi$, $\Psi = \widetilde{\Psi}\Xi$. Then the function $\Phi_x\Phi^{-1} = -\Psi^{-1}\Psi_x$ is meromorphically continued to the whole complex plane with the unique "pole" at $\lambda = 1$ and "zero" at $\lambda = \infty$. Consequently,

$$\Phi_x\Phi^{-1} = \frac{1}{1-\lambda}U(x,t), \;\; \Psi_t\Psi^{-1} = \frac{1}{1+\lambda}V(x,t).$$

for matrix functions $U, V(x,t)$ independent of λ. These functions U and V satisfy (0.2) (cf.(0.3)) and are equivalent to U_0 and V_0.

If the initial function S is subject to proper analytical requirements, then the functions U, V will approach to U_0, V_0 for $x \to \pm\infty$ (and for arbitrary t).

D) Let Γ be a Riemann surface of genus g on which there exists a meromorphic function λ of order n. Let us assume that the poles $R_1, \dots, R_n$ of the function λ and the zeroes $\{R_j^\mp\}$ of functions $1 \mp \lambda$ are all simple. Choose a divisor $D \geq 0$ of degree $g + n - 1$ (i.e., a set of $g + n - 1$ points) which does not containe the points from the set $\{R_j, R_j^\pm, 1 \leqq j \leqq n\}$. For x, t in a open subset (everywhere dense in $\mathbb{R}^2$), there exist unique *Baker functions* $\phi_1, \dots, \phi_n$ which are meromorphic apart from the points $\{R_j^\pm\}$ with the pole divisor D and have the following properties:
a) the functions $\exp(-\mu_i(1-\lambda)^{-1}x)\phi_j$ are holomorphic at the point R_i^+ for $1 \leq i \leq n$,
b) $\exp(-\nu_i(1+\lambda)^{-1}t)\phi_j$ are holomorphic at $\{R_i^-\}$,
c) $\phi_i(R_j) = \delta_i^j$ $(1 \leqq i, j \leqq n)$.

Let us denote the vector function ${}^t(\phi_1, \dots, \phi_n)$ by ϕ. Then there exist unique matrix functions $U, V(x,t)$ such that

$$\phi_x = \frac{1}{1-\lambda} U\phi, \qquad \phi_t = \frac{1}{1+\lambda} V\phi.$$

This pair $\{U, V\}$ satisfies (0.2) and is equivalent to the pair $\{U_0, V_0\}$ (see above). The matrices U, V (*algebraic-geometric currents*) can be expressed in terms of the Riemann theta function of the curve Γ.

0.5. Basic notations.

$\mathbb{C}, \mathbb{R}(\mathbb{C}^*, \mathbb{R}^*)$ — the (nonzero) complex, real numbers;
$\mathbb{Z}, (\mathbb{Z}_+), \mathbb{Z}_n$ — the (non-negative) integers, integers modulo n;
$\operatorname{Re}\alpha, \operatorname{Im}\alpha (\alpha \in \mathbb{C})$ — the real, imaginary part of α;

$GL_n, \mathfrak{gl}_n$ — the complex general linear group, its Lie algebra;
$GL_n(\mathbb{R}), \mathfrak{gl}_n(\mathbb{R})$ — the real general linear group, its Lie algebra;
$U_n, \mathfrak{u}_n$ — the unitary group, its Lie algebra;
$O_n, \mathfrak{o}_n$ — the orthogonal group, its Lie algebra;
$SL_n, \mathfrak{sl}_n$ — the special linear group, its Lie algebra;

$\otimes, \bigwedge^p$ — tensor product, wedge product;
$\operatorname{Sp}, \det$ — matrix trace, determinant;

$(p,q), \quad p,q \in \mathbb{C}^n, \mathbb{R}^n$ — the euclidean or hermitian scalar product;
${}^t(\cdot), \overline{(\cdot)}, (\cdot)^\dagger$ — transposition, complex, hermitian conjugation;
$(p \wedge q)z = (z,p)q - (z,q)p$ — the $\mathfrak{u}_n$-valued "vector" product;
$|z|^2 = z^\dagger z = (z,z)$ — the hermitian norm of $z \in \mathbb{C}^n$;

$\delta^j_i, \delta(x)$ — Kronecker's symbol, Dirac's delta ;
$A = (a^j_i) = \sum_{i,j} a^j_i I^j_i$ — a^j_i is the (i, j) element of a matrix A;
$\boldsymbol{a} = {}^t(a_i) = \sum_i a_i e^i$ — a_i is the i-th component of a vector $\boldsymbol{a}$;

$\operatorname{diag} A = \sum_i a^i_i I^i_i$ — the diagonal part of A;
$e^l = {}^t(\delta^l_i)$ — the standard bases of $\mathbb{C}^n$;
$I^l_k = (\delta^k_i \delta^j_l)$, $\mathbf{I}$ — the standard bases of $\mathfrak{gl}_n$, matrix unit;

$[A]_p = \sum^p_{i,j=1} a^j_i I^j_i$ — the cross of the first p rows and columns;
$l_p(A) = \mathrm{Sp}[A]_p$ — the trace of the matrix $[A]_p$;
$m_p(A) = \det[A]_p$ — the p-th principal minor of A;
$\boldsymbol{a}^p = {}^t(a^p_i)$ — the p-th column of A;

$g_x = \dfrac{\partial g}{\partial x}, g_t = \dfrac{\partial g}{\partial t}$ — the derivatives of g;

$\mathbb{C}[\lambda], \mathbb{C}[[\lambda]]$ — the ring of polynomials of λ, formal power series;
$\mathbb{C}(\lambda), \mathbb{C}((\lambda))$ — their quotient fields;

Φ, Ψ, ϕ, ψ — the λ-analitical solutions of spectral problem ;
E_+, E_- — the solutions normalized at $x \to \pm\infty$;
$T, t_p = m_p(T)$ — the monodromy matrix and its minors;

(1.2), §1.2 — the second formula, item of §1;
Theorem 1.2 — the second theorem of §1.