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1. On the Theory of Galvano-magnetic Effects

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It will be shown that one can derive from Bloch's calculations qualitatively correct conclusions about the galvano-magnetic effects: in particular, both signs are obtained for the Hall effect, which the Sommerfeld Theory had not been able to produce, and the order of magnitude of the changes in resistance is obtained.

Recently, Bloch* developed a theory of conductivity in metals which represents a refinement on Sommerfeld's Theory** by táking the interaction between electrons and lattices more exactly into account. Bloch's treatment will now be generalised to the case of galvano-magnetic effects. Our notation follows closely that of Bloch.

§1. In a metallic cube of side $K = aG$, where a is the interatomic distance, there exist G^3 translational states for each electron, characterized by the quantum numbers k, l, m. The values of energy and current corresponding to a particular state are then given by (*l.c.*, p. 566):

$$\left.\begin{aligned} E_{klm} &= E_0 - 2\beta\left(\cos\frac{2\pi k}{G} + \cos\frac{2\pi l}{G} + \cos\frac{2\pi m}{G}\right), \\ s^x_{klm} &= e\frac{a h \Phi^x}{\pi m}\sin\frac{2\pi k}{G}, \quad \text{etc.} \end{aligned}\right\} \tag{1}$$

β and Φ^x are constants; their relationship will be derived later. In addition, Bloch obtained the time derivative of the motion under the influence of an electric field F, by replacing the electron by a special wave packet. Let the electron wave function be expanded in terms of the eigenfunctions of the system in the absence of an external field. If $\psi = \sum c_{klm}\psi_{klm}$, then one obtains (*l.c.*, Eq. (48))

$$\frac{d}{dt}\left|c_{klm}\right|^2 = -\frac{KeF}{h}\cdot\frac{\partial}{\partial k}\left|c_{klm}\right|^2 . \tag{2}$$

This equation leads to the following peculiar result: For $k > G/4$, the current decreases with increasing k, according to (1), i.e. such an electron will be decelerated, rather than accelerated, in the presence of the field. This result is so counter-intuitive that its correctness would need to be proved, preferably without any special approximations and assumptions.

First of all, this effect is already present in the classical case, as long as the mutual dependence of energy and current is given by (1). For when the electron moves in the direction of the field, the potential energy decreases, leading to an increase in the kinetic energy. In other words, since l and m remain unchanged by symmetry, k has to increase. For $k > G/4$, therefore, s_x decreases.

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* F. Bloch, Z. Phys. **52**, 555, 1928. This will be quoted as *l.c.* in what follows.

** A. Sommerfeld, *ibid.* **47**, 1, 1928.

These considerations carry over very simply into quantum mechanics. Let the wave function be expanded in terms of eigenfunctions of the motion in the absence of an external field:

$$\psi = \sum c_k \psi_k \tag{3}$$

(where the number of degrees of freedom has been reduced to 1 for simplicity). Then, following Dirac*,

$$\dot{c}_k = -\frac{2\pi i}{h} eF \sum_i x_{ik} c_i \tag{3a}$$

with

$$x_{ik} = \int x \psi_i \bar{\psi}_k \, dx. \tag{3b}$$

In our case, the expression $x\psi_i$ cannot be expanded, since it does not satisfy the periodic boundary condition $[(x\psi_i)_0 \neq (x\psi_i)_K]$. If, instead, one assumes that the electron is initially in the interior of the metal, then (3) vanishes at the boundary, and $x\psi$ can be expanded. (3a) and (3b) are then easily seen to remain valid. If the field potential is not included in the energy, the system is not conservative, and we obtain for the change in energy

$$\frac{dE}{dt} = \frac{d}{dt}\sum c_k \bar{c}_k E_k = -\frac{2\pi i}{h} eF \sum_{i,k} (\bar{c}_k c_i x_{ik} - c_k \bar{c}_i x_{ki}) E_k$$

$$= -\frac{2\pi i}{h} eF \sum c_i \bar{c}_k x_{ik} (E_k - E_i). \tag{4}$$

From the Schrödinger equation and from (3a) it follows that

$$x_{ik}(E_k - E_i) = \frac{h^2}{8\pi^2 m} \int_0^K x \left(\bar{\psi}_k \frac{\partial^2 \psi_i}{\partial x^2} - \psi_i \frac{\partial^2 \bar{\psi}_k}{\partial x^2} \right) dx$$

$$= \frac{h^2}{8\pi^2 m} \left\{ \left[x \left(\bar{\psi}_k \frac{\partial \psi_i}{\partial x} - \psi_i \frac{\partial \bar{\psi}_k}{\partial x} \right) \right]_0^K - \int_0^K \left(\bar{\psi}_k \frac{\partial \psi_i}{\partial x} - \psi_i \frac{\partial \bar{\psi}_k}{\partial x} \right) dx \right\}. \tag{5}$$

If we substitute (5) into (4), the integrated term in the sum drops out, since $\sum c_k \psi_k$ vanishes at the boundary.

$$\frac{dE}{dt} = -\frac{h}{2\pi i m} eF \sum c_i \bar{c}_k \int_0^K \left(\bar{\psi}_k \frac{\partial \psi_i}{\partial x} - \psi_i \frac{\partial \bar{\psi}_k}{\partial x} \right) dx = -\frac{eF}{m} \cdot p,$$

where $-ep/m$ is the electron current. For the momentum in the direction of the field, the energy of the electron will increase, so that the momentum decreases, according to (1).

(1) contains just one independent statement, for we will show that in this case, too, the de Broglie relation holds**:

$$\frac{\partial E_{klm}}{\partial k} = -\frac{h}{m} \cdot p^x_{klm} \tag{6}$$

* P.A.M. Dirac, Proc.Roy.Soc., 112, 661, 1926

** L. de Broglie, Ann. de phys. (10) 3, 22, 1926

Let ψ_{klm} and $\psi_{k'lm}$ be two eigenfunctions for which $k - k' \ll G$. Later we shall go to the limit $G \to \infty$ and consider k as a continuous variable. According to the wave equation which is satisfied by ψ_k and $\psi_{k'}$,

$$\Delta\psi_k \cdot \overline{\psi}_{k'} - \psi_k \cdot \Delta\overline{\psi}_{k'} = \mu(E_{k'} - E_k)\psi_k\overline{\psi}_{k'} \qquad \left(\mu = \frac{8\pi^2 m}{h^2}\right) .$$

Integrating over an elementary parallelepiped,

$$\left[\frac{\partial \psi_k}{\partial x}\overline{\psi}_{k'} - \psi_k \frac{\partial \overline{\psi}_{k'}}{\partial x}\right] = \mu(E_{k'lm} - E_{klm})\int \psi_k\overline{\psi}_{k'}\, d\tau. \tag{7}$$

The square brackets indicate the difference of the integrals over the right and left surfaces. Because of the periodicity of the potential in the crystal, $\psi_{klm} = e^{\frac{2\pi i}{K}(kx+ly+mz)} \cdot u_{klm}$ where u_{klm} varies slowly with k, l, m and is periodic in the atomic distance (*l.c.*, p. 559). Evidently, for $(k - k')/G \ll 1$ one can put

$$\psi_{k'lm} = e^{2\pi i \frac{k'-k}{K}x}\psi_{klm} ,$$

$$\frac{\partial\psi_{k'lm}}{\partial x} = 2\pi i\frac{k'-k}{K}\psi_{k'lm} \cdot e^{2\pi i\frac{k'-k}{K}x} + \frac{\partial\psi_{klm}}{\partial x} \cdot e^{2\pi i\frac{k'-k}{K}x} .$$

(7) then becomes, up to terms of higher order in $(k - k')/K$,

$$\left(e^{2\pi i\frac{k'-k}{K}x} - 1\right)\int\left(\frac{\partial\psi_{klm}}{\partial x}\overline{\psi}_{klm} - \frac{\partial\overline{\psi}_{klm}}{\partial x}\psi_{klm}\right)df = \mu(E_{k'} - E_k)\int \psi_{klm}\overline{\psi}_{klm}\, dx\, dy\, dz .$$

The integrations are carried out, on the left over the plane x = const., on the right over an elementary parallepiped. Since

$$G^2 \cdot \frac{h}{4\pi i}\int\left(\frac{\partial\psi_{klm}}{\partial x}\overline{\psi}_{klm} - \frac{\partial\overline{\psi}_{klm}}{\partial x}\psi_{klm}\right)df = p^x_{klm}$$

and $G^3\int \psi_{klm}\overline{\psi}_{klm}d\tau = 1$,

$$-\frac{8\pi^2}{h}\frac{k'-k}{K}a p^x_{klm} = \frac{8\pi^2 m}{Gh^2}(E_{k'} - E_k)\, p^x_{klm} = \frac{m}{h}\cdot\frac{E_{k'} - E_k}{k'-k} ,$$

which turns into (6) in the limit $k - k' \to 0$. This formula is very closely connected with the de Broglie relation between wave length, phase velocity and group velocity. Besides, it also contains a relationship between the constants β and Φ^x of equation (1), which reduces the number of arbitrary constants. The only basic assumption left in (1) is that the electrons are strongly bound, i.e. that the exchange energy between lattice points is small compared with the excitation potential of the atoms.

Finally, we want to show that Eq. (2) is independent of the special choice of Kennard's wave packet used by Bloch. Because of Eq. (3a), we need only evaluate the integral $\int x\psi_{klm}\overline{\psi}_{k'l'm'}d\tau$. For better legibility, the formulae are again shown for a single degree of freedom:

$$\int_{-K/2}^{K/2} x\psi_k\overline{\psi}_{k'}\,dx = \int x e^{\frac{2\pi i}{K}(k-k')x} u_k\overline{u}_{k'}\,dx$$

$$= \sum_{\alpha=-G/2}^{G/2} e^{\frac{2\pi i}{G}(k-k')\alpha} \cdot \int_0^a (x+\alpha a) e^{\frac{2\pi i}{K}(k-k')x} u_k\overline{u}_{k'}\,dx\,.$$

Because $\sum_\alpha e^{\frac{2\pi i}{G}(k-k')\alpha} = 0$ for $k \neq k'$,

$$x_{kk'} = a\sum_\alpha \alpha\cdot e^{\frac{2\pi i}{G}(k-k')\alpha} \cdot \int_0^a e^{\frac{2\pi i}{K}(k-k')x} u_k\overline{u}_{k'}\,dx\,.$$

In the limit for large G ($a \to 0$), the integral reduces to $\int_0^a u_k\overline{u}_{k'}\,dx = \frac{1}{G}$, to give

$$x_{kk'} = \frac{a}{e^{\frac{2\pi i}{G}(k-k')} - 1} \approx \frac{K}{2\pi i(k-k')}\,.$$

When this is substituted into Eq.(3), it produces Eq.(2) for large G.

§2. Let us now assume that, in addition to the electric field F in the x-direction, there exists a magnetic field H in the z-direction. An extra term

$$\frac{eH}{mc}\left(xp_y - yp_x\right) + \left(\frac{eH}{mc}\right)^2 \cdot \left(x^2+y^2\right) \tag{8}$$

will then have to be included in the Hamiltonian, and the resulting velocity components become*

$$\left.\begin{aligned} v_x &= \frac{h}{4\pi i m}\int\left[\left(\psi\frac{\partial\overline{\psi}}{\partial x} - \overline{\psi}\frac{\partial\psi}{\partial x}\right) + \frac{eH}{mc}y\psi\overline{\psi}\right]d\tau\,, \\ v_y &= \frac{h}{4\pi i m}\int\left[\left(\psi\frac{\partial\overline{\psi}}{\partial y} - \overline{\psi}\frac{\partial\psi}{\partial y}\right) - \frac{eH}{mc}x\psi\overline{\psi}\right]d\tau\,. \end{aligned}\right\} \tag{9}$$

The explicit dependence on the absolute coordinates in Eq.(9) is only an apparent one. It is exactly compensated for by the last term in Eq.(8), which corresponds to the centrifugal force. Eq. (9) will only be needed to calculate the total current produced by all the electrons. But in that case $\psi\overline{\psi}$ will be a constant for the whole metal, on average, and the coordinate system can be chosen such that $\int x\psi\overline{\psi}\,d\tau = 0$, etc. A somewhat long-winded calculation shows that at the same time the second term in Eq. (8) will make no contribution to Eq. (12). Corresponding to (3a) and (3b) we now have to determine the integrals $\frac{h}{2\pi i}\int x\frac{\partial\psi}{\partial y}\overline{\psi}_{klm}d\tau$, with

$$x\psi = \sum a_{k'l'm'}\,\psi_{k'l'm'} \tag{10}$$

$$\frac{h}{2\pi i}\int x\,\frac{\partial\psi}{\partial y}\,\overline{\psi}_{klm}d\tau = \frac{h}{2\pi i}\sum a_{k'l'm'}\,\frac{\partial\psi_{k'l'm'}}{\partial y}\,\overline{\psi}_{klm}d\tau.$$

* Cf., e.g., B.O.Klein, Z. Phys. **41**, 407, 1927, Eq. (18).

But $\frac{\partial}{\partial y}$ is an operator associated with the group of translations by multiples of the lattice distance. The above integral therefore vanishes, provided $\psi_{k'l'm'}$ and ψ_{klm} belong to different representations of this group, i.e. provided we do not have $k=k'$, $l=l'$, $m=m'$. There remains

$$\frac{h}{2\pi i}\alpha_{klm}\int\frac{\partial\psi_{klm}}{\partial y}\overline{\psi}_{klm}\,d\tau=\alpha_{klm}p^{y}_{klm}\,. \tag{11}$$

But, from (10), (2) and (3a),

$$\alpha_{klm}=\frac{K}{h}\frac{\partial c_{klm}}{\partial k}.$$

If, apart from the magnetic field, we have an electric field in the x- and y-directions,

$$\frac{d}{dt}|c_{klm}|^2=\frac{Ke}{h}\Big[-F_x\frac{\partial|c_{klm}|^2}{\partial k}-F_y\frac{\partial|c_{klm}|^2}{\partial l}+\frac{H}{mc}\Big(p^{y}_{klm}\frac{\partial|c_{klm}|^2}{\partial k}-p^{x}_{klm}\frac{\partial|c_{klm}|^2}{\partial l}\Big)\Big]$$

or, for the distribution of all the electrons,

$$\frac{d}{dt}f(\xi,\eta,\zeta)=\frac{2\pi ea}{h}\Big\{-F_x\frac{\partial f}{\partial\xi}-F_y\frac{\partial f}{\partial\eta}+\frac{H}{mc}\Big(p^y\frac{\partial f}{\partial\xi}-p^x\frac{\partial f}{\partial\eta}\Big)\Big\} \tag{12}$$

with

$$\xi=\frac{2\pi k}{G},\quad \eta=\frac{2\pi l}{G},\quad \zeta=\frac{2\pi m}{G},$$

in analogy with the corresponding formula in classical mechanics*.

On the other hand, we shall have to consider the effect of thermal motion, and then determine $f(\xi,\eta,\zeta)$ by means of the condition $\frac{df}{dt}=0$. Bloch performs this calculation by making the simplifying assumption $s^x\propto\xi$, $E\propto\xi^2+\eta^2+\zeta^2$. This assumption either represents the limiting case of free electrons or the case where the electrons are essentially in states $|k|,|l|,|m|\ll G/4$ and excludes the phenomenon discussed in §1. A rigorous calculation becomes very complicated, and so we shall only carry it out for two limiting cases, the one treated by Bloch, the other the case where the relevant electrons are found in states $|G/2-k|\ll 1$, etc.

Let us take the distribution

$$f(\xi,\eta,\zeta)=f_0(\xi,\eta,\zeta)+\chi_1\sin\xi+\chi_2\sin\eta\,,$$

where f_0 is the Fermi distribution, χ_1 and χ_2 are functions that can only be assumed approximately to be dependent on the energy alone. Furthermore, χ_1 and χ_2 will be appreciably different from 0 only where $\frac{\partial f_0}{\partial E}\neq 0$, i.e. for low temperatures, corresponding to degenerate statistics in the neighbourhood of a critical value E_1. From (12) and *l.c.*

* See e.g. A. Sommerfeld, Z. Phys. 47, 43, 1928, Eq. (65)

Eq. (75), it becomes obvious that outside this range the condition for stationary values is satisfied by $\chi_1 = \chi_2 = 0$. We now distinguish between two limiting cases [cf. (1)],

$$\begin{array}{ll} \text{a)} & E_1 - E_0 \ll 3\beta , \\ \text{b)} & E_0 + 6\beta - E_1 \ll 3\beta , \end{array} \tag{13}$$

and make the following definitions for a) and b), respectively:

$$\left.\begin{array}{llll} \text{a)} & \bar{\xi} = \xi, & \bar{\eta} = \eta, & \bar{\zeta} = \zeta \\ \text{b)} & \bar{\xi} = \pi - \xi, & \bar{\eta} = \pi - \eta, & \bar{\zeta} = \pi - \zeta \end{array}\right\} . \tag{14}$$

Evidently, in both cases $\bar{\xi}, \bar{\eta}, \bar{\zeta} \ll \pi$. We replace E by the approximation

$$\left.\begin{array}{l} E(\xi,\eta,\zeta) = \text{ const. } + \bar{\beta}(\bar{\xi}^2 + \bar{\eta}^2 + \bar{\zeta}^2), \\ \text{with } \bar{\beta} = \begin{cases} \beta \text{ in case a)} \\ -\beta \text{ " " b)} \end{cases} \end{array}\right\} \tag{15a}$$

(from now on we shall omit the additive constant) and replace the current by

$$s^x(\xi,\eta,\zeta) = \frac{e\tau}{mc}\bar{\xi}, \text{ with } \tau = \frac{ah\Phi^x}{\pi} . \tag{15b}$$

We shall now anticipate the result that Bloch's integral (77) only contributes when $E+h\nu$ and $E-h\nu$ lie in the neighbourhood of the critical value, and we can therefore also assume that (15a) is valid for $E_{k'l'm'} = E_{klm} - h\nu$. Using the above notation, Bloch's derivation of Eq. (77) can then be followed step by step, and we obtain, in place of Bloch's (78), the following equation:

$$\begin{aligned} & B\cdot\left(\frac{T}{\Theta}\right)^3 \int_0^{\Theta/T} \Bigg\{ (\bar{\xi}\chi_1 + \bar{\eta}\chi_2)\cdot \frac{f_0(E+kTx)e^x + f_0(E-kTx)}{f_0(E)} \\ & \qquad - [\bar{\xi}\chi_1(E+kTx) + \bar{\eta}\chi_2(E+kTx)]\frac{f_0(E)}{f_0(E+kTx)} \\ & \qquad - [\bar{\xi}\chi_1(E-kTx) + \bar{\eta}\chi_2(E-kTx)]\frac{f_0(E)e^x}{f_0(E-kTx)} \Bigg\} \frac{x^2dx}{e^x-1} \\ & + \frac{B'}{E}\cdot\left(\frac{T}{\Theta}\right)^5 \int_0^{\Theta/T} \Bigg\{ [\bar{\xi}\chi_1(E+kTx) + \bar{\eta}\chi_2(E+kTx)]\frac{f_0(E)}{f_0(E+kTx)} \\ & \qquad + [\bar{\xi}\chi_1(E-kTx) + \bar{\eta}\chi_2(E-kTx)]\frac{f_0(E)e^x}{f_0(E-kTx)} \Bigg\} \frac{x^4dx}{e^x-1} \\ & = 2\beta\cdot\frac{\partial f_0}{\partial E}(F_x\bar{\xi} + F_y\bar{\eta}) + \frac{H\tau}{mc}(\bar{\xi}\chi_2\cos\eta - \bar{\eta}\chi_1\cos\xi) . \end{aligned} \tag{16}$$

It should be noted that it is β and not $\bar{\beta}$ which appears on the right-hand side. This is because, to form $\frac{df_0}{dE}$, one has to differentiate with respect to ξ, η, ζ, and not with respect to $\bar{\xi}, \bar{\eta}, \bar{\zeta}$. In particular, therefore, there is no distinction between cases a) and b) for a vanishing magnetic field, and the formula is then identical with that of Bloch. The same is true for the conductivity, in spite of the phenomenon discussed in §1. This can be visualized in the following way: we are dealing not with a single electron, but with the totality of electrons, filling all states between two values of k and being displaced in the x-direction under the influence of the field F with increasing values of k. True, the current produced by electrons with $k > G/4$ will be decreased, and that corresponding to $k < -G/4$ will be increased; but since the distribution is displaced as a single entity, the overall result is the same as if there were a transition for an electron from a state with largest negative k to one with largest positive k. To the approximation used here, the dependence of the current on k is largely irrelevant, as far as the conductivity is concerned. But one can also see from (16) that it is by no means irrelevant for the Hall effect, since $\cos\xi$ and $\cos\eta$ are negative for case b), and positive for case a). H is therefore multiplied by -1 in the second case. This fact will turn out to be the cause for the anomalous Hall effect (which we shall denote as 'positive', as usual in the literature).

When χ_1 and χ_2 are functions of E only, it is not possible to solve (16) rigorously, i.e. the underlying functional equation is not separable. But in the limiting cases considered here it will be correct, up to terms of second order, to replace $\cos\xi$ and $\cos\eta$ by δ, where

$$\delta = \begin{Bmatrix} +1, & \text{for case a)} \\ -1, & \text{" \quad " b)} \end{Bmatrix}. \tag{17}$$

After dividing by $\bar{\xi}$ and $\bar{\eta}$, resp., we obtain

$$B\cdot\left(\frac{T}{\Theta}\right)^3 \int_0^{\Theta/T} \left\{ \chi_1 \cdot \frac{f_0(E+kTx)e^x + f_0(E-kTx)}{f_0(E)} - \chi_1(E+kTx)\frac{f_0(E)}{f_0(E+kTx)} \right.$$
$$\left. -\chi_1(E-kTx)\frac{f_0(E)e^x}{f_0(E-kTx)} \right\} \frac{x^2\,dx}{e^x-1} + \frac{B'}{E}\cdot\left(\frac{T}{\Theta}\right)^5 \int_0^{\Theta/T} \left\{ \chi_1(E+kTx)\frac{f_0(E)}{f_0(E+kTx)} \right.$$
$$\left. + \chi_1(E-kTx)\frac{f_0(E)e^x}{f_0(E-kTx)} \right\} \frac{x^4\,dx}{e^x-1} = 2\beta\frac{\partial f_0}{\partial E}F_x + \delta\frac{H\tau}{mc}\chi_2 . \tag{18}$$

and a corresponding second equation. Solving these integral equations is very difficult for low temperatures. In order to obtain at least some qualitative information about these effects, we shall assume that $T \gg \Theta$.
Because $x \ll 1$, we can put

$$f_0(E+kTx) = f_0(E-kTx) = f_0(E)$$
$$\text{and} \qquad \chi_1(E+kTx) = \chi_1(E-kTx) = \chi_1(E), \text{ etc.}$$

in (18). An easy transformation then gives

$$\left.\begin{aligned} C\cdot\left(\frac{T}{\Theta}\right)\left(\frac{\beta}{E}\right)^{3/2}\cdot\chi_1 &= \frac{\partial f_0}{\partial E}\cdot F_x + \frac{\delta}{2\beta}\cdot\frac{H\tau}{mc}\chi_2, \\ C\cdot\left(\frac{T}{\Theta}\right)\left(\frac{\beta}{E}\right)^{3/2}\cdot\chi_2 &= \frac{\partial f_0}{\partial E}\cdot F_y - \frac{\delta}{2\beta}\cdot\frac{H\tau}{mc}\chi_1. \end{aligned}\right\} \tag{19}$$

To terms of order H^2, we obtain

$$\chi_1 = \frac{\partial f_0}{\partial E}\left[F_x \frac{1}{C}\frac{\Theta}{T}\left(\frac{E}{\beta}\right)^{3/2} + F_y \frac{H\tau}{mc}\frac{1}{C^2}\left(\frac{\Theta}{T}\right)^2\left(\frac{E}{\beta}\right)^3\frac{\delta}{2\beta}\right],$$

with a corresponding expression for χ_2. The current density in the x-direction then becomes

$$I_x = \frac{e\tau}{ma^3}\cdot\frac{1}{8\pi^3}\iiint \bar{\xi}^2\chi_1\cdot d\bar{\xi}\,d\bar{\eta}\,d\bar{\zeta}\,.$$

Now, for a slowly varying function $\varphi(E)$, $\int \frac{\partial f}{\partial E}\varphi(E)dE = \varphi(E_1)$, where E_1 is the 'critical value' of the Fermi distribution. It then follows that

$$I_x = \frac{e\tau}{ma^3}\frac{1}{6\pi^2}\left\{\frac{F_x}{C\beta}\frac{\Theta}{T}\cdot\left(\frac{E_1}{\beta}\right)^3 + \frac{\delta F_y}{2C^2\beta^2}\cdot\frac{H\tau}{mc}\cdot\left(\frac{\Theta}{T}\right)^2\left(\frac{E_1}{\beta}\right)^{9/2}\right\},$$

$$I_y = \frac{e\tau}{ma^3}\frac{1}{6\pi^2}\left\{\frac{F_y}{C\beta}\frac{\Theta}{T}\cdot\left(\frac{E_1}{\beta}\right)^3 - \frac{\delta F_x}{2C^2\beta^2}\cdot\frac{H\tau}{mc}\cdot\left(\frac{\Theta}{T}\right)^2\left(\frac{E_1}{\beta}\right)^{9/2}\right\}.$$

§3. For the Hall effect, $I_y = 0$, which produces an opposing field F_y,

$$F_y = F_x \cdot H \cdot \frac{\delta\tau}{2mc\beta C}\cdot\frac{\Theta}{T}\left(\frac{E_1}{\beta}\right)^{3/2}$$

or, because

$$F_x = \frac{6\pi^2 ma^3}{e\tau}C\beta\cdot\frac{T}{\Theta}\left(\frac{\beta}{E_1}\right)^3 I_x,$$

we have

$$F_y = \delta H I_x \cdot \frac{3\pi^2 a^3}{ec}\cdot\left(\frac{\beta}{E_1}\right)^{3/2}.$$

E_1 is determined by the condition that the number of quantum cells with $E < E_1$ must be equal to the number of electrons. Writing $n = N/G^3$ = the number of conducting electrons per atom, and λ = the number of quantum cells in an unperturbed atom, we obtain

$$\left(\frac{E_1}{\beta}\right)^{3/2} = \begin{cases} \dfrac{6\pi^2 n}{2\lambda} & \text{for the case a),} \\ 6\pi^2\left(1-\dfrac{n}{2\lambda}\right) & \text{" " " b).} \end{cases}$$

Thus the Hall constant is

$$R = \frac{F_y}{H\cdot I_x} = \left\{\begin{array}{ll} -\dfrac{2a^3}{ec\dfrac{n}{2\lambda}} & \text{(a)} \\ +\dfrac{2a^3}{ec\left(1-\dfrac{n}{2\lambda}\right)} & \text{(b)} \end{array}\right\}. \qquad (20)$$

This value is of the same order of magnitude as Sommerfeld's value, unless $n/2\lambda$ or $1-n/2\lambda$, respectively, happen to be very small. n and λ are both of order 1. (a^3 is the atomic volume. The n used by Sommerfeld corresponds to our n/a^3 .)

The sign depends essentially on the occupation number $x = n/2\lambda$. If x is small, we have case a), i.e. the Hall effect is negative; if x is approximately equal to 1, we have case b), and the Hall effect is positive (anomalous). Even without any approximation, the whole calculation is symmetric about $x = 1/2$, so that the Hall effect cannot appear in this case. This is connected with Pauli's reciprocity law* holding between free and "occupied" positions in the atom, but is only valid here for low temperatures, for which one can consider $\frac{\partial f_0}{\partial E}$ as symmetric in E_1. The conductivity vanishes for $x = 0$ and $x = 1$: in the former case, because there are no conducting electrons present; in the latter, because all positions are occupied.

There is very little that can be said *a priori* about these occupation numbers. Although they are known for free atoms, in the case of mutually interacting atoms two more factors emerge. First of all, β can in certain circumstances become comparable with the excitation energy, so that an 'escape' to excitation levels of the unperturbed atom becomes a possibility. Secondly, one of two originally equal terms could e.g. be shifted through some perturbation by an amount that is greater than β. Nothing can be stated about this without taking into account the interactions between the electrons themselves. But one could expect the following to happen: if by any chance the excitation potential is $> 6\beta$ but still comparable with β, then more electrons will go into the excited state with increasing temperature, so that the occupation number becomes smaller. In such a case, the Hall effect can change sign, from positive to negative. This would appear to be the case for bismuth, and perhaps for zinc**.

Finally, we want to compute the change in resistance in a magnetic field. For this, two factors have to be taken into account. First, the magnetic will have an effect on the spin orientation, and will thus change the distribution f_0. This is of course independent of the direction of the magnetic field relative to the current, and could therefore possibly dominate the parallel effect. The second effect, discussed by Sommerfeld, is the lengthening of the electron paths by virtue of their deflection in the magnetic field. This effect is direction-dependent and vanishes when current and field are parallel and can therefore be made to account for the difference between the parallel and transversal effects.

This latter effect can be obtained from our formula (19), if one goes up to terms of second order in H. But in the above approximation the change in resistance becomes zero, and one has to take the integrals over the Fermi distribution one step further in the approximation, exactly as in Sommerfeld's calculation. The details will be omitted here. For the change in conductivity one finds that

$$\frac{\Delta\sigma}{\sigma} = -R^2 \cdot \sigma^2 \cdot 12\pi^2 \left(\frac{kT}{E_1}\right)^2 \cdot H^2 . \tag{21}$$

* W. Pauli, ZS. f. Phys. **31**, 765, 1925

** Landolt-Börnstein, Phys.-chem. Tabellen, Ergänzungsband. Berlin 1927. p. 666

To get a preliminary idea, let us put into (21) the values of R and σ for Ag at 0°C. One obtains

$$\frac{\Delta\sigma}{\sigma} \approx -10^{-11}\left(\frac{kT}{E_1}\right)^2 \cdot H^2 .$$

In order to agree with experiment, the quantity $\ln A = E_1/kT$, which is responsible for the anomaly according to Sommerfeld, would have to be of order 10. It would seem that such values of E_1 are possible, and that the above result is correct, at least to a very rough approximation.

When $T \gg \Theta$, the case for which formula (21) was derived, one would find temperature-independence; but in general this assumption is not satisfied. I have not yet been able to establish, by comparison with reliable experimental data, whether the proportionality with $T^2 \cdot \sigma^2$ holds for low temperatures. In addition, (21) does not yet enable one to estimate the order of magnitude of the anomalous effects for bismuth and ferromagnetic materials.

Finally, I wish to thank Prof. Heisenberg most sincerely for his encouragement and help with his work, and Dr. Bloch for many interesting discussions.

Theor.-Phys. Institut der Universität Leipzig, 22 December 1928.

REP comments:
This paper and the next deal with the 'anomalous' Hall effect by which some metals show a positive sign as if the current was carried by positive carriers. This is explained in terms of 'holes' although this name for this is not used; however, the diagram of the next paper makes it clear that the hole concept is involved. To establish this, I had to prove that the current was proportional to dE/dk (E = energy; k = wave vector) and that the rate of change of k in an electric or magnetic field is the same as for a free electron.