# Chapter 1

# The Ongoing Quest For Three-Manifold Invariants

How much we can know about the universe in which we live depends very much on our ability to understand three-dimensional manifolds. This is a difficult task since three-manifolds are inherently complicated objects and there is a bewildering array of them. There are several methods currently available for constructing three-manifolds (e.g. the combinatorial approach, the Heegaard gluing approach, the Dehn surgery approach, etc.) What is lacking, however, are methods for determining when 3-manifolds are the same. Topologically speaking, two objects are said to be equivalent if there is a homeomorphism from one to the other. A homeomorphism is a one-to-one onto map that is continuous, and whose inverse is also continuous. In short, it is a topological isomorphism.

Over the last forty years, invariants have proven to be the most effective tools available for studying three-manifolds. The problem is the following: standard invariants, while useful, do not give us all of the information that we would like to have.

Nature, as we observed earlier, has blessed us with an abundant variety of 3-manifolds. According to Thurston's work [1], most 3-manifolds are hyperbolic. A hyperbolic 3-manifold is a three-manifold with a metric locally isometric to hyperbolic 3-space. A typical three-manifold is either topologically simple or otherwise possesses a hyperbolic structure. For instance, a knot in the 3-sphere possesses a hyperbolic complement , in which case it is referred to as a hyperbolic knot, when it is neither a torus knot nor a satellite knot. It should be noted that all but a finite number of the 3-manifolds obtained through performing Dehn surgery on a given hyperbolic knot have hyperbolic structures.

In essence, a 3-manifold must be topologically complicated to admit a hyperbolic structure. This may explain why the discovery of closed, hyperbolic 3-manifolds has been a slow process. Throughout this chapter and the book, *when referring to a closed manifold, we shall mean a manifold that* is compact and without boundary. Orientable shall mean that the manifold contains no mirror-reversing path.

The first cited construction of some hyperbolic 3-manifolds is credited to Löbell and dates as far back as 1931. Later, in 1933, Seifert and Weber discovered an easier way to describe hyperbolic 3-manifolds. But it would take about 38 years before Alan Best discovered, in 1971, several other hyperbolic 3-manifolds. Shortly thereafter, Bob Riley put hyperbolic structures on several knot and link complements, and Troels Jørgensen constructed hyperbolic three-manifolds that fiber over the circle. The Riley and Jørgensen examples were a major source of motivation for Thurston's seminal work on three-dimensional hyperbolic universes.

The evidence to date suggests that hyperbolic 3-manifolds are the most abundant, the most complicated, and the most important class of 3-manifolds. Therefore, for physicists and mathematicians alike, studying the 3-dimensional universe in which we live is best achieved by restricting our attention to topological three-manifolds which admit hyperbolic structures. Hyperbolic structures reveal many insights about 3-dimensional topology and physics. Among these is the discovery of a new class of 3-manifold invariants.

Below, we take up the quest for 3-manifold invariants. The latest The latest invariants to have surfaced in the past three years are presented in the next chapter.

Determining when two 3-manifolds are the same (that is, are homeomorphic) is a highly non-trivial procedure. In contrast, in dimension two, one can solve this problem by using a single invariant: the Euler characteristic. This invariant assigns a number to each closed, orientable 2-manifold. This, in turn, allows us to answer basic questions about such manifolds.

For instance, the problem of classifying 2-manifolds is made easier by the classical fact that two closed, orientable two-manifolds are homeomorphic if, and only if, they have the same Euler characteristic. Unfortunately, the Euler characteristic is not a viable invariant for 3-manifolds since all closed three-manifolds have Euler characteristic zero.

One may wonder at this point: are there are any other options? The answer is, yes. The fundamental group of a 3-manifold is one of those. It does contain far more information than the Euler characteristic, and is essentially straightforward to compute. It is, however, not considered to be a good invariant primarily because extracting information out of it is difficult. William Massey [2] provides us with a good example underlying this dilemma: given two group presentations (a presentation is a description of a group in terms of generators and relations), are they presentations of the same group or of different groups?

Homology groups, on the other hand, are often able to distinguish between 3-manifolds, and they are relatively easy to compute. As such, they are good candidates for invariants. But, there are several examples in which they fail to distinguish between 3-manifolds. All knot complements in the 3-sphere have the very same homology group, for instance.

Three of the most natural invariants of hyperbolic 3-manifolds are the volume, the Chern-Simons invariant, and the $\eta$-invariant.

## 1.1 Volume as a Three-Manifold Invariant

Consider a 3-manifold $M^\tau$ obtained via surgery from the manifold $M$. A theorem of Ruberman [3] states the following:

$$\mathrm{vol}\,(M^\tau) = \mathrm{vol}\,(M).$$

In essence, Ruberman's theorem implies that the volumes of hyperbolic 3-manifolds are qualitative invariants, since even a radical procedure such as surgery leaves them unchanged.

How does one compute the volume of a given hyperbolic 3-manifold? The starting point is perhaps to consider the analogous 2-dimensional case. In dimension two, the closest analogue to a hyperbolic structure is the Poincaré disk, a hyperbolic plane with infinitesimal metric

$$ds = \frac{\sqrt{(dx)^2 + (dy)^2}}{(1 - (x^2 + y^2))/2}.$$

An infinitesimal line segment parallel to the $x$-axis at the point $(x, y)$ has hyperbolic length

$$\frac{dx}{(1 - (x^2 + y^2))/2};$$

similarly, a segment parallel to the $y$-axis has hyperbolic length

$$\frac{dy}{(1 - (x^2 + y^2))/2}.$$

Combining these, we see that an infinitesimal rectangle has hyperbolic area

$$\frac{dx\, dy}{(1 - (x^2 + y^2))^2/4}.$$

Thus, the hyperbolic area of a region in the Poincaré disk is computed by integrating the area form:

$$dA = \frac{dx\, dy}{(1 - (x^2 + y^2))^2/4}$$

over the region.

In dimension three, the Poincaré ball of hyperbolic 3-space has infinitesimal metric

$$ds = \frac{\sqrt{(dx)^2 + (dy)^2 + (dz)^2}}{(1 - (x^2 + y^2 + z^2))/2},$$

and volume form

$$dV = \frac{dx\, dy\, dz}{(1 - (x^2 + y^2 + z^2))^3/8}.$$

The volume of a tetrahedron can be computed by integration, and we obtain a formula for the volume of the tetrahedron in terms of its dihedral angles, as shown by Thurston [1]. As such, the problem of computing the volume of a hyperbolic 3-manifold reduces to the problem of decomposing it.

Is the volume a useful invariant for three-manifolds three-manifolds? It is in the sense that it is effective in distinguishing between 3-manifolds. Unfortunately, the volume is far from being a complete invariant for (hyperbolic) three-manifolds: there are examples of non-homeomorphic (i.e. distinct) hyperbolic 3-manifolds which nonetheless have equal volumes.

Provided that the volume fails, two remaining natural invariants for three-manifolds are the Chern-Simons invariant and the $\eta$-invariant. Both play a central role in various aspects of 3-dimensional physics, as we shall see in due course. Mostow's theorem is a sufficient criteria for proving that they are topological invariants for closed, hyperbolic 3-manifolds as well.

## 1.2 The Chern-Simons Invariant

In 1974, Chern and Simons [4] defined a certain 3-form $Q$ on the oriented frame bundle $F(M)$. Owing to the fact that any orientable 3-manifold is parallelizable, one can focus on sections of the frame bundle, which act to pull $Q$ back to $M$. We refer to this operation as a pull-back. Integrating $s^\star Q$ over $M$ produces a real number; this number depends a priori on the choice of section of $F(M)$. Given one section, any other differs from it by a map from $M$ to $SO(3)$. This means that the integral of the Chern-Simons form changes by $8\pi^2$, the degree of the map.

Thus, we can write the Chern-Simons invariant as:

$$CS(M) = \frac{1}{8\pi^2}\int_M s^\star Q \ \ (mod\, 1).$$

This invariant grew out of attempts by Chern and Simons to develop a combinatorial formula for the first Pontryagin number! of a given 4-manifold.

Though it has given us an enormous number of successful applications both in physics and mathematics, a drawback of the Chern-Simons invariant is that it is very difficult to compute. As a matter of fact, it was several years after its inception before we knew whether or not it was a trivial invariant for hyperbolic 3-manifolds. The first sign that the Chern-Simons invariant was not trivial for this class of manifolds surfaced in 1981. But it wasn't until

1986 that any strong evidence emerged corroborating its non-triviality; this was done by Meyerhoff in reference [5]. His approach can be summarized as follows: in the circle $\mathbb{R}/\mathbb{Z}$, the Chern-Simons invariant takes on a dense set of values. The trick is then to investigate the geometrical behavior of the set in question. We refer the interested reader to [5] for more details.

There exists an analytic relation between the volume invariant and the Chern-Simons invariant for hyperbolic 3-manifolds. This property was discovered in 1986 by Walter Neumann and Don Zagier [6].

Just how useful is the Chern-Simons 3-form as a three-manifold invariant? Or, put differently, can the Chern-Simons invariant enable one to distinguish large classes of hyperbolic 3-manifolds that have equal volumes? In answer, Meyerhoff and Ruberman [7] offer the following:

**Theorem 1.1 (Meyerhoff-Ruberman)** *Consider the circle $\mathbb{R}/\mathbb{Z}$ in which the Chern-Simons invariant takes its value. Given any rational number in $\mathbb{R}/\mathbb{Z}$, there exist hyperbolic! three-manifolds with equal volumes whose Chern-Simons invariants differ by this rational number.*

This is undoubtedly a nice result: one can appreciate the degree of refinement for which a difference in values between the two invariants reduces to a computable, yet easily quantifiable and manageable norm. The problem, however, is simply that a systematic understanding of manifolds with equal volumes and different Chern-Simons invariants seems a long way off.

## 1.3 The $\eta$-invariant

Most of the invariants we just have just encountered have, to some degree, some drawbacks. As such, they are in no way the ultimate sought 3-manifold invariants. Another possibilty is to look at the $\eta$-invariant. This invariant was introduced by Atiyah, Patodi and Singer [8]. In its original formulation for odd-dimensional manifolds, it was given in terms of the eigenvalues of the Laplace operator. It was, furthermore, initially introduced to measure the extent to which the Hirzebruch signature formula fails for geometric 4-manifolds with boundary.

Atiyah, Patodi and Singer gave the following remarkable formula [8], which we take as the definition of the $\eta$-invariant.

**Theorem 1.2 (Atiyah-Patodi-Singer)** *Consider a 4-dimensional manifold $W$ whose boundary is a 3-dimensional manifold $M$. Choose a framing $\alpha$ on $M$, such that it gives rise to the Pontrjagin number $p_1(W)$. Define the signature defect $\sigma(M,\alpha)$ to be the integer $\frac{1}{3}p_1(W) - \operatorname{sign}(W)$. We can then write the $\eta$-invariant as*

$$\eta(M) = \frac{1}{12\pi^2}\int \alpha^\star Q + \sigma(M,\alpha). \qquad (1.1)$$

The $\eta$-invariant, it should be noted, is closely related to the Chern-Simons invariant; specifically,

$$3\eta(M) = 2CS(M) \bmod \mathbb{Z}.$$

The $\eta$-invariant contains information that the Chern-Simons invariant does not have. For instance, there are examples of hyperbolic 3-manifolds with Chern-Simons invariants equal but different $\eta$-invariants. Tomoyoshi Yoshida has made substantial progress toward a systematic computation of the $\eta$-invariant for hyperbolic 3-manifolds [9].

## 1.4 The Chern-Simons Invariant Revisited

The discovery by Vaughan Jones of a new polynomial invariant of links in the 3-sphere in 1985 [13] was an important breakthrough which has led to the introduction of a whole range of new techniques in three-dimensional topology. The original Jones polynomial, a Jones polynomial in one variable, was obtained via a braid description of a link, utilizing the remarkable properties of some representations of the braid group which arose in the theory of von Neumann algebras. Early developments were largely combinatorial, leading to alternative definitions of the invariant and to generalizations, including a two-variable polynomial which specializes in both the Jones polynomial and the classical Alexander polynomial after appropriate substitutions.

The new invariants are comparatively easy to calculate and have had many concrete applications, but for some time no satisfactory conceptual definition of the invariants was known-one not relying on the special combinatorial presentations of a link. It was not clear, for example, whether such invariants could be defined for links in other 3-manifolds. While there were many intriguing connections between the Jones theory and statistical mechanics, for instance through the Yang-Baxter equation and the newly developed theory of quantum groups, it was a major problem to find the correct geometric setting for the Jones theory.

In July of 1988, Witten proposed a scheme which largely resolved this problem. He showed that the invariants (including the Kauffman polynomial) should be obtained from a quantum field theory with a Lagrangian involving the Chern-Simons invariant of connections. Witten's approach [10] provided a truly natural definition of the invariants, and indeed allowed considerable generalization to links in arbitrary three-manifolds.

Taking in particular the empty link, he obtained a new invariant of closed 3-manifolds. The challenge in this approach arose from the notorious difficulties of quantum field theory in attaching a real meaning to the functional integral over the space of connections which is involved.

The principal source of interest in this new class of theories was the realization that despite these foundational difficulties, a new class of invariants could be constructed and yet make concrete predictions which could be verified on a more elementary level.

The definition of these invariants is given in terms of a functional integral, namely

$$Z_{k,G}(M,g) = \int exp(ik\,\mathrm{CS}(A))\,\mathcal{D}[A], \tag{1.2}$$

where $CS$ denotes the Chern-Simons functional, and $G$ is a compact semisimple, simply connected Lie group, $g$ is a metric on $M$; the integration is done over all gauge equivalence classes of connections on a principal $G$-bundle over the 3-manifold $M$.

The functional integral (1.2) is, however, not a mathematically well defined object, and this very fact underscores the discovery of a new class of 3-manifold invariants–some of which are presented in the next chapter.

How do physicists avoid this problem? Mainly by using the following two standard methods. The first method consists of giving a proper definition of functional integrals via perturbation theory. With regard to equation (1.2), this would be the limit $k \to \infty$. In this limit, one can try to compute the integral by the stationary phase approximation method. Witten, in reference [10], gives the following formula for the large $k$ limit of (1.2):

$$\begin{aligned} Z_{k,G}(M,g) &= \exp i\pi \dim G \left(\frac{\eta_{\text{grav}}}{2} + \frac{1}{12}\frac{\text{CS}(A^g)}{2\pi}\right) \\ & \Sigma_{[A^0]} \exp i \left(k + \frac{c_2(G)}{2}\right) \text{CS}(A^0)\, T_{A^0}; \end{aligned} \tag{1.3}$$

where the sum is taken over the gauge equivalence classes of flat connections on the principal $G$-bundle over $M$, $A^g$ is the corresponding Levi-Civita connection, $\eta_{\text{grav}}$ is the $\eta$-invariant of the operator $\star D^g + D^g\star$ ($D^g$ being the exterior derivative twisted by $A^g$), $c_2$ is the value of the quadratic Casimir operator in the adjoint representation of $G$, and finally, $T_{A^0}$ is the Ray-Singer torsion of $A^0$.

The sum in (1.3) is defined if $\{[A^{(0)}]\}$ is a set of isolated points; otherwise the sum should be replaced by an integral over the classes of flat connections, with a suitable measure. A coherent description of the asymptotics of (1.2) in this case is an interesting and, as far as we know, open problem. It has been shown by Witten that the right hand side of (1.2) should depend only on the 2-framing of the 3-manifold $M$. On the other hand, Atiyah's canonical framing [11] should yield a similar Atiyah's canonical framing 3-manifold invariant.

The second way to rigourously define the functional integral (1.2) is to use some phenomenological formula for studying (or rather to find) transformation properties of (2) under certain natural transformations. In most cases, these properties uniquely fix the object on the left hand side of (1.2), and give an independent rigorous definition of it. Reshetikhin and Turaev's definition of 3-manifold invariants via surgery on a link in $S^3$ [12] is perhaps the most readily available realization of this program for the functional integral (1.2).

Equation (1.3) tells us that in order to find the limit of (1.2) for $k \to \infty$, we have, at least, to sum over all classes of flat connections. On the other hand, it is clear that each individual term in (1.2) should be an invariant of the pair $\left(M, [A^{(0)}]\right)$. After appropriate normalization, it becomes an element of $\mathbb{C}\left[[\frac{1}{k}]\right]$, which we write as $Z_k(M, [A^{(0)}])$. Axelrod and Singer [14], and Kontsevich [15] found that this power series exists if $M$ is a rational homology sphere. There are additional factors to be taken into account, namely, the de Rham complex twisted by $A^{(0)}$ is acyclic (in Kontsevich's paper [15], $A^{(0)} = 0$; furthermore, the coefficients are a linear combination of integrals $\int_{M \times \cdots \times M} \omega$ for some suitable forms $\omega$. These are invariants of pairs $\left(M, [A^{(0)}]\right)$. The term $Z_k\ (M, 0) \in \mathbb{C}\left[[\frac{1}{k}]\right]$ turns out to be an invariant of 3-manifolds.

This term differs from the 3-manifold invariants obtained via canonical framing, and for $k \in \mathbb{N}$ seems to exist for rational homology spheres. Reconciling these two invariants is at the core of understanding perturbative Chern-Simons-Witten theories.

## 1.5 Outlook and Summary

Let us review the dilemma with which we are faced. We live in a three-dimensional universe of which very little is known in terms of its mathematical and physical properties. Although this universe is allowed to have infinitely many shapes, the message from Thurston is clear: hyperbolic 3-universes are the most abundant, important and yet complicated collection of all types of 3-manifolds.

In order to understand the world we live in, we need some objects whose primary role is to take the unwieldy collection of information that defines the universe, and distill it into a manageable packet. Such objects are called invariants.

*For hyperbolic 3-manifolds, we have seen that neither the homology nor* the fundamental group are satisfactory invariants. New invariants are crucially needed. If we focus our attention on 3-dimensional hyperbolic universes with finite volume, then a fundamental theorem of Mostow tells us that such invariants ought to be topological invariants.

Hyperbolic 3-manifolds have natural invariants: the volume, the Chern-Simons invariant, and the $\eta$-invariant. The volume, we have seen, has proven successful at distinguishing between manifolds with the same homology, while the Chern-Simons invariant and the $\eta$-invariant can distinguish between many hyperbolic 3-universes with the same volume. In addition, these invariants should be able to yield information about the underlying universe. For instance, the volume appears to be a good measure of complexity, while the Chern-Simons invariant appears to measure handedness.

A fundamental issue is to measure to what extent the volume, the Chern-Simons and the $\eta$-invariant, when taken together, determine a given 3-dimensional universe. Unfortunately, as is often the case with central questions there is as yet no known answer.

To further complicate our quest to understand the universe in which we live, some examples have recently surfaced of 3-manifolds that are not distinguishable by the three invariants we just mentioned. Cusped hyperbolic 3-manifolds fall into this category. These are hyperbolic mutant manifolds [7]; they do share among themselves equal volumes and Chern-Simons invariant. But the nature of the $\eta$-invariant is problematic, that is to say, hard to compute. A mutation is the result of cutting a 3-manifold $M$ along a genus two surface $\Sigma_g$, and regluing via the (unique) involution in the center of $\Sigma_g$.

It is a likely possibility that mutant manifolds are insensitive to the above-mentioned three invariants, in part because of our poor knowledge of 3-dimensional mapping class groups. The unique involution defining the pasting of the mutant manifold originates from the center of the mapping class group. What is not known at this point, however, is the nature of this involution in terms of the mapping class group itself. Depending on what subgroup of the mapping class group determines the mutation, we may have an impetus to investigate 3-dimensional mapping class groups and subgroups more aggressively.

Mapping class groups enhance our interest in quantum topology because of their central roles in various forms of quantization, operator ordering, global anomalies [16], etc. A great many of these issues will be discussed throughout this book. The next chapter is devoted to an invariant of 3-manifolds obtained using mapping class groups, while the relation between mapping class groups, Teichmüller space and moduli space is thoroughly presented in following chapters.

We should point out that mutant hyperbolic three-manifolds are not insensitive only to the $\eta$-invariant: the volume, the Chern-Simons and $\eta$-invariant do not provide a complete set of invariants for closed, hyperbolic, mutant 3-manifolds. Certain mutations, such as the one described by Meyerhoff and Ruberman in reference [7], leave the volume, the Chern-Simons invariant (mod 1), and the $\eta$-invariant unchanged.

Perhaps it comes across to the reader that we critically need new invariants of three-manifolds. If so, then the objective of this chapter to convey the efforts behind the continuous quest for three-manifold invariants has been achieved.

## 1.6 References

[1] Thurston, William: *The Geometry and Topology of 3-Manifolds*, Princeton University Press (1978).

— *Three-Dimensional Manifolds, Kleinian groups and hyperbolic geometry*, Bull. American Mathematical Society (2) 6 (1982) 357-381.

[2] Massey, Williams: **Algebraic Topology: An Introduction**, Harcourt, Brace, and World Inc., (1967) New York, p. 106.

[3] Ruberman, Daniel: *Mutations and Volumes of Knots in* $S^3$, Inventiones Math. 90 (1987), 189-216.

[4] Chern, S. S. and Simons, J.: *Characteristic Forms and Geometric Invariants*, Annals Math. 2, 99 (1974) 48-69.

[5] Meyerhoff, Robert: *Density of the Chern-Simons Invariant for Hyperbolic 3-manifolds*,in **Low Dimensional Topology and Kleinian Groups**, Editor: D. B. A. Epstein, Cambridge University Press, (1986) 217-239.

[6] Neumann, Walter and Zagier, Don: *Volumes of Hyperbolic Three-manifolds*, Topology 24 (1985) 307-332.

[7] Meyerhoff, Robert and Ruberman, Daniel: *Mutation and the* $\eta$*-invariant*, Journal of Differential Geometry 31 (1990) 101-130.

[8] Atiyah, M. F., Patodi, V. K. and Singer, I.: *Spectral Asymmetry and Riemannian Geometry* I: Math. Proc. Camb. Phil. Soc. 78 (1975) 405-432.

[9] Yoshida, Tomoyoshi: *The* $\eta$*-invariant of Hyperbolic 3-Manifolds*, Invent. Math. 81 (1985) 473-514.

[10] Witten, Edward: *Quantum Field Theory and the Jones Polynomial*, Comm. Mathematical Physics 121 (1989) 351-399.

[11] Atiyah, M. F.: *On Framings of 3-Manifolds*, Topology 29 (1990) 1-7.

[12] Reshetikhin, N. Y. and Turaev, V. G.: *Invariants of 3-Manifolds via Link Polynomials and Quantum Groups*, Invent. Math. 103 (1991) 547-597.

[13] Jones, Vaughan: *A New Polynomial Invariant for Links via von Neumann Algebras*, Bull. Amer. Math. Soc. 129 (1985) 103-112.

[14] Axelrod, S. and Singer, I.: *Perturbation Theory for Chern-Simons Theory*, in **Proc.** *XX* **Int. Conf. on Diff. Geom. Methods in Physics**, World Scientific (1991) 3-45.

[15] Kontsevich, M.: *Feynman Diagrams and Low-dimensional Topology*, prepint, 1992. To appear in the Proceedings of the First European Congress of Mathematicians.

[16] Baadhio, R. A.: *Mapping Class Groups for* $D = 2{+}1$ *Quantum Gravity and Topological Quantum Field Theories*, Nuclear Physics B441 (1995) 383-401.