

CHAPTER 1

FOUNDATIONS OF SPACE–TIME THEORIES

1.1. Introduction

The first step for constructing a flat space–time theory is to define an inertial reference frame (an inertial system of reference) in vacuum, which will be denoted by, e.g., $F(xyzt)$ with (x, y, z) and t being space and time coordinates, respectively. To define an inertial reference frame, the following three problems must be solved:

- (i) What are the space coordinates (x, y, z) ?
- (ii) What is the time coordinate t ?
- (iii) What is “inertial”?

Responding the first and third problems will be simple and unique in spite of the definition of the vacuum. However the answer to the second problem will be complex and variant. The main difference among Newtonian mechanics, Einstein’s theory and test theories of special relativity is just the different answer to the second problem. In this chapter, we shall mainly clarify some definitions of the time coordinate, i.e., some definitions of simultaneity. In Einstein’s theory of special relativity, the second postulate, i.e., the so-called principle of the constancy of the velocity of light, establishes the definition of Einstein simultaneity. As shown in chapter 8, it is the two-way speed but not the one-way speed of light has been measured.

The second step is to assume the principle of relativity. In Einstein’s theory of special relativity the so-called Einstein’s principle of relativity is just a simple generalization of the Galilean principle of relativity in classical mechanics. This principle makes us possible to establish transformation equations of any physical quantities in the well-defined inertial frames.

1.2. Definition of Inertial Reference Frame

Let $F(xyzt)$ denote an inertial reference frame, where x, y, z are called the space coordinates which are given in a Cartesian coordinate system $S(xyz)$ in a three dimensional vacuum space and t is the time coordinate representing a “common” time within the whole system S . In this section we shall give the definition of the inertial reference frame.

1.2.1. Definition of Space Coordinates

We assume that the vacuum, a three-dimensional Euclidean space, exists. Imagine that a framework of three orthogonal rods (i.e., x -, y -, and z -axis) which extends

into the space can be rigidly attached to a reference body. This conceptual framework is regarded as a rectangular Cartesian coordinate system, $S(xyz)$. Using this Cartesian system, we characterize any space point, e.g., P_1 by three numbers, the coordinates x_1, y_1, z_1 of that space point. We use $P_1(x_1, y_1, z_1)$ to denote that point. Let $P_2(x_2, y_2, z_2)$ be another point. The coordinate differences between these two points are denoted by $\Delta x, \Delta y, \Delta z$. The distance Δs between the two points is given by

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2, \quad (1.2.1)$$

or

$$(\Delta s)^2 = \delta_{ij} \Delta x_i \Delta x_j, \quad i, j = 1, 2, 3, \quad (1.2.2)$$

where

$$\Delta x_1 \equiv \Delta x = x_2 - x_1,$$

$$\Delta x_2 \equiv \Delta y = y_2 - y_1,$$

$$\Delta x_3 \equiv \Delta z = z_2 - z_1,$$

and δ_{ij} is the so-called Kronecker symbol defined by the equations

$$\delta_{ij} = 1, \quad i = j,$$

$$\delta_{ij} = 0, \quad i \neq j. \quad (1.2.3)$$

The convention is used: Each index that occurs once in a product (i.e., an open index) assumes all its values; Each index which occurs twice in a product (i.e., a contracted index) is a summation index, where the summation is to be carried over all possible values.

Consider another Cartesian system $S'(x'y'z')$. Let the origin point of S' coincide with that of S . The differences between the corresponding coordinates (x'_1, y'_1, z'_1) , and (x'_2, y'_2, z'_2) of the two points P_1 and P_2 as measured in the system S' are denoted by $\Delta x'_i \equiv x'_2 - x'_1$, $\Delta x'_2 \equiv y'_2 - y'_1$, and $\Delta x'_3 \equiv z'_2 - z'_1$. The coordinate transformations between $\Delta x'_i$ and Δx_i are given by

$$\Delta x'_i = D_{ij} \Delta x_j, \quad (1.2.4)$$

where D_{ij} are constants in x_i and t . These equations keep the three-dimensional space interval (1.2.1) invariant:

$$(\Delta s)^2 = (\Delta s')^2, \quad (1.2.5)$$

with

$$(\Delta s')^2 = (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2. \quad (1.2.6)$$

The transformations (1.2.4) are the orthogonal transformations where the transformation matrix (D_{ij}) satisfies the condition:

$$D_{ki} D_{li} = D_{ik} D_{il} = \delta_{kl}. \quad (1.2.7)$$

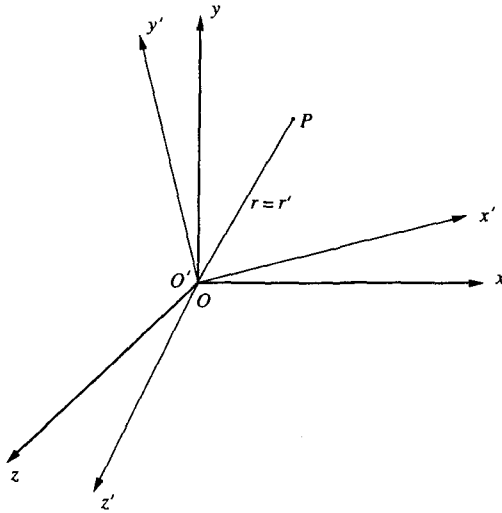


Fig. 1. Special rotation of Cartesian coordinate systems.

The inverse matrix $(D_{ij})^{-1}$ gives the inverse transformations of Eq. (1.2.4). Equations (1.2.4) together with Eqs. (1.2.7) define the group of orthogonal transformation, the $O(3)$ group.

It is easy to prove that the determinant of the matrix D_{ij} is equal to ± 1 :

$$\det(D_{ij}) \equiv |(D_{ij})| = \pm 1. \tag{1.2.8}$$

The value $+1$ of the determinant belongs to the proper rotations, the $SO(3)$ group, while the value -1 belongs to orthogonal transformations involving a reflection.

The transformations (1.2.4) give the general transformation law of vectors with respect to orthogonal transformations. In other words, a vector \mathbf{A} is defined as a set of three quantities A_i which transform like coordinate differences:

$$A'_j = D_{ji}A_i. \tag{1.2.9}$$

Therefore, when the vector components are given with respect to any one Cartesian coordinate system, they can be calculated with respect to every other Cartesian coordinate system.

The norm of the vector is defined as the sum of the squared vector components: $A = |\mathbf{A}| = (A_iA_i)^{1/2}$. The norm is an invariant with respect to the orthogonal transformations,

$$A'_iA'_i = A_iA_i. \tag{1.2.10}$$

The scalar product of two vectors $\mathbf{A} = (A_x, A_y, A_z) = (A_i, i = 1, 2, 3)$ and $\mathbf{B} = (B_x, B_y, B_z) = (B_i, i = 1, 2, 3)$ is defined as the sum of the products of corresponding vector components,

$$\mathbf{A} \cdot \mathbf{B} = A_i B_i = AB \cos \theta, \quad (1.2.11)$$

where $B = |\mathbf{B}|$, and θ is the angle between the directions of \mathbf{A} and \mathbf{B} . We see that the norm of a vector is the scalar product of the vector by itself. Thus the scalar product of two vectors is an invariant too.

In general, a tensor has N indices, all of which take all values 1 to 3. Thus, the tensor has 3^N components, which transform according to the transformation law

$$T'_{ijk\dots} = D_{il} D_{jm} D_{kn} \dots T_{lmn\dots} \quad (1.2.12)$$

The number of index, N , is referred to its rank. Scalars may be called tensors of rank 0. Vectors are tensors of rank 1. Quantities with $N \geq 2$ are called tensors of rank N .

The values of the Kronecker symbol (the three-dimensional Euclidean metric) keep invariant under the orthogonal transformations (1.2.4):

$$\delta'_{ij} = D_{ik} D_{jl} \delta_{kl} = D_{ik} D_{jk} = \delta_{ij}. \quad (1.2.13)$$

One can define other kinds of quantities, the tensor densities, which transform like the tensors, except that they are also multiplied by the transformation determinant (1.2.8). As long as this determinant equals +1, that is, when the transformation is a proper orthogonal transformation without reflection, there is no difference between a tensor and a tensor density. But a density undergoes a change of sign (compared with a tensor) when a reflection of the coordinate frame is carried out. A typical tensor density is the constant tensor density of rank 3, the Levi-Civita tensor density, defined as follows. ϵ_{ijk} is skewsymmetric in its three indices; therefore, all those components which have at least two indices equal vanish. The values of the non-vanishing components are ± 1 , the sign depending on whether (i, j, k) is an even or an odd permutation of $(1, 2, 3)$. The contraction of two Levi-Civita tensor densities is related to Kronecker symbols:

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}. \quad (1.2.14)$$

The vector product of \mathbf{A} and \mathbf{B} can be expressed in terms of the Levi-Civita tensor density as follows:

$$(\mathbf{A} \times \mathbf{B})_i = \epsilon_{ijk} A_j B_k. \quad (1.2.15)$$

1.2.2. Definitions of Time Coordinate

We have already defined a (space) frame of reference $S(xyz)$, and hence the location of a body can be designated by its three coordinates. However, in order to describe the equation of motion for the body, we must define a “common” time within the whole system. For this reason, assume that at each one of space points in the Cartesian frame $S(xyz)$ there is a standard clock at rest. A reading of the clock at each point defines a “local time” at that point. The name “local time” means that comparison among the readings of different clocks at different space points will have no physical meaning because those clocks have not yet been synchronized one another. The synchronization among the clocks at different space points is called the definition of simultaneity.

Consider two physical events $E_1(x_1, y_1, z_1, t_1)$ and $E_2(x_2, y_2, z_2, t_2)$ which occur at two different points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, respectively. Note that t_1 and t_2 are given by the two clocks C_{P_1} and C_{P_2} at the two points P_1 and P_2 , respectively. In other words, t_1 and t_2 represent only the so-called “local times” before C_{P_1} and C_{P_2} are synchronized each other. Thus, to compare the times of the two events, we must firstly synchronize the two clocks at those two points. A reading of a synchronized clock represents the “common” time of the frame, which is called the time coordinate t .

After completed the above steps, we can say that we have already constructed a general frame of reference $F(xyzt)$ which consists of a Cartesian frame $S(xyz)$ involving a well-defined “common” time t . Of course, definition of simultaneity, i.e., clock synchronization, is a complex problem to be discussed in section 1.3.

1.2.3. Definition of “Inertial” Frames

Definition of an “inertial” frame is based on the law of inertia. The law of inertia can be stated as follows: *Bodies when removed from interaction with other bodies will continue in their states of rest or straight-line uniform motion.*

A frame of reference $F(xyzt)$ is regarded as an *inertial* frame of reference if the law of inertia is valid within it. In other words, the motion of a free particle in the inertial frame of reference F is the inertial motion: The particle is unaccelerated,

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = 0, \quad \frac{d^2z}{dt^2} = 0,$$

or equivalently of the constant velocity,

$$\frac{dx}{dt} = \text{const}, \quad \frac{dy}{dt} = \text{const}, \quad \frac{dz}{dt} = \text{const}.$$

Note that for given any particle we can always introduce a frame of reference with respect to which it is at rest or of constant velocity and, therefore, unaccelerated. So the key point is that the particle is “free”, i.e., removed from any interactions with other bodies.

We know from the law of inertia that if F is an inertial frame of reference, then another frame F' moving at a constant velocity relative to F is an inertial frame of reference too.

1.3. Simultaneity and Clock Synchronization

We emphasize again that the key point for constructing an inertial frame is the clock synchronization, i.e., the definition of simultaneity. In the present section we shall discuss in detail clock synchronization.

Let us first imagine that we could find an idea signal, the velocity of which in a given frame had a certain value not depending upon the states of motion of its emitter. Therefore, one could use such a signal in clock synchronizing. We assume that the one-way velocity of that signal in the direction \mathbf{r}/r is c_r as seen in the frame F . Let C_O and C_P denote two standard clocks at rest at the origin $O(0, 0, 0)$ and the point $P(x, y, z)$, respectively. Consider that the signal is emitted from the origin O at the reading t_O of the clock C_O , and then reaches the point P at the reading t_P of the clock C_P . The clocks C_P and C_O are regarded as synchronization if and only if

$$t_P = t_O + \frac{r}{c_r}, \quad (1.3.1)$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance between the two points O and P . Of course, one can synchronize the two clocks by means of a signal propagating from P at the time t_P to O at the time t'_O . Similarly, C_O and C_P are regarded as synchronization if and only if

$$t'_O = t_P + \frac{x}{c_{-r}}, \quad (1.3.2)$$

where c_{-r} is the one-way velocity of the signal in the direction $-\mathbf{r}/r$. The time given by the synchronized clock is just the common time, the time coordinate of the frame.

In another point of view, the clock synchronization equations, (1.3.1) and (1.3.2), are equivalent to the definitions of the one-way velocities,

$$c_r = \frac{r}{t_P - t_O}, \quad c_{-r} = \frac{r}{t'_O - t_P}, \quad (1.3.3)$$

where the differences $(t_P - t_O)$ and $(t'_O - t_P)$ are called *the coordinate time intervals* which are given by the two clocks at the two different points and, hence, dependent on the definition of simultaneity. Therefore we can explain the clock synchronization equations (1.3.1) and (1.3.2) as follows: *The clocks C_O and C_P are synchronized in such a way that the one-way velocities of the signal as measured by use of the two clocks are c_r and c_{-r} .*

We now introduce the two-way velocity \bar{c}_r of the signal traveling from the point O to the point P and, by reflection, back to O . By definition we have

$$\bar{c}_r \equiv \frac{2r}{(t'_O - t_O)}, \quad (1.3.4)$$

where $(t'_O - t_O)$ is the difference between the two readings t'_O and t_O of the same clock C_O and, therefore, called *the proper time interval* not relevant to the definition of simultaneity. By using Eqs. (1.3.3), equation (1.3.4) becomes

$$\bar{c}_r = \frac{2r}{(t'_O - t_P) + (t_P - t_O)} = \frac{2}{(1/c_{-r}) + (1/c_r)} = \frac{2c_r c_{-r}}{c_r + c_{-r}}, \quad (1.3.5a)$$

or

$$\frac{1}{c_r} + \frac{1}{c_{-r}} = \frac{2}{\bar{c}_r}. \quad (1.3.5b)$$

This gives the relationship between the one-way and the two-way velocities of the signal. In contrast to the one-way speed, the two-way speed \bar{c}_r is a measurable quantity because it is related to the proper time interval $(t'_O - t_O)$.

The one-way velocity may be parameterized by introducing a directional parameter \mathbf{q} as follows:

$$c_r = \frac{\bar{c}_r}{1 - q_r}, \quad c_{-r} = \frac{\bar{c}_r}{1 + q_r}, \quad (1.3.6)$$

where $q_r = \mathbf{q} \cdot \mathbf{r}/r$.

Equation (1.3.5) implies that the choices of c_r and c_{-r} are restricted in such a way that the sense of cause is preserved. In other words, the signal starting at O cannot reach P before it leaves O . Since $(t_P - t_O)$ and $(t'_O - t_P)$ must be positive, so must c_r and c_{-r} be positive, we see from (1.3.5b) that neither can be smaller than $\bar{c}_r/2$. This leads to the restriction

$$\frac{\bar{c}_r}{2} \leq c_r(c_{-r}) \leq \infty. \quad (1.3.7)$$

Using (1.3.6) in (1.3.7), we get the limit on the directional parameter

$$-1 \leq q_r \leq +1. \quad (1.3.8)$$

This is to say that we have the following conclusion: If the one-way velocity of the signal were known in advance, then one could use it in clock synchronizing. In the above example, when the observer at the point $P(x, y, z)$ receives the signal, he should adjust the reading t_P of his clock C_P to the value that satisfies the clock synchronization equation (1.3.1); Conversely, if the clocks C_O and C_P were synchronized in advance by means of some given method, then they could be used for the measurement of the one-way velocity according to equation (1.3.3).

1.3.1. Newtonian Absolute Simultaneity

Newtonian absolute simultaneity is equivalent to postulating the existence of an instantaneously propagating signal with an infinite velocity, i.e., $c_r = c_{-r} = \bar{c}_r = \infty$. This, according to Eq. (1.3.1), implies that when the signal reaches the point P the reading t_P of the clock C_P should be adjusted to be the same as its emitting time t_O ,

$$t_P = t_O. \quad (1.3.9)$$

Similarly, when that signal reflected from P returns to O , we obtain from (1.3.2)

$$t'_O = t_P. \quad (1.3.10)$$

It is well known that in classical mechanics the time coordinate within an inertial frame of reference is defined by *Newtonian absolute simultaneity*. Such a frame denoted by $N(xyzt)$ will be called a Newtonian inertial frame of reference or a *Newtonian frame* for short, so as to distinguish it from an Einstein frame $F(xyzt)$.

We have known that there is no any *instantaneous signal* in nature and, therefore, the absolute simultaneity cannot be realized in any laboratory.

1.3.2. Einstein's Definition of Simultaneity

According to Einstein's second postulate [1], the (one-way) velocity of light in every inertial reference frame is the same constant c independent of the motion of the light source. In other words, Einstein chose a vanishing directional parameter, $\mathbf{q} = 0$, in each one of inertial frames. Thus, we have

$$c \equiv c_r = c_{-r} = \bar{c}_r. \quad (1.3.11)$$

This is the definition of *Einstein simultaneity*. Therefore, Einstein's synchronization between the clocks C_O and C_P at, respectively, the points $O(0, 0, 0)$ and $P(x, y, z)$ is obtained from Eq. (1.3.1)

$$t = t_O + \frac{r}{c}, \quad (1.3.12a)$$

or from Eq. (1.3.2)

$$t'_O = t + \frac{r}{c}, \quad (1.3.12b)$$

where the subscript " P " of t_P is omitted. Such a frame $F(xyzt)$, in where the time coordinate t is the reading of the clock C_P synchronized with the clock C_O , will be called an Einstein's inertial frame of reference, or an *Einstein frame* for short.

The clock synchronization equations (1.3.12) can be also explained as follows: *The clocks C_O and C_P are synchronized in such a way that the one-way velocity of light is the constant c as measured by the two clocks.*

It is well-known that one always use a light signal for the clock synchronization in a laboratory. Therefore Einstein's simultaneity can be directly realized in experiments. What we want to stress here is that only the two-way speed, but not the one-way speed, of light has been already measured in the experimental measurements, and hence the isotropy of the one-way velocity of light is just a postulate.

We shall see from Chap. 6 that a more general postulate, a choice of the anisotropy of the one-way velocity of light, together with the principle of relativity, would give the same physical predictions as Einstein's theory of special relativity.

1.3.3. *Edwards' Definition of Simultaneity*

A more general definition of simultaneity is the postulate made by Edwards (1963) [2]: *The two-way speed of light in a vacuum as measured in two coordinate systems moving with constant relative velocity is the same constant regardless of any assumptions concerning the one-way speed.*

Therefore, in the general inertial frame the directional parameter \mathbf{q} is chosen to be an arbitrary constant vector and the two-way velocity \bar{c} is a universal constant which is assumed by Edwards to be the same as c in Einstein's theory. Thus, we have from (1.3.6)

$$c_r = \frac{c}{1 - q_r}, \quad c_{-r} = \frac{c}{1 + q_r}, \quad \bar{c}_r = c. \tag{1.3.13}$$

This is the definition of *Edwards simultaneity*.

Edwards simultaneity may be explained as follows. Let \tilde{C}_O and \tilde{C}_P denote two standard clocks at rest at the origin $O(0, 0, 0)$ and the point $P(x, y, z)$, respectively, in the three-dimensional Cartesian system S . A light signal starting from the origin O at the reading \tilde{t}_O of the clock \tilde{C}_O reaches the point P at the reading \tilde{t}_P of the clock \tilde{C}_P , and then return to P at the reading \tilde{t}'_O of \tilde{C}_O . The clocks \tilde{C}_P and \tilde{C}_O are regarded as synchronization if and only if

$$\tilde{t} = \tilde{t}_O + \frac{r}{c_r}, \tag{1.3.14a}$$

or

$$\tilde{t}'_O = \tilde{t} + \frac{r}{c_{-r}}, \tag{1.3.14b}$$

where $\tilde{t} \equiv \tilde{t}_P$, and $r = \sqrt{x^2 + y^2 + z^2}$ is the distance between the two points O and P . The general inertial frame of reference with the Edwards (time) coordinate \tilde{t} is called *Edwards frame* $\tilde{F}(xyz\tilde{t})$.

It will be clear in section 6.5 that Edwards simultaneity is physically equivalent to Einstein simultaneity.

1.3.4. *Coordinate Transformation Between Einstein and Edwards Frames*

The relation between Einstein frame $F(xyzt)$ and Edwards frame $\tilde{F}(\tilde{x}\tilde{y}\tilde{z}\tilde{t})$ with $\tilde{x}_i = x_i$ can be obtained from their definitions. Let $S(xyz)$ be a Cartesian system of reference in a three-dimensional (vacuum) space. There are two standard clocks at the location of the origin $O(0, 0, 0)$ of the frame, which are denoted by C_O and \tilde{C}_O . It is similar that other two standard clocks, C_P and \tilde{C}_P , are resided at another point $P(x, y, z)$. Note that all the clocks are at rest within this Cartesian frame. Consider a light signal propagating from O to P . Let the readings of C_O and \tilde{C}_O be, respectively, t_O and \tilde{t}_O when the signal leaves O , and the readings of C_P and \tilde{C}_P be, respectively, t and \tilde{t} when the signal reaches P (see Fig. 2).

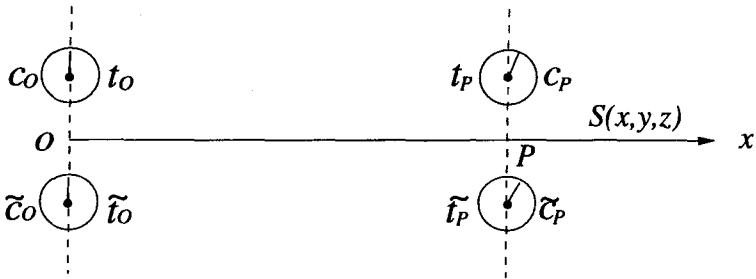


Fig. 2. Einstein clocks and Edwards clocks.

The clocks C_O and C_P are synchronized according to Einstein's definition of simultaneity, i.e., Eq. (1.3.12a),

$$t = t_O + \frac{r}{c}. \quad (1.3.15)$$

These clocks are named *Einstein clocks*. An *Einstein frame* $F(xyzt)$ is just the system $S(xyz)$ involving an *Einstein clock*, which defines the time coordinate t , at each one of space points.

On the other hand, the clocks \tilde{C}_O and \tilde{C}_P are synchronized according to Edwards' definition of simultaneity, i.e., Eq. (1.3.14a),

$$\tilde{t} = \tilde{t}_O + \frac{r}{c_r}. \quad (1.3.16)$$

We refer the clocks synchronized by use of (1.3.16) to *Edwards clocks*. An *Edwards frame* $\tilde{F}(\tilde{x}\tilde{y}\tilde{z}\tilde{t})$ with $\tilde{x}_i = x$ is just a three-dimensional Cartesian system $S(xyz)$ involving an *Edwards clock*, which give the time coordinate \tilde{t} , at every point.

We now discuss the necessary condition under which different definitions of simultaneity can be compared one another. Let us now consider the light signal being reflected from the point P back to the origin O . When the signal comes back to the point O , the readings of C_O and \tilde{C}_O are

$$t'_O = t + \frac{r}{c} \quad (1.3.17)$$

and

$$\tilde{t}'_O = \tilde{t} + \frac{r}{c-r}, \quad (1.3.18)$$

respectively.

It is emphasized here that the difference between the readings of the two clocks C_O and \tilde{C}_O must always be a constant because they are all the standard clocks and at rest at the same point O . For instance, we can choose the initial condition:

$$t_O - \tilde{t}_O = f, \tag{1.3.19}$$

where f is a constant. This is to say that the reading of C_O differs from the reading of \tilde{C}_O by the value f when the light signal leaves O . Then, the difference between the readings of C_O and \tilde{C}_O must be the same forever. This means that when the signal comes back to O we must have

$$t'_O - \tilde{t}'_O = f. \tag{1.3.20}$$

Then Eqs. (1.3.19) and (1.3.20) lead to

$$t_O - \tilde{t}_O = t'_O - \tilde{t}'_O = f$$

or

$$t'_O - t_O = \tilde{t}'_O - \tilde{t}_O. \tag{1.3.21}$$

In other words, the difference between the two successive readings of the clock C_O must be equal to that between the corresponding readings of the clock \tilde{C}_O . This implies that the two-way velocities of light in the two definitions of simultaneity must be the same. Thus, *the necessary condition under which different definitions of simultaneity can be compared each other is as follows: The two-way velocities of light obtained from the one-way velocities must have the same value.*

It is obvious that Eqs. (1.3.13)–(1.3.15) satisfy the condition (1.3.21), because the two-way speeds of light in Einstein's and Edwards' postulates are the same constant c . Therefore we can consider the difference between the Einstein and Edwards time coordinates t and \tilde{t} . To this end, we obtain from Eqs. (1.3.15) and (1.3.16)

$$t = \tilde{t} + r \left(\frac{1}{c} - \frac{1}{c_r} \right), \tag{1.3.22}$$

where we choose the initial condition $t_O - \tilde{t}_O = f = 0$. Using Eq. (1.3.13), we can express (1.3.22) in terms of the directional parameter as

$$t = \tilde{t} + q_r \frac{r}{c}. \tag{1.3.23a}$$

Of course, the space coordinates are the same:

$$x = \tilde{x}, \quad y = \tilde{y}, \quad z = \tilde{z}. \tag{1.3.23b}$$

Equations (1.3.23) are just the coordinate transformations between Einstein frame F and Edwards frame \tilde{F} . We shall see in section 6.5 that the difference between t and \tilde{t} given by Eq. (1.3.23a) is unobservable in experiments.

It is useful for understanding the definitions of simultaneity to note the following comparing: The transformation (1.3.23) between Einstein and Edwards frames is similar to the transformation (1.2.4) between any two three-dimensional Cartesian frames in the sense that the relative velocities of these frames are all zero.

1.3.5. Robertson's Definition of Simultaneity

In Einstein's and Edwards' postulates, the two-way velocity of light as measured in any one of inertial frames is the same constant c . A more general postulate relaxes this limit: The two-way velocity \bar{c}_r of light in a general inertial frame of reference, which is called the *Robertson frame* $\bar{F}(\bar{x}\bar{y}\bar{z}\bar{t})$, is equal to the one-way velocity of light, and is anisotropic and independent of the motion of the light source, i.e.,

$$\bar{c}_r = c_r = c_{-r}, \quad (1.3.24a)$$

where c_r and c_{-r} are the one-way velocities of light in the directions $\pm\bar{r}/\bar{r}$ with $\bar{r} = (\bar{z}, \bar{y}, \bar{x})$ and $\bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2}$. Furthermore we assume that the two-way velocity can be expressed as

$$\bar{c}_r = \frac{\bar{c}_{\parallel}\bar{c}_{\perp}}{\sqrt{\bar{c}_{\parallel}^2 + (\bar{c}_{\perp}^2 - \bar{c}_{\parallel}^2)\cos^2\alpha}}, \quad (1.3.24b)$$

where $\bar{c}_{\parallel} \equiv \bar{c}_x$ is the two-way velocity of light along the path of the \bar{x} -axis, \bar{c}_{\perp} is the two-way velocity in the direction perpendicular to the \bar{x} -axis, and α is the angle between the directions of \bar{r}/\bar{r} and the \bar{x} -axis, and in general $\bar{c}_{\parallel} \neq \bar{c}_{\perp}$.

The *Robertson frame* $\bar{F}(\bar{x}\bar{y}\bar{z}\bar{t})$ is just such a three-dimensional Cartesian inertial frame of reference $S(\bar{x}\bar{y}\bar{z})$ involving an *Robertson clock*, which will be defined below, at each point $(\bar{x}, \bar{y}, \bar{z})$, and \bar{t} are called Robertson coordinates).

Let a light signal starting from the origin point $O(0, 0, 0)$ at the reading \bar{t}_O of the clock \bar{C}_O at the origin reach the point $P(\bar{x}, \bar{y}, \bar{z})$ at the reading \bar{t} of the clock \bar{C}_P at that point, and then return to O at the reading \bar{t}'_O of \bar{C}_O . The clocks \bar{C}_P and \bar{C}_O are regarded as synchronization if and only if

$$\bar{t} = \bar{t}_O + \frac{\bar{r}}{\bar{c}_r}, \quad (1.3.25a)$$

or

$$\bar{t}'_O = \bar{t} + \frac{\bar{r}}{\bar{c}_r}, \quad (1.3.25b)$$

where \bar{c}_r is given by Eq. (1.3.24b) in terms of the two parameters \bar{c}_{\parallel} and \bar{c}_{\perp} . These clocks synchronized according to Eq. (1.3.25) are called *Robertson clocks*, which define the time coordinate \bar{t} of the frame \bar{F} . In other words, *Robertson simultaneity* [3] in the frame \bar{F} is defined in such a way that the velocities of light are given by Eq. (1.4.24).

It is obvious that we can distinguish the Robertson simultaneity from the Einstein simultaneity by experiments, because any possible anisotropy of the two-way speeds of light could be tested.

1.3.6. Mansouri–Sexl's Definition of Simultaneity

$F^*(x^*y^*z^*t^*)$ is called a Mansouri–Sexl inertial system of reference or a *MS frame* in short, where (x^*, y^*, z^*) are the coordinates of a Cartesian frame of reference $S(x^*y^*z^*)$ and t^* is the time coordinate defined by the Mansouri–Sexl (MS) simultaneity [4]. The MS simultaneity is a generalization of the Robertson simultaneity, just as the Edwards simultaneity is a generalization of the Einstein simultaneity.

Therefore, in the MS frame F^* the velocity of light is given by

$$c_r = \frac{\bar{c}_r}{1 - q_r}, \quad c_{-r} = \frac{\bar{c}_r}{1 + q_r}, \quad (1.3.26a)$$

$$\bar{c}_r = \frac{\bar{c}_{\parallel}\bar{c}_{\perp}}{\sqrt{\bar{c}_{\parallel}^2 + (\bar{c}_{\perp}^2 - \bar{c}_{\parallel}^2) \cos^2 \alpha}}, \quad (1.3.26b)$$

$$-1 \leq q_r \leq +1, \quad (1.3.26c)$$

where c_r and c_{-r} are the one-way velocities of light in the directions $\pm \mathbf{r}^*/r^*$ respectively, $\mathbf{r}^* = (x^*, y^*, z^*)$, $r^* = \sqrt{x^{*2} + y^{*2} + z^{*2}}$, \bar{c}_r is the corresponding two-way speed which is expressed as a function of the two parameters \bar{c}_{\parallel} and \bar{c}_{\perp} , and $q_r = \mathbf{q} \cdot \mathbf{r}/r$ is a directional parameter. In case of $q_r = 0$, equation (1.3.26a) reduces to Eq. (1.3.24a), i.e., the MS simultaneity reduces to the Robertson simultaneity. So we can compare the difference between the two definitions of simultaneity.

The MS simultaneity or MS clock synchronization may be stated as follows: As it does in the previous sections, let a light signal starting from the point $O(0, 0, 0)$ at the reading t_O^* of the clock C_O^* at that point reach the point $P(x^*, y^*, z^*)$ at the reading t^* of the clock C_P^* at that point, and then return to O at the reading $t_O'^*$ of C_O^* . The clocks C_P^* and C_O^* are regarded as synchronization if and only if

$$t^* = t_O^* + \frac{r^*}{c_r}, \quad (1.3.27a)$$

or

$$t_O'^* = t^* + \frac{r^*}{c_{-r}}, \quad (1.3.27b)$$

where c_r and c_{-r} are given by Eq. (1.2.26a). These equations are called the *MS clock synchronization* equations or the definition of the *MS simultaneity*, and C_O^* and C_P^* are called the *MS clocks*.

Therefore, the definition of the MS frame $F^*(x^*y^*z^*t^*)$ can be stated as follows: *The MS frame F^* is just such a Cartesian inertial frame of reference $S(x^*y^*z^*)$ involving the MS clocks which define the Mansouri–Sexl time coordinate t^* .*

In other words, the space coordinates are given by the usual Cartesian frame $S(x^*y^*z^*)$, and the time coordinate t^* is defined in such a way that the one-way velocity of light should be expressed by Eqs. (1.3.26).

1.3.7. Coordinate Transformation Between the Robertson and MS Frames

It is known that the two-way velocity of light in the Robertson frame is the same as that in the MS frame. Thus, we can consider the difference between these two kinds of frames.

As done in section 1.3.4, let the space coordinates of the Robertson frame $\tilde{F}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$ coincide with those of the MS frame $F^*(x^*y^*z^*t^*)$, i.e.,

$$\tilde{x} = x^*, \quad \tilde{y} = y^*, \quad \tilde{z} = z^*. \quad (1.3.28a)$$

The time coordinates \tilde{t} and t^* are defined by (1.3.25) and (1.3.27) respectively, and thus the transformation between them is

$$\tilde{t} = t^* + \bar{r} \left(\frac{1}{\bar{c}_r} - \frac{1}{c_r} \right) = t^* + q_r \frac{r^*}{\bar{c}_r}, \quad (1.3.28b)$$

where $r^* = \bar{r}$, and the initial condition $\tilde{t}_O = t^*_O$ is chosen for simplicity. Equations (1.3.28) are called the coordinate transformations between the Robertson and MS frames. Comparing the transformation (1.3.28) with the transformation (1.3.23), we can see that the only difference between them is the different two-way velocities of light, i.e., $\bar{c}_r \neq c$. So that the MS simultaneity is a generalization of Robertson simultaneity, just as Edwards simultaneity is a generalization of Einstein simultaneity.

1.4. Principle of Relativity

Let us first recall Galilean principle of relativity. It is well known that the choice of a frame of reference determines the form of a law of nature. Using the inertial frame of reference defined in the previous sections, the law of inertia takes its simple form: In the absence of forces, the space coordinates of a mass point are linear functions of the time coordinate; Inversely, of course, that the space coordinates of a free particle takes linear form of time coordinate is just the definition of an inertial frame of reference.

Galilean principle of relativity says that all of the laws of mechanics take the same form when stated in terms of any one of the Newtonian frames.

In Einstein's theory of special relativity, the first postulate, the principle of special relativity, is simply a generalization of the Galilean principle of relativity. Of course, the definition of the inertial frames of reference in special relativity differs from that in the Newton-Galileo space-time theory. Einstein's principle of relativity says that all of the laws of physics take the same form when stated in terms of any one of Einstein frames.

1.5. Velocity and Simultaneity

In the previous sections, we have given the definitions of the inertial frames. Thus, we can now describe the motion of any body, and define any physical quantities relevant to the body. An important quantity is the velocity of the body, which is denoted by $\mathbf{u} = (u_x, u_y, u_z)$. By definitions, we have

$$u_x = \frac{dx}{dt}, \quad u_y = \frac{dy}{dt}, \quad u_z = \frac{dz}{dt}. \quad (1.5.1)$$

We want to stress that dt is a *coordinate time interval* that depends upon the definition of simultaneity. This implies that the velocity of any given body would have different values corresponding to different definitions of simultaneity. Consider the following example. Let a particle move from the point $O(0, 0, 0)$ to another point $P(dx, 0, 0)$, and the distance between O and P be dx . Let two clocks C_O and \tilde{C}_O be at rest at O , while two other clocks C_P and \tilde{C}_P be at rest at P . The clocks C_O and C_P are two Einstein clocks, which have been synchronized by use of a light signal according to Eq. (1.3.12a),

$$t_P = t_O + \frac{dx}{c}. \quad (1.5.2a)$$

On the other hand, the clocks \tilde{C}_O and \tilde{C}_P are two Edwards clocks, which have been synchronized by use of a light signal according to equation (1.3.14a),

$$\tilde{t}_P = \tilde{t}_O + \frac{dx}{c_x}. \quad (1.5.2b)$$

For simplicity, we choose the initial condition $t_O = \tilde{t}_O$. This means that the readings of the two clocks C_O and \tilde{C}_O are equal to, and furthermore will be forever equal to, each other. However, the readings of the clocks C_P and \tilde{C}_P are different, the difference between them being the value from (1.5.2),

$$t_P - \tilde{t}_P = dx \left(\frac{1}{c} - \frac{1}{c_x} \right) = q_x \frac{dx}{c}, \quad (1.5.3)$$

where Eq. (1.3.13) is used. Note that after the two clocks C_P and \tilde{C}_P are synchronized according to Eqs. (1.5.2a) and (1.5.2b) respectively, the difference between them will forever take the same value given by Eq. (1.5.3). Let us now establish the time coordinates of a moving body by means of these synchronized clocks. Consider the body moving from the point O to the point P . Let the time $t_O = \tilde{t}_O = 0$ when the body leaves O , and the readings of C_P and \tilde{C}_P be dt and $d\tilde{t}$, respectively, when the body reaches P . Then, the difference between $d\tilde{t}$ and dt should be equal to the difference $(t_P - \tilde{t}_P)$ given by Eq. (1.5.3), because, as mentioned above, the two synchronized clocks will preserve the relative difference between their readings. Thus, we have

$$dt - d\tilde{t} = q_x \frac{dx}{c}. \quad (1.5.4)$$

By definitions, the velocities of the body are given by

$$u_x \equiv \frac{dx}{dt}, \quad \tilde{u}_x \equiv \frac{dx}{\tilde{dt}}, \quad (1.5.5)$$

where u_x and \tilde{u}_x are, respectively, called *Einstein velocity* and *Edwards velocity*, which are measured by use of Einstein clocks and Edwards clocks, respectively. Using (1.5.4) in (1.5.5), we obtain the relationship between u_x and \tilde{u}_x ,

$$\tilde{u}_x = \frac{u_x}{1 - q_x u_x / c}, \quad (1.5.6a)$$

or inversely,

$$u_x = \frac{\tilde{u}_x}{1 + q_x \tilde{u}_x / c}. \quad (1.5.6b)$$

Similarly, if the body moves from $P(dx, 0, 0)$ to $O(0, 0, 0)$, then using Eqs. (1.3.12b) and (1.3.14b) we get

$$u_{-x} = \frac{\tilde{u}_{-x}}{1 - q_x \tilde{u}_{-x} / c}, \quad (1.5.6c)$$

where $u_{-x} = |\mathbf{u}_{-x}|$ and $\tilde{u}_{-x} = |\tilde{\mathbf{u}}_{-x}|$ are, respectively, Einstein velocity and Edwards velocity.

This is to say that the velocity for a given moving body would take different values according to different definitions of simultaneity. A similar conclusion is also valid for any quantity related to the definition of simultaneity. For instance, the validity of the reciprocity of relative velocities depends upon the definitions of simultaneity.

The reciprocity of relative velocities can be stated as follows: For any two inertial frames, e.g., \tilde{F} and \tilde{F}' , the relative velocity of \tilde{F}' to \tilde{F} is equal to v if and only if the relative velocity of \tilde{F} to \tilde{F}' is equal to $-v$.

We now discuss the *condition of reciprocity of relative velocities*. At first we know that for any two Einstein frames, F and F' , the reciprocity of relative velocities is valid (see Lorentz transformation in section 2.2):

$$\mathbf{v}' = -\mathbf{v}, \quad (1.5.7)$$

where \mathbf{v} is the relative velocity of F' to F as seen from F , and \mathbf{v}' is the relative velocity of F to F' as seen from F' . In terms of the norms of \mathbf{v}' and \mathbf{v} , equation (1.5.7) becomes

$$v' = v. \quad (1.5.8)$$

Using the relations between the Einstein velocities (v and v') and the Edwards velocities (\tilde{v} and \tilde{v}') given by Eqs. (1.5.6b,c), equation (1.5.8) becomes

$$\frac{\tilde{v}'}{1 - q'_x \tilde{v}' / c} = \frac{\tilde{v}}{1 + q_x \tilde{v} / c}, \quad (1.5.9)$$

where q_x and q'_x are the directional parameters within \tilde{F} and \tilde{F}' , respectively. The reciprocity of the relative velocities implies

$$\tilde{v}' = \tilde{v}. \quad (1.5.10)$$

Substituting (1.5.10) into (1.5.9), we get the condition of reciprocity of relative velocities,

$$q'_x = -q_x. \quad (1.5.11)$$

This result can be also obtained from Edwards transformations (6.3.4). This shows that the reciprocity of the relative velocities must not require the vanishing directional parameters; In other words, the constancy of the velocity of light (i.e., $\mathbf{q}' = \mathbf{q} = 0$) is simply a special case that the reciprocity of the relative velocities is valid. (John A. Winnie (1970)[5] assumed $q' = q$, so that he omitted the above result).

We have considered above the relations between Edwards' quantities and Einstein's quantities, while relations between Mansouri-Sexl's (MS') and Robertson's quantities can be found in Sec. 7.3.