Chapter 1

Manifolds

1.1 Preliminaries

§1.1.1 Space and Coordinatization

Mathematics is a natural science with a special modus operandi. It replaces concrete natural objects with mental abstractions which serve as intermediaries. One studies the properties of these abstractions in the hope they reflect facts of life. So far, this approach proved to be very productive.

The most visible natural object is the Space, the place where all things happen. The first and most important abstraction of math is the notion of number. Loosely speaking, the aim of this book is to illustrate how these two concepts, Space and Number, fit together.

It is safe to say that geometry as a rigorous science is a creation of ancient Greeks. Euclid proposed a method of research which was later adopted by the entire mathematics. We refer of course to the axiomatic method. He viewed the space as a collection of points and distinguished some basic objects in the space such as lines, planes etc. He then postulated certain (natural) relations between them. All other properties were derived from these simple axioms.

Euclid's work is a masterpiece of mathematics and has produced many interesting results but it has its own limitations. For example, the most complicated figures one could reasonably study using this method are the conics and/or quadrics and the Greeks certainly did this. A major breakthrough in geometry was the discovery of **coordinates** by René Descartes in the 17th century. Numbers were put to work in the study of space. Their idea of producing what is now commonly referred to as Cartesian coordinates is familiar to any undergraduate. These coordinates are obtained using a very special method (in this case using three concurrent, pairwise perpendicular lines each one endowed with an identification with $\mathbb{R}$). What is important here is that they produced an one-to-one mapping

$$\text{Euclidian Space} \to \mathbb{R}^3 \qquad\qquad P \mapsto (x(P), y(P), z(P)).$$

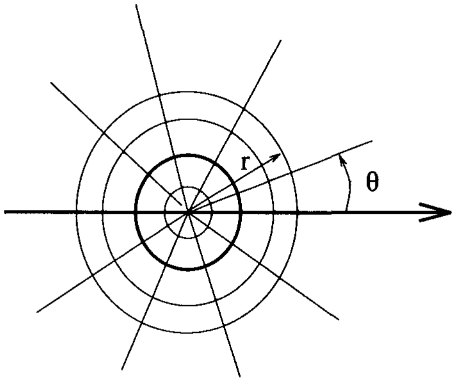

Figure 1.1: *Polar coordinates*

We call such a process **coordinatization**. The corresponding map is called (in this case) *Cartesian system of coordinates.* A line or a plane becomes via coordinatization an algebraic object (more precisely an equation).

In general, any coordinatization replaces geometry by algebra and we get a two-way correspondence

$$\text{Study of Space} \longleftrightarrow \text{Study of Equations.}$$

The shift from geometry to numbers is beneficial to geometry as long as one has efficient tools do deal with numbers and equations. Fortunately, about the same time with the introduction of coordinates Newton created the differential and integral calculus which opened new horizons in the study of equations.

The Cartesian system of coordinates is by no means the unique or the most useful coordinatization. Concrete problems dictate other choices. For example, the polar coordinates represent another coordinatization of (a piece of the plane) (see Figure 1.1).

$$P \mapsto (r(P), \theta(P)) \in (0, \infty) \times (-\pi, \pi).$$

This choice is related to the Cartesian choice by the well known formulae

$$x = r\cos\theta \quad y = r\sin\theta. \tag{1.1.1}$$

A remarkable feature of (1.1.1) is that $x(P)$ and $y(P)$ depend smoothly upon $r(P)$ and $\theta(P)$.

As science progressed so did the notion of Space. One can think of Space as a *configuration set*, i.e. the collection of all possible states of a certain phenomenon. For example, we know from the principles of dynamics that the motion of a particle in the ambient space can be completely described if we know the position and the velocity of the particle at a given moment. The space associated with this problem consists of all pairs *(position, velocity)* a particle can possibly have. We can coordinatize this space using 6 functions: three of them will describe the position and the other three of them will describe the velocity. We say the configuration space is 6-dimensional. We cannot visualize this space but it helps to think of it as an Euclidian space, only "roomier".

There are many ways to coordinatize the configuration space of a motion of a particle and for each choice of coordinates we get a different description of the motion. Clearly, all these descriptions must "agree" in some sense sense since they all reflect the same phenomenon. In other words, these descriptions should be **independent of coordinates**. Differential geometry studies the objects which are independent of coordinates.

The coordinatization process had been used by people centuries before mathematicians accepted it as a method. For example, sailors used it to travel from one point to another on Earth. Each point has a latitude and a longitude which completely determines its position on Earth. This coordinatization is not a global one. There exist four domains delimited by the Equator and the Greenwich meridian and each of is then naturally coordinatized. The points on the equator for example admit two different coordinatizations which are smoothly related.

The manifolds are precisely those spaces which can be piecewise coordinatized (with smooth correspondence on overlaps) and the intention of this book is to introduce the reader to the problems and the methods which arise in the study of manifolds. The next section is a technical interlude. We will review the implicit function theorem which will be one of the basic tools for detecting manifolds.

§1.1.2 The implicit function theorem

We gather here, without proofs, a collection of classical analytical facts. For more details one can consult [22].

Let X and Y be two Banach spaces and denote by $L(X,Y)$ the space of bounded linear operators $X \to Y$.

Definition 1.1.1 Let $F : U \subset X \to Y$ be a continuous function (U is an open subset of X). F is said to be (Frechet) differentiable at $u \in U$ if there exists $T \in L(X,Y)$ such that

$$\|F(u_0 + h) - F(u_0) - Th\|_Y = o(\|h\|_X) \ \text{ as } h \to 0.$$

Loosely speaking, a continuous function is differentiable at a point if near that point it admits a " best approximation " by a linear map.

When F is differentiable at $u_0 \in U$, the operator T in the above definition is uniquely determined by

$$Th = \frac{d}{dt}\,|_{t=0}\, F(u_0 + th) = \lim_{t \to 0} \frac{1}{t}\left(F(u_0 + th) - F(u_0)\right).$$

We will use the notation $T = D_{u_0}F$ and we will call T the **Frechet derivative** of F at u_0. Assume $F : U \to Y$ is differentiable at each point $u \in U$. Then F is said to be of class C^1 if the map $u \mapsto D_u F \in L(X,Y)$ is continuous. F is said to be of class C^2 if $u \mapsto D_u F$ is of class C^1. One can define inductively C^k and C^∞ (or **smooth**) maps.

Example 1.1.2 Consider $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$. Using Cartesian coordinates $x = (x^1, \cdots, x^n)$ in $\mathbb{R}^n$ and $u = (u^1, \cdots, u^m)$ in $\mathbb{R}^m$ we can think of F as a collection of m functions on U

$$u^1 = u^1(x^1, \cdots, x^n), \cdots, u^m = u^m(x^1, \cdots, x^n).$$

F is differentiable at a point $x \in U$ if and only if the functions u^i are differentiable at x in the usual sense of calculus. The Frechet derivative of F at x is the linear operator $D_x F : \mathbb{R}^n \to \mathbb{R}^n$ given by the Jacobian matrix

$$D_x F = \frac{\partial(u^1, \cdots, u^m)}{\partial(x^1, \cdots, x^n)} = \left(\frac{\partial u^i}{\partial x^j}\right)_{1 \le i \le m,\, 1 \le j \le n}.$$

F is smooth if and only if the functions $u^i(x)$ are smooth.

Exercise 1.1.1 (a) Let $\mathcal{U} \subset L(\mathbb{R}^n, \mathbb{R}^n)$ denote the set of invertible $n \times n$ matrices. Show that $\mathcal{U}$ is an open set.
(b) Let $F : \mathcal{U} \to \mathcal{U}$ be defined as $A \to A^{-1}$. Show that $D_A F(H) = -A^{-1} H A^{-1}$ for any $n \times n$ matrix H.
(c) Show the Frechet derivative of the map $\det : L(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}$ ($A \mapsto \det A$) at $\mathbf{1}_{\mathbb{R}^n} \in L(\mathbb{R}^n, \mathbb{R}^n)$ is given by

$$D_A \,|_{A=1} \det(H) = \operatorname{tr} H, \;\; \forall H \in L(\mathbb{R}^n, \mathbb{R}^n).$$

Theorem 1.1.3 (Inverse function theorem) Let X, Y be two Banach spaces and $F : U \subset X \to Y$ a smooth function. If at a point $u_0 \in U$ the derivative $D_{u_0}F \in L(X,Y)$ is invertible, then there exits a neighborhood U_1 of u_0 in U such that $F(U_1)$ is an open neighborhood of $v_0 = F(u_0)$ in Y and $F : U_1 \to F(U_1)$ is bijective, with smooth inverse.

The spirit of the theorem is very clear: the invertibility of the derivative $D_{u_0}F$ "propagates" (locally) to F because $D_{u_0}F$ is a very good local approximation for F.
More formally, if we set $T = D_{u_0}F$, then

$$F(u_0 + h) = F(u_0) + Th + r(h)$$

where $r(h) = o(\|h\|)$ as $h \to 0$. The theorem states that for every v sufficiently close to v_0 the equation $F(u) = v$ has a unique solution $u = u_0 + h$ with h very small. To prove the theorem one has to show that for $\|v - v_0\|_Y$ sufficiently small the equation below

$$v_0 + Th + r(h) = v$$

has a unique solution. We can rewrite the above equation as

$$Th = v - v_0 - r(h)$$

or

$$h = T^{-1}(v - v_0 - r(h)).$$

The last equation is a fixed point problem which can be approached successfully via the Banach fixed point theorem.

Theorem 1.1.4 (Implicit function theorem) Let X, Y, Z be Banach spaces and $F : X \times Y \to Z$ a smooth map. Let $(x_0, y_0) \in X \times Y$ and set $z_0 = F(x_0, y_0)$. Set $F_2 : Y \to Z$, $F_2(y) = F(x_0, y)$. Assume $D_{y_0}F_2 \in L(Y, Z)$ is invertible. Then there exist neighborhoods U of $x_0 \in X$, V of $y_0 \in Y$ and a smooth map $G : U \to V$ such that $S = F^{-1}(z_0) \cap (U \times V)$ is the graph of G i.e.

$$\{(x, y) \in U \times V \; ; \; F(x, y) = z_0\} = \{(x, G(x)) \in U \times V \; ; \; x \in U\}.$$

Proof Consider the map $H : X \times Y \to X \times Z$, $\xi = (x, y) \mapsto (x, F(x, y))$. H is a smooth map and at $\xi_0 = (x_0, y_0)$ its derivative $D_{\xi_0}H : X \times Y \to X \times Z$ has the block decomposition

$$D_{\xi_0}H = \begin{bmatrix} \mathbf{1}_X & 0 \\ D_{\xi_0}F_1 & D_{\xi_0}F_2 \end{bmatrix}.$$

Above, DF_1 (resp. DF_2) denotes the derivative of $x \mapsto F(x, y_0)$ (resp. the derivative of $y \mapsto F(x_0, y)$). $D_{\xi_0}H$ is invertible and its inverse has the block decomposition

$$(D_{\xi_0}H)^{-1} = \begin{bmatrix} \mathbf{1}_X & 0 \\ (D_{\xi_0}F_2)^{-1} D_{\xi_0}F_1() & (D_{\xi_0}F_2)^{-1} \end{bmatrix}.$$

Thus, by the inverse function theorem the equation

$$(x, F(x, y)) = (x, z_0)$$

has a unique solution $(\tilde{x}, \tilde{y}) = H^{-1}(x, z_0)$ in a neighborhood of (x_0, y_0). It obviously satisfies $\tilde{x} = x$ and $F(\tilde{x}, \tilde{y}) = 0$. Hence, the set $\{(x, y) \; ; \; F(x, y) = z_0\}$ is locally the graph of $x \mapsto H^{-1}(x, z_0)$. □

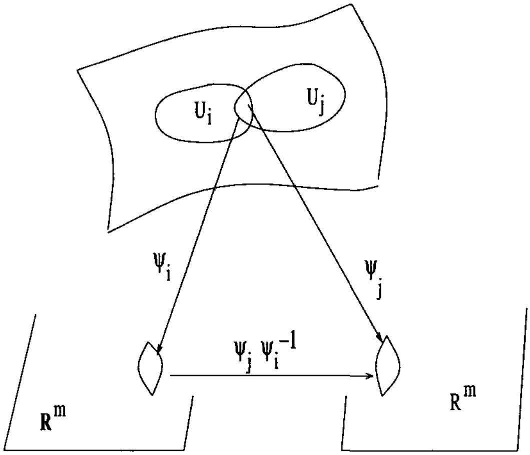

Figure 1.2: *Transition maps*

1.2 Smooth manifolds

§1.2.1 Basic definitions

We now introduce the object which will be the main focus of this book, namely we will define the concept of (smooth) manifold. It formalizes the general principles outlined in Sec. 1.1.1.

Definition 1.2.1 A **smooth manifold** of dimension m is a locally compact Hausdorff space M together with the following collection of data (henceforth called **atlas** or **smooth structure**) consisting of:
(a) an open cover $\{U_i\}_{i\in I}$ of M;
(b) continuous, injective maps $\Psi_i : U_i \to \mathbb{R}^m$ (called **charts** or **local coordinates**) such that if $U_i \cap U_j \neq \emptyset$ the transition map

$$\Psi_j \circ \Psi_i^{-1} : \Psi_i(U_i \cap U_j) \subset \mathbb{R}^m \to \Psi_j(U_i \cap U_j) \subset \mathbb{R}^m$$

is smooth. (We say the various charts are compatible; see Figure 1.2).

The chart Ψ_i can be viewed as a collection of m functions $(x^1, \ldots, x^m)$ on U_i and similarly, we can view Ψ_j as a collection of functions $(y^1, \ldots, y^m)$. The transition map $\Psi_j \circ \Psi_i^{-1}$ can then be interpreted as a collection of maps

$$(x^1, \ldots, x^m) \mapsto (y^1(x^1, \ldots, x^m), \cdots, y^m(x^1, \ldots, x^m)).$$

The first and the most important example of manifold is $\mathbb{R}^n$ itself. The natural smooth structure consists of an atlas with a single chart $\mathbf{1}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$. To construct more examples we will use the implicit function theorem .

Definition 1.2.2 (a) Let M, N be two smooth manifolds of dimensions m and respectively n. A continuous map $f : M \to N$ is said to be **smooth** if for any local chart ϕ on M and ψ on N the composition $\psi \circ f \circ \phi^{-1}$ (whenever this makes sense) is a smooth map $\mathbb{R}^m \to \mathbb{R}^n$.
(b) A smooth map $f : M \to N$ is called a **diffeomorphism** if it is invertible and its inverse is also a smooth map

Example 1.2.3 The map $t \mapsto e^t$ is a diffeomorphism $(-\infty, \infty) \to (0, \infty)$. The map $t \mapsto t^3$ is a homeomorphism $\mathbb{R} \to \mathbb{R}$ but it is not a diffeomorphism!

If M is a smooth manifold we will denote by $C^\infty(M)$ the linear space of all smooth functions $M \to \mathbb{R}$.

Remark 1.2.4 Let U be an open subset of the smooth manifold M ($\dim M = m$) and $\psi : U \to \mathbb{R}^m$ a smooth, one-to one map with open image and smooth inverse. Then ψ defines local coordinates over U compatible with the existing atlas of M. Thus (U, ψ) can be added to the original atlas and the new smooth structure is diffeomorphic with the initial one. Using Zermelo's Axiom we can produce a **maximal atlas** (no more compatible local chart can be added to it).

Our next result is a general recipe for producing manifolds. Historically, this is how manifolds entered mathematics.

Proposition 1.2.5 Let M be a smooth manifold of dimension m and $f_1, \ldots, f_k \in C^\infty(M)$. Define

$$\mathcal{Z} = \mathcal{Z}(f_1, \ldots, f_k) = \{p \in M \ ; \ f_1(p) = \ldots = f_k(p) = 0\}.$$

Assume the functions $f_1, \ldots, f_k$ are functionally independent along $\mathcal{Z}$ i.e., for each $p \in \mathcal{Z}$ the matrix $\left(\frac{\partial f_i}{\partial x^j}\right)_{1 \leq i \leq k, 1 \leq j \leq m}$ has rank k. Here $x = (x^1, \ldots, x^m)$ denotes a local chart of M near p. Then $\mathcal{Z}$ has a natural structure of smooth manifold of dimension $m - k$.

Proof **Step 1: Constructing the charts** Let $p_0 \in \mathcal{Z}$ and denote by $(x^1, \ldots, x^m)$ local coordinates near p_0 such that $x^i(p_0) = 0$. One of the $k \times k$ minors of the matrix $\left(\frac{\partial f_i}{\partial x^j}\right)_{1 \leq i \leq k, 1 \leq j \leq m}$ is non degenerate. Assume this is determined by the last k columns (and all the k lines). We can now think of the functions $f_1, \ldots, f_k$ as defined on an open subset U of $\mathbb{R}^m$. Split $\mathbb{R}^m$ as $\mathbb{R}^{m-k} \times \mathbb{R}^k$ and set $x' = (x^1, \ldots, x^{m-k})$, $x'' = (x^{m-k+1}, \ldots, x^m)$. We are now in the setting of the implicit function theorem with $X = \mathbb{R}^{m-k}$, $Y = \mathbb{R}^k$, $Z = \mathbb{R}^k$ and $F : X \times Y \to Z$ is $x \mapsto (f_1(x), \ldots, f_k(x))$. In this case $DF_2 = \left(\frac{\partial F}{\partial x''}\right)$ is invertible since it corresponds to our non degenerate minor. Thus, in a neighborhood U of p_0, the set $\mathcal{Z}$ is the graph of some function $g : \mathbb{R}^{m-k} \to \mathbb{R}^k$, $x' \mapsto x'' = g(x')$

$$\mathcal{Z} \cap U = \{(x', g(x')) \ ; \ |x'| \text{ small}\}.$$

We now define $\psi_{p_0} : \mathcal{Z} \cap U \to \mathbb{R}^{m-k}$ by

$$p = (x', g(x')) \mapsto x' \in \mathbb{R}^{m-k}.$$

ψ_{p_0} is a local chart of $\mathcal{Z}$ near p_0.

Step2 The transition maps for the charts constructed above are smooth. The details are left to the reader. □

Exercise 1.2.1 Complete Step 2 in the proof of Proposition1.2.5.

Definition 1.2.6 Let M be a m-dimensional manifold. A **codimension k submanifold** of M is a subspace $N \subset M$ locally defined as the common zero locus of functionally independent functions $f_1, \cdots, f_k \in C^\infty(M)$.

Proposition1.2.5 shows that any submanifold has a natural smooth structure so it becomes a manifold *per se.*

§1.2.2 Partitions of unity

This is a very brief technical subsection describing a trick we will extensively use in this book.

Definition 1.2.7 Let M be a smooth manifold and $(U_\alpha)_{\alpha\in\mathcal{A}}$ an open cover of M. A (smooth) **partition of unity** subordinated to this cover is a family $(f_\beta)_{\beta\in\mathcal{B}} \subset C^\infty(M)$ satisfying the following conditions.
(i) $0 \leq f_\beta \leq 1$
(ii) $\exists \phi : \mathcal{B} \to \mathcal{A}$ such that $\operatorname{supp} f_\beta \subset U_{\phi(\beta)}$.
(iii) The family $(\operatorname{supp} f_\beta)$ is locally finite i.e. any point $x \in M$ admits an open neighborhood intersecting only finitely many supports $\operatorname{supp} f_\beta$.
(iv) $\sum_\beta f_\beta(x) = 1$ for all $x \in M$.

We include here for the reader's convenience the basic existence result concerning partitions of unity. For a proof we refer to [73].

Proposition 1.2.8 (a) For any open cover $\mathfrak{U} = (U_\alpha)_{\alpha\in\mathcal{A}}$ of a smooth manifold M there exists at least one smooth partition of unity $(f_\beta)_{\beta\in\mathcal{B}}$ subordinated to $\mathfrak{U}$ such that $\operatorname{supp} f_\beta$ is compact for any β.
(a) If we do not require compact supports then we can find a partition of unity in which $\mathcal{B} = \mathcal{A}$ and $\phi = \mathbf{1}_\mathcal{A}$.

Exercise 1.2.2 Let M be a smooth manifold and $S \subset M$ a **closed** submanifold. Prove that the restriction map

$$r : C^\infty(M) \to C^\infty(S) \quad f \mapsto f|_S$$

is surjective.

§1.2.3 Examples

Manifolds are everywhere and in fact, to many physical phenomena which can be modeled mathematically one can naturally associate a manifold. On the other hand, many problems in mathematics find their most natural presentation using the language of manifolds. To give the reader an idea of the scope and extent of modern geometry we present here a short list of examples of manifolds. This list will be enlarged as we enter deeper into the study of manifolds.

Example 1.2.9 (The round n-dimensional sphere) This is the codimension 1 submanifold of $\mathbb{R}^{n+1}$ given by the equation

$$|x|^2 = \sum_{i=0}^{n}(x^i)^2 = r^2, \quad x = (x^0, \ldots, x^n) \in \mathbb{R}^{n+1}.$$

One checks that, along the sphere, the differential of $|x|^2$ is nowhere zero so by Proposition 1.2.5 S^n is indeed a smooth manifold. In this case one can explicitly construct an atlas (consisting of two charts) which is useful in many applications. The construction relies on **stereographic projections**. Let N and S denote the north and resp. south pole of S^n ($N = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$, $S = (0, \ldots, 0, -1) \in \mathbb{R}^{n+1}$). Consider the open sets $U_N = S^n \setminus \{N\}$ and $U_S = S^n \setminus \{S\}$. They form an open cover of S^n. The stereographic projection from the north pole is the map $\sigma_N : U_N \to \mathbb{R}^n$ such that for any $P \in U_N$, $\sigma_N(P)$ is the intersection of the line NP with the hyperplane $\{x^n = 0\} \cong \mathbb{R}^n$. The stereographic projection from the south pole is defined similarly. For $P \in U_N$ denote by $(y^1(P), \cdots, y^n(P))$ the coordinates of $\sigma_N(P)$ and for $Q \in U_S$ denote by $(z^1(Q), \cdots, z^n(Q))$ the coordinates of $\sigma_S(Q)$. A simple argument shows the map $(y^1(P), \cdots, y^n(P)) \mapsto (z^1(P), \cdots, z^n(P))$ ($P \in U_N \cap U_N$) is smooth (see the exercise below. Hence $\{(U_N, \sigma_N),\ (U_S, \sigma_S)\}$.

Exercise 1.2.3 Show that the functions y^i, z^j constructed in the above example satisfy

$$z^i = \frac{y^i}{\left(\sum_{j=1}^{n}(y^j)^2\right)}, \quad \forall i = 1, \cdots, n.$$

Example 1.2.10 (The n-dimensional torus) This is the codimension n submanifold of $\mathbb{R}^{2n}(x_1, y_1; \ldots; x_n, y_n)$ defined as the zero locus

$$x_1^2 + y_1^2 = \cdots = x_n^2 + y_n^2 = 1.$$

Note that T^1 is diffeomorphic with the 1-dimensional sphere S^1 (unit circle). As a set T^n is a direct product of n circles $T^n = S^1 \times \cdots \times S^1$ (see Figure 1.3)

The above example suggests the following general construction.

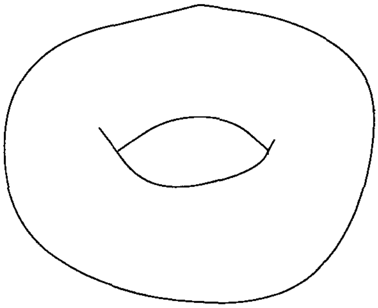

Figure 1.3: *The 2-dimensional torus*

Example 1.2.11 Let M and N be smooth manifolds of dimension m and respectively n. Then their topological direct product has a natural structure of smooth manifold of dimension $m+n$.

Example 1.2.12 (The connected sum of two manifolds) Let M_1 and M_2 be two manifolds of the same dimension m. Pick $p_i \in M_i$ ($i = 1, 2$), choose small open neighborhoods U_i of p_i in M_i and then local charts ψ_i identifying each of these neighborhoods with the ball of radius 2 in $\mathbb{R}^m$, $B_2(0)$. Let $V_i \subset U_i$ correspond (via ψ_i) to the annulus $\{1/2 < |x| < 2\} \subset \mathbb{R}^m$. Consider

$$\phi : \{1/2 < |x| < 2\} \to \{1/2 < |x| < 2\}, \quad \phi(x) = \frac{x}{|x|^2}.$$

The action of ϕ is clear: it switches the two boundary components of $\{1/2 < |x| < 2\}$ and reverses the orientation of the radial directions. Now " glue" V_1 to V_2 using the "prescription" given by $\psi_2^{-1} \circ \phi \circ \psi_1 : V_1 \to V_2$. In this way we obtain a new topological space with a natural smooth structure induced by the smooth structures on M_i. Up to a diffeomeorphism, the new manifold thus obtained is independent of the choices of local coordinates ([16]) and it is called **the connected sum of** M_1 **and** M_2 and is denoted by $M_1 \# M_2$ (see Figure 1.4).

Example 1.2.13 (The real projective space $\mathbb{RP}^n$) As a topological space $\mathbb{RP}^n$ is the quotient of $\mathbb{R}^{n+1}$ modulo the equivalence relation

$$x \sim y \stackrel{\text{def}}{\Longleftrightarrow} \exists \lambda \in \mathbb{R}^* : \quad x = \lambda y.$$

The equivalence class of $x = (x^0, \ldots, x^n) \in \mathbb{R}^{n+1} \setminus \{0\}$ is usually denoted by $[x^0 : \ldots : x^n]$. Alternatively, $\mathbb{RP}^n$ is the set of all lines (directions) in $\mathbb{R}^{n+1}$. Traditionally one attaches to each direction in $\mathbb{R}^{n+1}$ a point at infinity so that $\mathbb{RP}^n$ can be thought as the collection of all points at infinity along all directions.

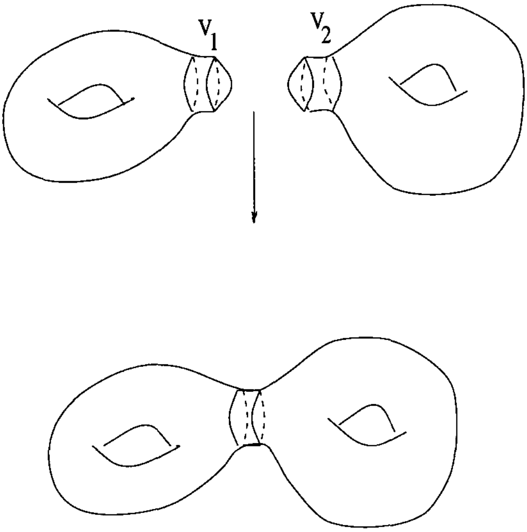

Figure 1.4: *Connected sum of tori*

$\mathbb{RP}^{n+1}$ has a natural structure of smooth manifold. To describe it consider the sets

$$U_k = \{[x^0 : \ldots : x^n] \in \mathbb{RP}^n \ ; \ x^k \neq 0\}, \quad k = 0, \ldots, n.$$

Now define

$$\psi_k : U_k \to \mathbb{R}^n \qquad [x^0 : \ldots : x^n] \mapsto (x^0/x^k, \ldots, x^{k-1}/x^k, x^{k+1}/x^k, \ldots x^n).$$

The maps ψ_k define local coordinates on the projective space. The transition map on the overlap region $U_k \cap U_m = \{[x^0 : \ldots : x^n] \ ; \ x^k x^m \neq 0\}$ can be easily described. Set

$$\psi_k([x^0 : \ldots : x^n]) = (\xi_1, \ldots, \xi_n), \ \ \psi_m([x^0 : \ldots : x^n]) = (\eta_1, \ldots, \eta_n).$$

The equality

$$[x^0 : \ldots : x^n] = [\xi_1 : \ldots : \xi_{k-1} : 1 : \xi_k : \ldots : \xi_n] = [\eta_1 : \ldots : \eta_{m-1} : 1 : \eta_m : \ldots : \eta_n]$$

immediately implies (assume $k < m$)

$$\begin{cases} \xi_1 = \eta_1/\eta_k, & \cdots, & \xi_{k-1} = \eta_{k-1}/\eta_k & \xi_{k+1} = \eta_k \\ \xi_k = \eta_{k+1}/\eta_k, & \cdots, & \xi_{m-2} = \eta_{m-1}/\eta_k & \xi_{m-1} = 1/\eta_k \\ \xi_m = \eta_m\eta_k, & \cdots, & \xi_n = \eta_n/\eta_k \end{cases} \tag{1.2.1}$$

This shows the map $\psi_k \circ \psi_m^{-1}$ is smooth and proves that $\mathbb{RP}^n$ is a smooth manifold. Note that when $n = 1$, $\mathbb{RP}^1$ is diffeomorphic with S^1. One way to see this is to observe that the projective space can be alternatively described as the quotient space of S^n modulo the equivalence relation which identifies antipodal points .

Example 1.2.14 (The complex projective space $\mathbb{CP}^n$) The definition is formally identical to that of $\mathbb{RP}^n$. $\mathbb{CP}^n$ is the quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$ modulo the equivalence relation

$$x \sim y \stackrel{\text{def}}{\Longleftrightarrow} \exists \lambda \in \mathbb{C}^* : \quad x = \lambda y.$$

The open sets U_k are defined similarly and so are the local charts $\psi_k : U_k \to \mathbb{C}^n$. They satisfy transition rules similar to (1.2.1) so that $\mathbb{CP}^n$ is a smooth manifold of dimension $2n$.

In the above example we encountered a special (and very pleasant) situation: the gluing maps not only are smooth, they are also holomorphic as maps $\psi_k \circ \psi_m^{-1} : U \to V$ where U and V are open sets in $\mathbb{C}^n$. This type of gluing induces a "rigidity" in the underlying manifold and it is worth distinguishing this situation.

Definition 1.2.15 (Complex manifolds)A complex manifold is a smooth, $2n$-dimensional manifold M which admits an atlas $\{(U_i, \psi_i) : U_i \to \mathbb{C}^n\}$ such that all transition maps are holomorphic.

The complex projective space is a complex manifold. Our next example naturally generalizes the projective spaces described above.

Example 1.2.16 (The grassmannians $G_{k,n}(\mathbb{R})$ and $G_{k,n}(\mathbb{C})$) As a set $G_{k,n}(\mathbb{R})$ consists of all k-dimensional linear subspaces of $\mathbb{R}^n$. To topologize it introduce (following [41]) the notion of **gap** between two subspaces $U, V \subset \mathbb{R}^n$ as

$$\delta(U, V) = \sup\{\text{dist}(u, V) \; ; \; u \in U, \; |u| = 1\}.$$

The distance between two subspaces is then defined as

$$\hat{\delta}(U, V) = \max\{\delta(U, V), \; \delta(V, U)\}.$$

One can show easily that $(G_{k,n}(\mathbb{R}), \; \hat{\delta})$ is a metric space (exercise).

Let $\{e_1, \cdots, e_n\}$ denote the standard basis of $\mathbb{R}^n$. For any subset $I \subset \{1, \cdots, n\}$ with $\#I = k$ define $E_I = \text{span}\,\{e_i \; ; \; i \in I\}$ so that $E_i \in G_{k,n}(\mathbb{R})$. Now define

$$\mathcal{D}_I = \{V \in G_{k,n}(\mathbb{R}) \; ; \; u \cap E_I^\perp = 0\}.$$

Here $E_I^\perp$ is the orthogonal complement of E_I in $\mathbb{R}^n$ with respect to the Euclidian inner product. We leave the reader as an exercise the proofs of the following facts.
(a) $\mathcal{D}$ is a an open subset of $G_{k,n}(\mathbb{R})$.
(b) The family $(\mathcal{D}_I)$ covers $G_{k,n}(\mathbb{R})$.
(c) Any $U \in \mathcal{D}_I$ can be uniquely represented as the graph of a linear operator $T = T(U) : E_I \to E_I^\perp$ i.e.

$$U = \{(x, Tx) \in E_I \times E_I^\perp \cong \mathbb{R}^n \; ; \; x \in E_I\}.$$

We obtain continuous (*why ?*) maps $\Psi_I : \mathcal{D}_I \to L(E_I, E_I^\perp) \cong M_{(n-k)\times k}(\mathbb{R}) \cong \mathbb{R}^{k(n-k)}$. One can show that
(d) the maps Ψ_I define an atlas on $G_{k,n}(\mathbb{R})$. In particular $G_{k,n}(\mathbb{R})$ has dimension $k(n-k)$.

$G_{k,n}(\mathbb{C})$ is defined as the space of complex k-dimensional subspaces of $\mathbb{C}^n$. It can be structured as above as a smooth manifold of dimension $2k(n-k)$. Note that $G_{1,n}(\mathbb{R}) \cong \mathbb{RP}^{n-1}$ and $G_{1,n}(\mathbb{C}) \cong \mathbb{CP}^{n-1}$. The grassmannians have important applications in many classification problems.

Exercise 1.2.4 Prove the statements (a)-(d) in the above example. Show that $G_{k,n}(\mathbb{C})$ is a complex manifold (complex dimension $k(n-k)$).

Example 1.2.17 (Lie groups) A Lie group is a smooth manifold G together with a group structure on it such that the map

$$G \times G \to G \quad (g,h) \mapsto g \cdot h^{-1}$$

is smooth. These structures provide an excellent way to formalize the notion of symmetry .
(a) $(\mathbb{R}^n, +)$ is a commutative Lie group.
(b) The unit circle S^1 can be alternatively described as the set of complex numbers of norm one and the complex multiplication defines a Lie group structure on it. This is a commutative group. More generally the torus T^n is a Lie group as a direct product of n circles. One can show that any connected commutative Lie group has the from $T^n \times \mathbb{R}^m$.
(c) The general linear group $GL(n, \mathbb{K})$ defined as the group of invertible $n \times n$ matrices with entries in the field $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ is a Lie group. Indeed, $GL(n, \mathbb{K})$ is an open subset (see Exercise 1.1.1) in the linear space of $n \times n$ matrices with entries in $\mathbb{K}$. It has dimension $d_{\mathbb{K}}n^2$ where $d_{\mathbb{K}}$ is the dimension of $\mathbb{K}$ as a linear space over $\mathbb{R}$.
(d) The orthogonal group $O(n)$ is the group of real $n \times n$ matrices satisfying

$$T \cdot T^t = \mathrm{id}.$$

To describe its smooth structure we will use the Cayley transform trick as in [65] (see also the classical [74]). Set

$$M_n(\mathbb{R})^\# = \{T \in M_n(\mathbb{R}) \ ; \ \det(1+T) \neq 0\}.$$

The matrices in $M_n(\mathbb{R})^\#$ are called non exceptional. Clearly $1 \in O(n)^\# = O(n) \cap M_n(\mathbb{R})^\#$ so that $O(n)^\#$ is a **nonempty** open subset of $O(n)$. The Cayley transform is the map $\# : M_n(\mathbb{R})^\# \to M_n(\mathbb{R})$ defined by

$$A \mapsto A^\# = (1-A)(1+A)^{-1}.$$

The Cayley transform has some very nice properties.

(i) $A^{\#} \in M_n(\mathbb{R})^{\#}$ for every $A \in M_n(\mathbb{R})^{\#}$.
(ii) $\#$ is involutory i.e. $(A^{\#})^{\#} = A$ for any $A \in M_n(\mathbb{R})^{\#}$.
(iii) For every $T \in O(n)^{\#}$ the matrix $T^{\#}$ is skew-symmetric and conversely if $A \in M_n(\mathbb{R})^{\#}$ is skew-symmetric then $A^{\#} \in O(n)$.

Thus the Cayley transform is a homeomorphism from $O(n)^{\#}$ to the space of non-exceptional, skew-symmetric, matrices. The latter space is an open subset in the linear space of real $n \times n$ skew-symmetric matrices, $\underline{o}(n)$.

Any $T \in O(n)$ defines a self-homeomorphism of $O(n)$ by **left translation in the group**

$$L_T : O(n) \to O(n) \quad S \mapsto L_T(S) = T \cdot S.$$

We obtain an open cover of $O(n)$:

$$O(n) = \bigcup_{T \in O(n)} T \cdot O(n)^{\#}.$$

Define $\Psi_T : T \cdot O(n)^{\#} \to \underline{o}(n)$ by $S \mapsto (T^{-1} \cdot S)^{\#}$. One can show that $\left(T \cdot O(n)^{\#}, \Psi_T\right)_{T \in O(n)}$ defines a smooth structure on $O(n)$. In particular we deduce

$$\dim O(n) = n(n-1)/2.$$

Inside $O(n)$ lies a normal subgroup (the special orthogonal group)

$$SO(n) = \{T \in O(n) \; ; \; \det T = 1\}.$$

$SO(n)$ is a Lie group as well and $\dim SO(n) = \dim O(n)$.
(e) The unitary group $U(n)$ is defined as

$$U(n) = \{T \in GL(n, \mathbb{C} \; ; \; T \cdot T^* = 1\}$$

T^* denotes the conjugate transpose (adjoint) of T. To prove that $U(n)$ is a manifold one uses again the Cayley transform trick. This time we coordinatize the group using the space $\underline{u}(n)$ of skew-adjoint (skew-Hermitian) $n \times n$ complex matrices ($A = -A^*$). Thus $U(n)$ is a smooth manifold of dimension

$$\dim U(n) = \dim \underline{u}(n) = n^2.$$

Inside $U(n)$ sits the normal subgroup $SU(n)$, the kernel of the group homomorphism $\det : U(n) \to S^1$. $SU(n)$ is also called the **special unitary group**. This a smooth manifold of dimension $n^2 - 1$. In fact the Cayley transform trick allows one to coordinatize $SU(n)$ using the space

$$\underline{su}(n) = \{A \in \underline{u}(n) \; ; \; \operatorname{tr} A = 0\}.$$

Exercise 1.2.5 (a)Prove the properties (i)-(iii) of the Cayley transform and then show that $\left(T \cdot O(n)^{\#}, \Psi_T\right)_{T \in O(n)}$ defines a smooth structure on $O(n)$.
(b) Prove that $U(n)$ and $SU(n)$ are manifolds.
(c) Show that $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ are compact spaces.
(d) Prove that $SU(2)$ is diffeomorphic with S^3 (Hint: think of S^3 as the group of unit quaternions.)

Exercise 1.2.6 Let $SL(n;\mathbb{K})$ denote the group of $n \times n$ matrices of determinant 1 with entries in the field $\mathbb{K}$=$\mathbb{R}$, $\mathbb{C}$. Using the Cayley trick show that $SL(n;\mathbb{K})$ is a smooth manifold modeled on the linear space

$$\underline{sl}(n, \mathbb{K}) = \{A \in M_{n \times n}(\mathbb{K}) \; ; \; \operatorname{tr} A = 0\}.$$

In particular it has dimension $d_{\mathbb{K}}(n^2 - 1)$, where $d = \dim_{\mathbb{R}} \mathbb{K}$.

Exercise 1.2.7 (a)Let G be a **connected** Lie group and denote by U a neighborhood of $1 \in G$. If H is the subgroup algebraically generated by U show that H is dense in G.
(b) Let G be a **compact** Lie group and $g \in G$. Show that $1 \in G$ lies in the closure of $\{g^n \; ; \; n \in \mathbb{Z} \setminus \{0\}\}$.

§1.2.4 How many manifolds are there?

The list of examples in the previous subsection can go on for ever so one may ask whether there is any coherent way to organize the collection of all possible manifolds. This is too general a question to expect a clear cut answer. We have to be more specific. For example we can ask

Question 1: *Which are the* **compact** *manifolds of a given dimension d?*

For $d = 1$ the answer is very simple: the only compact 1-dimensional manifold is the circle S^1. So we can raise the stakes and try the same problem for $d = 2$. Already the situation is more elaborate. We know at least two surfaces: the sphere S^2 and the torus T^2. They clearly look different but we have not yet proved rigorously that they are indeed not diffeomorphic. This is not the end of the story. We can connect sum two tori, three tori or any number g of tori. We obtain doughnut-shaped surface as in Figure 1.5

Again we face the same question: do we get non-diffeomorphic surfaces for different choices of g? Figure 1.5 suggests that this may be the case but this is no rigorous argument.

We know another example of compact surface, the projective plane $\mathbb{RP}^2$ and we naturally ask whether it looks like one of the surfaces constructed above. Unfortunately, we cannot visualize the real projective plane (one can prove rigorously it

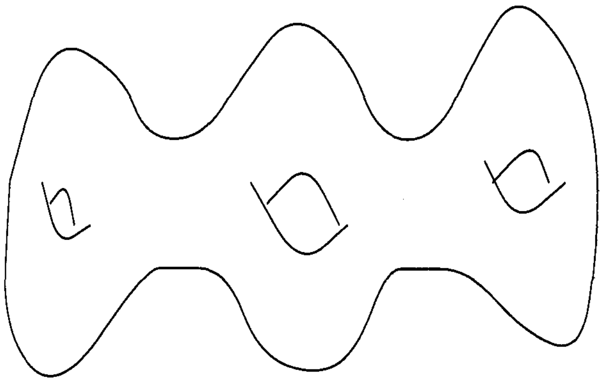

Figure 1.5: *Connected sum of 3 tori*

does not have enough room to exist inside our 3-dimensional Universe). We have to decide this question using a little more than the raw geometric intuition provided by a picture. To kill the suspense, we mention that $\mathbb{RP}^2$ does not belong to the family of doughnuts. The reason is that a torus for example has two faces: an inside face and an outside face (think of a car rubber tube). $\mathbb{RP}^2$ has a weird behavior: it has "no inside" and "no outside". It has only *one* side! One says the torus is **orientable** while the projective plane is not.

We can now connect sum any numbers of $\mathbb{RP}^2$'s to any doughnut an thus obtain more and more surfaces, which we cannot visualize and we have yet no idea if they are pairwise distinct. A classical result in topology says that all compact surfaces can be obtained in this way (see [52]) but in the above list some manifolds are diffeomorphic and we have to describe which. In dimensions ≥ 3 things are not settled and to make things worse in dimension ≥ 4 Question 1 is algorithmically undecidable .

We can reconsider our goals and look for all the manifolds with a given property X. In many instances one can give fairly accurate answers. Property X may refer to more than the (differential) topology of a manifold. Real life situations suggest the study of manifolds with additional structure. The following problem may give the reader a taste of the types of problems we will be concerned with in this book.

Question 2 *Can we wrap a planar piece of canvas around a metal sphere in a one-to-one fashion? (The canvas is flexible but not elastic).*

A simple do-it-yourself experiment is enough to convince anyone that this is not possible. Naturally, one asks for a rigorous explanation of what goes wrong. The best explanation of this phenomenon is contained the the celebrated Teorema Egregium (Golden Theorem) of Gauss. Canvas surfaces have additional structure (they are made of a special material) and for such objects there is a rigorous way to measure "how curved" are they. One then realizes that the problem in Question 2 is impossible since a (canvas) sphere is curved in a different way than a plane canvas.

There are many other structures Nature forced us into studying them but they may not be so easily described in elementary terms.

A word to the reader. The next two chapters are the most arid in geometry but, keep in mind that behind each construction lies a natural motivation and, even if we do not always have the time to show it to the reader, it is there, and it may take a while to reveal itself. Most of the constructions the reader will have to "endure" in the next two chapters constitute not just some difficult to "swallow" formalism but the basic language of geometry. Learning this language may not be the most pleasant thing, but surely enough, it is a very rewarding enterprise.