

Contents

Introduction	i
1 Manifolds	1
1.1 Preliminaries	1
§1.1.1 Space and Coordinatization	1
§1.1.2 The implicit function theorem	3
1.2 Smooth manifolds	6
§1.2.1 Basic definitions	6
§1.2.2 Partitions of unity	8
§1.2.3 Examples	9
§1.2.4 How many manifolds are there?	15
2 Natural Constructions on Manifolds	18
2.1 The tangent bundle	18
§2.1.1 Tangent spaces	18
§2.1.2 The tangent bundle	21
§2.1.3 Vector bundles	23
§2.1.4 Some examples of vector bundles	26
2.2 A linear algebra interlude	30
§2.2.1 Tensor products	30
§2.2.2 Symmetric and antisymmetric tensors	33
§2.2.3 The “super” slang	39
§2.2.4 Duality	42
§2.2.5 Some complex linear algebra	49
2.3 Tensor fields	52
§2.3.1 Operations with vector bundles	52
§2.3.2 Tensor fields	53
§2.3.3 Fiber bundles	56
3 Calculus on Manifolds	61
3.1 The Lie derivative	61
§3.1.1 Flows on manifolds	61
§3.1.2 The Lie derivative	63

§3.1.3	Examples	67
3.2	Derivations of $\Omega^*(M)$	69
§3.2.1	The exterior derivative	69
§3.2.2	Examples	74
3.3	Connections on vector bundles	75
§3.3.1	Covariant derivatives	75
§3.3.2	Parallel transport	80
§3.3.3	The curvature of a connection	82
§3.3.4	Holonomy	85
§3.3.5	Bianchi identities	88
§3.3.6	Connections on tangent bundles	89
3.4	Integration on manifolds	91
§3.4.1	Integration of 1-densities	91
§3.4.2	Orientability and integration of differential forms	95
§3.4.3	Stokes formula	101
§3.4.4	Representations and characters of compact Lie groups	105
§3.4.5	Fibered calculus	112
4	Riemannian Geometry	116
4.1	Metric properties	116
§4.1.1	Definitions and examples	116
§4.1.2	The Levi-Civita connection	120
§4.1.3	The exponential map and normal coordinates	125
§4.1.4	The minimizing property of geodesics	128
§4.1.5	Calculus on Riemann manifolds	134
4.2	The Riemann curvature	144
§4.2.1	Definitions and properties	144
§4.2.2	Examples	147
§4.2.3	Cartan's moving frame method	150
§4.2.4	The geometry of submanifolds	154
§4.2.5	The Gauss-Bonnet theorem	160
5	Elements of the calculus of variations	169
5.1	The least action principle	169
§5.1.1	1-dimensional Euler-Lagrange equations	169
§5.1.2	Noether's conservation principle	175
5.2	The variational theory of geodesics	179
§5.2.1	Variational formulæ	179
§5.2.2	Jacobi fields	183

6	The fundamental group and covering spaces	191
6.1	The fundamental group	192
§6.1.1	Basic notions	192
§6.1.2	Of categories and functors	196
6.2	Covering spaces	198
§6.2.1	Definitions and examples	198
§6.2.2	Unique lifting property	200
§6.2.3	Homotopy lifting property	201
§6.2.4	On the existence of lifts	202
§6.2.5	The universal cover and the fundamental group	204
7	Cohomology	206
7.1	DeRham cohomology	206
§7.1.1	Speculations around the Poincaré lemma	206
§7.1.2	Čech vs. DeRham	210
§7.1.3	Very little homological algebra	213
§7.1.4	Functorial properties of DeRham cohomology	218
§7.1.5	Some simple examples	222
§7.1.6	The Mayer-Vietoris principle	224
§7.1.7	The Künneth formula	227
7.2	The Poincaré duality	230
§7.2.1	Cohomology with compact supports	230
§7.2.2	The Poincaré duality	234
7.3	Intersection theory	239
§7.3.1	Cycles and their duals	239
§7.3.2	Intersection theory	243
§7.3.3	The topological degree	249
§7.3.4	Thom isomorphism theorem	251
§7.3.5	Gauss-Bonnet revisited	253
7.4	Symmetry and topology	258
§7.4.1	Symmetric spaces	258
§7.4.2	Symmetry and cohomology	261
§7.4.3	The cohomology of compact Lie groups	265
§7.4.4	Invariant forms on grassmannians and Weyl's integral formula	267
§7.4.5	The Poincaré polynomial of a complex grassmannian	273
7.5	Čech cohomology	279
§7.5.1	Sheaves and presheaves	279
§7.5.2	Čech cohomology	284
8	Characteristic classes	293
8.1	Chern-Weil theory	293
§8.1.1	Connections in principal G-bundles	293

§8.1.2	G-vector bundles	297
§8.1.3	Invariant polynomials	298
§8.1.4	The Chern-Weil theory	300
8.2	Important examples	304
§8.2.1	The invariants of the torus T^n	304
§8.2.2	Chern classes	305
§8.2.3	Pontryagin classes	308
§8.2.4	The Euler class	310
§8.2.5	Universal classes	313
8.3	Computing characteristic classes	320
§8.3.1	Reductions	320
§8.3.2	The Gauss-Bonnet-Chern theorem	325
9	Elliptic equations on manifolds	335
9.1	Partial differential operators: algebraic aspects	335
§9.1.1	Basic notions	335
§9.1.2	Examples	341
§9.1.3	Formal adjoints	344
9.2	Functional framework	350
§9.2.1	Sobolev spaces in \mathbb{R}^N	350
§9.2.2	Embedding theorems: integrability properties	358
§9.2.3	Embedding theorems: differentiability properties	362
§9.2.4	Functional spaces on manifolds	367
9.3	Elliptic partial differential operators: analytic aspects	370
§9.3.1	Elliptic estimates in \mathbb{R}^N	371
§9.3.2	Elliptic regularity	376
§9.3.3	An application: prescribing the curvature of surfaces	381
9.4	Elliptic operators on compact manifolds	392
§9.4.1	The Fredholm theory	392
§9.4.2	Spectral theory	402
§9.4.3	Hodge theory	407
10	Dirac operators	411
10.1	The structure of Dirac operators	411
§10.1.1	Basic definitions and examples	411
§10.1.2	Clifford algebras	414
§10.1.3	Clifford modules: the even case	417
§10.1.4	Clifford modules: the odd case	422
§10.1.5	A look ahead	423
§10.1.6	$Spin$	425
§10.1.7	$Spin^c$	433
§10.1.8	Low dimensional examples	436

§10.1.9 Dirac bundles	441
10.2 Fundamental examples	445
§10.2.1 The Hodge-DeRham operator	445
§10.2.2 The Dolbeault operator	451
§10.2.3 The <i>spin</i> Dirac operator	457
§10.2.4 The <i>spin</i> ^c Dirac operator	462
Bibliography	468
Index	473