$|A|^2 \cdot |B|^2 \geq 0$, in compact form. So Hamilton simply wanted to generalize this, to deal with the real world, since we have three dimensions and need (x,y,z) sets of numbers to locate positions in a room (or in the galaxy). He struggled about 10 years and finally found a pretty, new pattern that worked. He carved the discovered rule into the railing of a foot bridge at the University of Dublin, Ireland: $XY = -Z = -YX$, or something like that. The new thing he had to allow was $XY = -YX$. These weird new numbers *anti*-commute! No one apparently thought of such objects before about 1843. They were not needed for algebra solutions. Hamilton was forced to them in order to keep a close parallel with the complex numbers. These same anti-commuting numbers play a central role in modern physics.

HAMILTON'S QUATERNIONS

The notation which Pauli used when he *reinvented* the quaternions in the 1920's has now become standard in physics, if not in mathematics, so we shall use it. We have a set of 8 objects, $\{\sigma_\mu, i\sigma_\mu\}$ where $\mu = 0,1,2$, or 3 and $i = \sqrt{-1}$. The σ_0 object is the identity. The rules of multiplication are: $\sigma_1\sigma_2 \equiv +i\sigma_3 \equiv -\sigma_2\sigma_1$ and cyclic permutations. Also, $\sigma_1\sigma_1 \equiv \sigma_2\sigma_2 \equiv \sigma_3\sigma_3 \equiv \sigma_0$. Simple and elegant! We see Hamilton's original, four part quaternions as a 4-element subset, such as $\{\sigma_0, -i\sigma_k\}$, k = 1, 2, or 3. We will always include $\pm\sigma_\mu$, when we say σ_μ is in a set, and always assume $(-1)(-1) \equiv (+1)$ as usual. The $i\sigma_\mu$ terms are easily dealt with because $i\sigma_\mu \equiv \sigma_\mu i$. The i's can be 'collected' and then combined. For example: $i\sigma_1 i\sigma_2 = ii\sigma_1\sigma_2 = -\sigma_1\sigma_2 = -i\sigma_3$, *etc.* The pattern here is like that for 2×2 complex matrices; such matrices were the form that Pauli used to first describe electrons with spin, in the 1920's. They are <u>not</u> matrices, however. They are like i. We don't replace i by a matrix, although we could! Better to think of $A \equiv a^\mu\sigma_\mu + b^\mu i\sigma_\mu = (a^\mu + ib^\mu)\sigma_\mu \equiv c^\mu\sigma_\mu$ as just a 'hypercomplex' number. The repeated μ in a product means sum on $\mu = 0,1,2,3$, so A has 8 distinct parts, where a^μ and b^μ are real numbers. For example: $A_1 = 3\sigma_0 - 2\sigma_1 + 0\sigma_2 + 0\sigma_3 + -\sqrt{2}i\sigma_0 + 0i\sigma_1 + 4i\sigma_2 - 7i\sigma_3$. We can easily describe our universe's space and time in the form

$$x \equiv ct\sigma_0 + x\sigma_1 + y\sigma_2 + z\sigma_3 \equiv x^\mu\sigma_\mu$$

or the momentum and energy of a moving mass as

$$P \equiv (E/c)\sigma_0 + P_x\sigma_1 + P_y\sigma_2 + P_z\sigma_3 \equiv P^\mu\sigma_\mu$$

There is a totally natural match between quaternions and 1+3 spacetime. (Hamilton even noticed this 1+3 match!) I would go so far as to say that our universe's spacetime is 'built' on quaternions, and that is why we have *three* large space dimensions and *one* large time dimension!! (Just a guess of course.) Four space dimensions don't fit, in a natural way, with quaternions, but see later chapters. We need to generalize the usual complex conjugation next. We have $(a + ib)^* \equiv a - ib$, or $i^* \equiv -i$. It is useful to think of this now as $(a\sigma_0 + bi\sigma_0)$ and we have $\sigma_0^* \equiv +\sigma_0$, $(i\sigma_0)^* \equiv -i\sigma_0$. Or, we could instead think of $(a\sigma_0 + bi\sigma_1)$ as representing complex numbers; therefore, we have $\sigma_0^* \equiv \sigma_0$ and $(i\sigma_1)^* \equiv -i\sigma_1$. Clearly, $\sigma_\mu^* \equiv +\sigma_\mu$ is natural. We then find that $(AB)^* = B^*A^*$, where A and B are any two complex quaternions with *eight* parts each. (Notice the order reversal on the right side.) This is beautiful and takes some patience to prove. (Grind out each side then compare, term by term.) The complex quaternions can be mathematically thought of as being 'generated' from a so-called 'direct product' of the 'pure' quaternions, $\{\sigma_0, -i\sigma_k\}$, with $\{\sigma_0, -i\sigma_1\}$. (Using $i\sigma_2$ or $i\sigma_3$ here, instead of $i\sigma_1$, would serve just as well.) But once we have the *complex* quaternions, we find another closed subset $\{\sigma_0, i\sigma_1, \sigma_2, \sigma_3\}$. Now we can form a new 'direct product' of this closed set with the 'pure' quaternion set, $\{\sigma_0, -i\sigma_1, -i\sigma_2, -i\sigma_3\}$, and get a 16 element number system $\{e_\mu, ie_\mu, f_\mu, if_\mu\}$ which has turned out to be the foundation of our relativistic quantum physics! There is obviously great wonder and 'truth' in these number systems. Generalizing them further may well lead to presently unknown future truths. That is the subject of a later chapter of this book.

Returning to the conjugation analysis, we find that there is a useful, second conjugation for the complex quaternions. It could be defined by

$$\sigma_0^{\neq} \equiv +\sigma_0,\ \sigma_k^{\neq} \equiv -\sigma_k,\ i^{\neq} \equiv +i$$

(It only affects σ_1, σ_2, σ_3 and $i\sigma_1$, $i\sigma_2$, $i\sigma_3$.) I call it the quaternion conjugation, but I think mathematicians call it the symplectic conjugation.

Both of these conjugations, A^* and $A^{\neq}$, are very useful in physical descriptions. We can laboriously find that $(AB)^{\neq} = B^{\neq}A^{\neq}$, in general, for complex quaternions. Again, this is very tedious to check, for A and B with 8 parts each. Grind out both sides, for practice, and compare the 64 terms which reduce to 8 terms on each side. Notice that $\{\sigma_0, i\sigma_k\}^* = \{\cdots\}^{\neq}$ and $\{i\sigma_0, \sigma_k\}^* =$

$-\{\cdots\}^{\neq}$, and we know $(\cdots)^*$ is antiautomorphic. This simplifies the proof that $(AB)^{\neq} = B^{\neq}A^{\neq}$; all 64 elements do not have to be dealt with. Antiautomorphic means $(AB)^{conj.} = B^{conj.}A^{conj.}$.

Since we have generalized to complex quaternions from pure quaternions, we naturally wonder if spacetime is more than 1+3 dimensional. People have examined 2+6 for example (complex spacetime). So far it has not proven fruitful, that I know of, but the complex quaternions could probably handle it.

GROUPS

The quaternions have natural groups associated with them and these are very important in the physical universe. For example, consider the pure quaternions:

$$A \equiv a^0\sigma_0 + b^k i\sigma_k$$

There are 4 arbitrary parts: $\{a^0, b^1, b^2, b^3\}$. If we choose $AA^* \equiv 1\sigma_0$, then we get one equation linking these four parts. Therefore, only three parts are now independent. Consider a special case

$$A = 1\sigma_0 + \epsilon^k i\sigma_k \equiv A_\epsilon$$

where $\epsilon^1\epsilon^2 \approx 0$, $\epsilon^1\epsilon^1 \approx 0$, *etc.* In other words, the ϵ's are all very small (like 10^{-20}, so that $\epsilon\epsilon$ is like 10^{-40} and much, much smaller). Then

$$AA^* = (1\sigma_0 + \epsilon^k i\sigma_k)(1\sigma_0 + \epsilon^j i\sigma_j)^*$$

$$= \cdots = 1\sigma_0 + (\epsilon^k - \epsilon^k)i\sigma_k + \epsilon^k\epsilon^j\sigma_k\sigma_j$$

$$= 1\sigma_0 + \epsilon^k\epsilon^j\sigma_k\sigma_j \approx 1\sigma_0 + 0$$

These quaternion elements, A_ϵ, are called 'infinitesimal' members of a group. They are very 'close' to the identity element, σ_0. By multiplying them together, over and over, we can build up 'large' members of the group. The reason that we say A_ϵ is located 'close' to the identity member of the group, σ_0, is because $\sigma_0\sigma_0^* = \sigma_0$, so it is in the group also. A large member of the group is, for example, $A_\theta = \cos\theta\ \sigma_0 + \sin\theta\ i\sigma_1$, $A_\theta^* = \cos\theta\ \sigma_0 - \sin\theta\ i\sigma_1$; check that $A_\theta A_\theta^*$

$= \cdots = \sigma_0$, using $sin^2\theta + cos^2\theta = 1$. The infinitesimal member of this family of members is $A_\epsilon = cos\epsilon\ \sigma_0 + sin\epsilon\ i\sigma_1 \approx \sigma_0 + \epsilon i\sigma_1$, (from $cos\theta = 1 - \theta^2/2! + \theta^4/4! - \cdots$ and $sin\theta = \theta - \theta^3/3! + \theta^5/5! - \cdots$). There is an infinity of θ values, so there is an infinity of members of these groups. They are called Lie groups. We say $AA^* \equiv 1\sigma_0$ is a three parameter Lie group, because $A_\epsilon = 1\sigma_0 + \epsilon^k i\sigma_k$ has three free parameters: ϵ^1, ϵ^2, and ϵ^3.

The new group, $BB^{\neq} \equiv 1\sigma_0$, is the same group for the pure quaternions, since * and $\neq$ do the same thing there. For the complex quaternions, however, these groups are distinct. We examine their infinitesimal members to see how many free parameters they can have. Since $\epsilon\epsilon \approx 0$, we find that A^* must change the sign of each term in the infinitesimal members, (besides the σ_0 term of course). Therefore, we have

$$A_\epsilon^* = \sigma_0 - \epsilon^k i\sigma_k - \epsilon^0 i\sigma_0$$

and

$$B_\epsilon^{\neq} = \sigma_0 - \epsilon^k i\sigma_k - \delta^k\sigma_k$$

The A group obviously has 4 free parameters, if $AA^* \equiv 1\sigma_0$, and the B group has 6 free parameters, if $BB^{\neq} \equiv 1\sigma_0$. They have a common subgroup, $CC^* \equiv CC^{\neq} \equiv 1\sigma_0$, *i.e.*, $C^* = C^{\neq}$ and this is the 3 parameter group we found earlier in the pure quaternions:

$$C_\epsilon^* = (\sigma_0 + \epsilon^k i\sigma_k)^* = \sigma_0 - \epsilon^k i\sigma_k$$

Notice that these groups appear naturally in the number system, once that system is defined! The quaternion system was defined by Hamilton because nature is 3 dimensional and because complex numbers seem beautiful and powerful in applied calculus. Therefore, these groups are also natural to the real world, if the quaternions are, and they are! These groups historically have fancy names:

$$\{\epsilon^k\} \leftrightarrow SU(2)$$

$$\{\epsilon^k,\ \epsilon^0\} \leftrightarrow U(1)\otimes SU(2)$$

$$\{\epsilon^k,\ \delta^k\} \leftrightarrow SL(2,C) \approx Lorentz$$

We have not done any physics yet! We already have natural number systems and natural groups that follow from the 1+3 assumed spacetime split.

All of this algebraic stuff is a package and it goes with the 1+3 spacetime. This could have all been done by Hamilton before 1850! But Lorentz didn't find his group until 1900 and realize that it applies to the design of our real world.

GROUP REPRESENTATIONS

The group idea here is very straightforward: $AA^* \equiv 1\sigma_0$, $A = a^\mu\sigma_\mu + b^\mu i\sigma_\mu = c^\mu\sigma_\mu$ with only four of the eight numbers as independent, and they define the member A. Clearly, if $AA^* = 1\sigma_0$ and $BB^* = 1\sigma_0$, then $(AB)(AB)^* = (AB)(B^*A^*) = A(BB^*)A^* = A\sigma_0 A^* = AA^* = \sigma_0$, so the product of any two group members is also in the group here. That is why the collection is called a group. Notice that we have used **associativity** to get this result. We don't have commutivity, but we do have associativity! If nature has any *non*-associative math required, then groups will fall from grace or be generalized. See later for speculations on this.

Besides the group, we can invent things that 'change' because of a group's existence. For example: $P' \equiv A^*PA$ turns some given P into P$'$ by 'hitting' it with a group member A, $AA^* \equiv 1\sigma_0$. We can also invent $F' \equiv A^{\neq}FA$ as well, for some given F. Even though $AA^* \equiv 1\sigma_0$ here, remember $AA^{\neq} \neq \sigma_0$ in general for this group. The obvious next question is why do this complicated business? It turns out to relate to the physical world, of course, or we would not bother with it. (This is a physics book, not a math book, though you must be beginning to wonder by now.)

Notice that if $P \equiv P^*$ and $P' \equiv A^*PA$ then $P'^* = (A^*PA)^* = A^*P^*A^{**} = A^*PA = P'$, so $P' = P'^*$ and P$'$ is 'like' P, in a sense. So what? Have patience. Notice that $F^{\neq} \equiv -F$ and $F' = A^{\neq}FA$ means $F'^{\neq} = (A^{\neq}FA)^{\neq} = A^{\neq}F^{\neq}A^{\neq\neq} = A^{\neq}F^{\neq}A = A^{\neq}(-F)A = -F'$. So $F' = -F'^{\neq}$, and F$'$ is 'like' F, in a sense. All of this is true, regardless of which group A belongs to. These P's and F's are called representations of the group {A}. (Don't ask me why that name is chosen. Manifestations of the group might be a better name.) Now consider $P^{\neq}P$ and $P'^{\neq}P'$. Are they 'alike' in general? Let's see:

$$P'^{\neq}P' = (A^*PA)^{\neq}(A^*PA) = A^{\neq}P^{\neq}A^{*\neq}A^*PA = A^{\neq}P^{\neq}(AA^{\neq})^*PA,$$

so we need $AA^{\neq} \equiv 1\sigma_0$ to keep going. Then

$$P'^{\neq}P' = A^{\neq}P^{\neq}PA =?= (P^{\neq}P)A^{\neq}A = P^{\neq}P(\sigma_0) = P^{\neq}P$$

They are 'alike' if $AA^{\neq} \equiv 1\sigma_0$ and also $P^{\neq}P$ needs to be proportional to σ_0, so