

All of this algebraic stuff is a package and it goes with the 1+3 spacetime. This could have all been done by Hamilton before 1850! But Lorentz didn't find his group until 1900 and realize that it applies to the design of our real world.

GROUP REPRESENTATIONS

The group idea here is very straightforward: $AA^* \equiv I\sigma_0$, $A = a^\mu\sigma_\mu + b^\mu i\sigma_\mu = c^\mu\sigma_\mu$ with only four of the eight numbers as independent, and they define the member A. Clearly, if $AA^* = I\sigma_0$ and $BB^* = I\sigma_0$, then $(AB)(AB)^* = (AB)(B^*A^*) = A(BB^*)A^* = A\sigma_0A^* = AA^* = \sigma_0$, so the product of any two group members is also in the group here. That is why the collection is called a group. Notice that we have used **associativity** to get this result. We don't have commutivity, but we do have associativity! If nature has any *non*-associative math required, then groups will fall from grace or be generalized. See later for speculations on this.

Besides the group, we can invent things that 'change' because of a group's existence. For example: $P' \equiv A^*PA$ turns some given P into P' by 'hitting' it with a group member A, $AA^* \equiv I\sigma_0$. We can also invent $F' \equiv A^\neq FA$ as well, for some given F. Even though $AA^* \equiv I\sigma_0$ here, remember $AA^\neq \neq \sigma_0$ in general for this group. The obvious next question is why do this complicated business? It turns out to relate to the physical world, of course, or we would not bother with it. (This is a physics book, not a math book, though you must be beginning to wonder by now.)

Notice that if $P \equiv P^*$ and $P' \equiv A^*PA$ then $P'^* = (A^*PA)^* = A^*P^*A^{**} = A^*PA = P'$, so $P' = P'^*$ and P' is 'like' P, in a sense. So what? Have patience. Notice that $F^\neq \equiv -F$ and $F' = A^\neq FA$ means $F'^\neq = (A^\neq FA)^\neq = A^\neq F^\neq A^\neq = A^\neq F^\neq A = A^\neq (-F)A = -F'$. So $F' = -F'^\neq$, and F' is 'like' F, in a sense. All of this is true, regardless of which group A belongs to. These P's and F's are called representations of the group {A}. (Don't ask me why that name is chosen. Manifestations of the group might be a better name.) Now consider $P^\neq P$ and $P'^\neq P'$. Are they 'alike' in general? Let's see:

$$P'^\neq P' = (A^*PA)^\neq (A^*PA) = A^\neq P^\neq A^{*\neq} A^*PA = A^\neq P^\neq (AA^\neq)^*PA,$$

so we need $AA^\neq \equiv I\sigma_0$ to keep going. Then

$$P'^\neq P' = A^\neq P^\neq PA = ? = (P^\neq P)A^\neq A = P^\neq P(\sigma_0) = P^\neq P$$

They are 'alike' if $AA^\neq \equiv I\sigma_0$ and also $P^\neq P$ needs to be proportional to σ_0 , so

it then commutes through A^\neq , on its left above. Proof that $P^\neq P$ commutes goes as follows: $(P^\neq P)^\neq = P^\neq P^\neq \neq = P^\neq P$ and since \neq changes the signs of all σ_k terms, $(P^\neq P) = (B\sigma_0 + Ci\sigma_0)$ at most. It therefore does commute with any A .

We have found that $(P^\neq P)$ is an 'invariant' representation of only the group $AA^\neq \equiv I\sigma_0$, in the sense that $P^\neq P'$ and $P^\neq P$ are identical for this group. We say this $P^\neq P$ is an invariant of the group $AA^\neq = I\sigma_0$, which is called the $SL(2,C)$ group. We could find other representations and invariants for these groups. Another very useful representation is the spinor $\psi' \equiv A^\neq \psi$. We find $\psi'^* \psi' = (A^\neq \psi)^* (A^\neq \psi) = \psi^* A^\neq^* A^\neq \psi = \psi^* (AA^*)^\neq \psi = \psi^* \psi$, if $AA^* \equiv I\sigma_0$. So this is an invariant of another group, called $SU(2) \otimes U(1)$. But next consider

$$\begin{aligned} \psi'^* P' \psi' &= (A^\neq \psi)^* (A^* P A) (A^\neq \psi) = \psi^* A^\neq^* A^* P A A^\neq \psi = \psi^* (AA^\neq)^* P (AA^\neq) \psi \\ &= \psi^* P \psi \end{aligned}$$

if $AA^\neq \equiv I\sigma_0$. This is an invariant of the $SL(2,C)$ group. Clearly P , F , and ψ are all quite different. All are complex quaternions and all relate back to the same two groups, $AA^* \equiv I\sigma_0$ and $BB^\neq \equiv I\sigma_0$. They are like decorations in the picture around the central groups. The groups in turn are manifestations of the number system itself. All of this stuff 'must' be at the core of the physical world, if 1+3 dimensional spacetime is fundamental to the real universe!

OTHER CONJUGATIONS

So far we have $\{\sigma_0, i\sigma_k, i\sigma_0, \sigma_k\}$ as the basic number system. There are two basic conjugations: $()^\neq$ and $()^*$. The \neq changes $\sigma_1, \sigma_2, \sigma_3$ and $*$ changes i . These are both antiautomorphic: $(AB)^{conj.} = B^{conj.} A^{conj.}$. Are there other antiautomorphic conjugations? If so, could they have physical significance as well?

We can invent many new conjugations by the procedure:

$$A^{conj.} \equiv \sigma A^*(\sigma)^{-1}, \quad \sigma \sigma^{-1} \equiv 1\sigma_0$$

where A is any of the 8 elements and σ is any one of the 8 elements. Clearly, $\sigma \rightarrow \sigma_0$ and $\sigma \rightarrow i\sigma_0$ gives $A^{conj.} = A^*$, which we already have. For $\sigma \rightarrow \sigma_1$, we get $A^{\#1} = (\sigma_1) A^* (\sigma_1)$. Calculating $A^{\#1}$, for A equal to each basis element, is easy and good practice for you. You will find

$$\{\sigma_0, i\sigma_k, i\sigma_0, \sigma_k\}^{\#1} = \{\sigma_0, -i\sigma_1, i\sigma_2, i\sigma_3, -i\sigma_0, \sigma_1, -\sigma_2, -\sigma_3\}$$