it then commutes through $A^{\neq}$, on its left above. Proof that $P^{\neq}P$ commutes goes as follows: $(P^{\neq}P)^{\neq} = P^{\neq}P^{\neq\neq} = P^{\neq}P$ and since $\neq$ changes the signs of all σ_k terms, $(P^{\neq}P) = (B\sigma_0 + Ci\sigma_0)$ at most. It therefore does commute with any A.

We have found that $(P^{\neq}P)$ is an 'invariant' representation of only the group $AA^{\neq} \equiv 1\sigma_0$, in the sense that $P'^{\neq}P'$ and $P^{\neq}P$ are identical for this group. We say this $P^{\neq}P$ is an invariant of the group $AA^{\neq} = 1\sigma_0$, which is called the SL(2,C) group. We could find other representations and invariants for these groups. Another very useful representation is the spinor $\psi' \equiv A^{\neq}\psi$. We find $\psi'^{*}\psi' = (A^{\neq}\psi)^{*}(A^{\neq}\psi) = \psi^{*}A^{\neq *}A^{\neq}\psi = \psi^{*}(AA^{*})^{\neq}\psi = \psi^{*}\psi$, if $AA^{*} \equiv 1\sigma_0$. So this is an invariant of another group, called $SU(2)\otimes U(1)$. But next consider

$$\psi'^{*}P'\psi' = (A^{\neq}\psi)^{*}(A^{*}PA)(A^{\neq}\psi) = \psi^{*}A^{\neq *}A^{*}PAA^{\neq}\psi = \psi^{*}(AA^{\neq})^{*}P(AA^{\neq})\psi$$
$$= \psi^{*}P\psi$$

if $AA^{\neq} \equiv 1\sigma_0$. This is an invariant of the SL(2,C) group. Clearly P, F, and ψ are all quite different. All are complex quaternions and all relate back to the same two groups, $AA^{*} \equiv 1\sigma_0$ and $BB^{\neq} \equiv 1\sigma_0$. They are like decorations in the picture around the central groups. The groups in turn are manifestations of the number system itself. All of this stuff 'must' be at the core of the physical world, if 1+3 dimensional spacetime is fundamental to the real universe!

OTHER CONJUGATIONS

So far we have $\{\sigma_0, i\sigma_k, i\sigma_0, \sigma_k\}$ as the basic number system. There are two basic conjugations: $(\,)^{\neq}$ and $(\,)^{*}$. The $\neq$ changes σ_1, σ_2, σ_3 and * changes i. These are both antiautomorphic: $(AB)^{conj.} = B^{conj.}A^{conj.}$. Are there other antiautomorphic conjugations? If so, could they have physical significance as well?

We can invent many new conjugations by the procedure:

$$A^{conj.} \equiv \sigma A^{*}(\sigma)^{-1}, \quad \sigma\sigma^{-1} \equiv 1\sigma_0$$

where A is any of the 8 elements and σ is any one of the 8 elements. Clearly, $\sigma \rightarrow \sigma_0$ and $\sigma \rightarrow i\sigma_0$ gives $A^{conj.} = A^{*}$, which we already have. For $\sigma \rightarrow \sigma_1$, we get $A^{\#1} = (\sigma_1)A^{*}(\sigma_1)$. Calculating $A^{\#1}$, for A equal to each basis element, is easy and good practice for you. You will find

$$\{\sigma_0, i\sigma_k, i\sigma_0, \sigma_k\}^{\#1} = \{\sigma_0, -i\sigma_1, i\sigma_2, i\sigma_3, -i\sigma_0, \sigma_1, -\sigma_2, -\sigma_3\}$$

We get the same results for $(i\sigma_1)A^*(-i\sigma_1)$. For $\sigma_2 A^* \sigma_2$ we get: $\{\sigma_0, i\sigma_1, -i\sigma_2, i\sigma_3, -i\sigma_0, -\sigma_1, \sigma_2, -\sigma_3\}$. You can now guess what $\sigma_3 A^* \sigma_3$ gives without doing the calculation.

Are these antiautomorphic conjugations? Yes! They are if A^* is antiautomorphic itself. By the way, $\sigma A^{\neq} \sigma^{-1}$ would also generate antiautomorphic conjugations then too. Are they the same or different? Find them and see for yourself. Here is the proof that $\sigma A^* \sigma^{-1}$ is antiautomorphic:

$$(\sigma A^* \sigma^{-1})(\sigma B^* \sigma^{-1}) = \sigma A^* (\sigma^{-1}\sigma) B^* \sigma^{-1} = \sigma A^* B^* \sigma^{-1}$$

and

$$\sigma (BA)^* \sigma^{-1} = \sigma A^* B^* \sigma^{-1} = (BA)^{conj.}$$

We see that $A^{conj.} B^{conj.} = (BA)^{conj.}$ in general, where A and B are any two of the 8 basis elements.

It turns out that these conjugations are apparently not so useful. Perhaps because they treat the σ_1, σ_2, σ_3 parts differently. They single out one of them and nature does not have a favorite direction in space. The * and $\neq$ conjugations don't do this. Also notice again that

$$\{\sigma_0,\ i\sigma_k\}^{\neq} = \{\sigma_0,\ i\sigma_k\}^*$$

but

$$\{i\sigma_0,\ \sigma_k\}^{\neq} = -\{i\sigma_0,\ \sigma_k\}^*$$

The $\{\sigma_0, i\sigma_k\}$ subset is closed under multiplication, whereas the other one is not. This pattern between the closed and the non-closed segments of the algebra will occur again in the larger algebras that follow in later chapters. There, we shall find more useful conjugations besides the two here.

The conjugation $\sigma_1 \to -\sigma_1$ but $\sigma_2 \to \sigma_2$ and $\sigma_3 \to \sigma_3$ is called space inversion or parity. It does play a role in the details of quantum theory. Thus we have not invented a totally useless conjugation here.

This has been a depressing start for most readers. The rest of the book is not all this tedious. But if we ask deep questions, then we must expect complicated answers about nature. That has been the pattern over time in our advancement and mastery. The deeper we go the harder it gets to still understand what is needed to be understood. At some point we will reach our limits and just not be able to go farther. How soon? No one can say at present.