CHAPTER 1

BASIC NOTIONS AND DEFINITIONS

1.1 Signals and Systems

In everyday practice electrical and electronic engineers working on a variety of different problems are confronted with observations and measurements of different variables, parameters and signals coming from physical and natural systems. These systems are either powered by electrical energy and their principle of operation is electric in nature or electric signals and electronic equipment arise in the techniques which are used to sense and control them. Electrical signals may originate from industrial processes, household appliances, radio, TV, telecommunication equipment, biomedical test equipment or even natural phenomena like wind, weather or seismic activity. In virtually all domains of science and engineering we use electrical signals (voltages, currents, field intensity etc.) in one way or another. Typically these signals are sensed using probes and sensors connected to measurement equipment which in most modern approaches is computerized and serves as a data storage and processing unit. Electrical signals may also be produced as specific outputs during computer experiments. By means of transducers, electrical signals may be used to influence the behavior of the "object" of our interest; they can supply energy necessary for its operation or cause a specific type of behavior to occur.

From the analysis point of view we need to understand the physical meaning of these signals and consider mathematical models of the processes producing such signals. We need to put our observations into a more abstract framework. We need mathematical models which will be able to describe with a needed accuracy the behavior of a real process or object. To be able to analyze the nature of the signals we will use a number of mathematical notions which precisely describe properties that are of interest to the user.

For the purpose of this work we will assume that signals which we observe in practice are produced by systems.

Definition 1.1 (System – intuitive definition) *We will consider a system as an object, a device or a mechanism (phenomenon) performing certain operations or producing some phenomena, and having a number of connections*

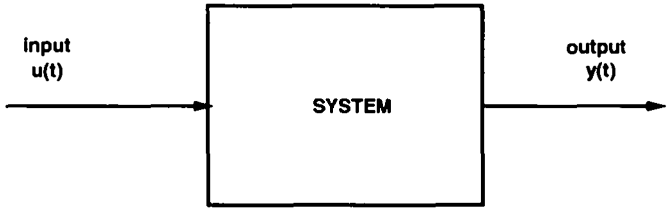

Figure 1.1: Schematic representation of the system.

with the outside world—some of these connections are classified as inputs (by means of which the outside world influences the system behavior) and some as outputs (which give the reaction of the system to the inputs). The system performs operations on the input signals to produce the output signals. It is often denoted symbolically in the form of a block diagram shown in Fig.1.

Definition 1.2 (System - proper mathematical setting) *From the mathematical point of view a system is described by a triplet* (U, Y, F), *where* U *is the set of input signals,* Y *is the set of output signals and the operation (law)* $F : U \to Y$ *describes the correspondence between given inputs and outputs (reaction of the system in terms of the outputs to given inputs).*

• **Remark:**
Usually u and y are functions of time (we call then the system *temporal system*) $u = u(t), y = y(t)$).

It should be stressed here that this definition is very general and does not impose any limitations on the classes of the input and output signals (these could be electrical, mechanical or any other variables, continuous in time or discontinuous, corrupted by noise etc.). Furthermore, the operator describing the "action" of the system can take a very general form starting from summation or multiplication, through nonlinear functions, to time averages, probabilistic measures, time-delay operations and integro-differential operators etc..

Simple electronic examples of systems could be amplifiers, filters, A/D and D/A converters, transmission channels etc.

Following the type of signals on which the system operates we can classify systems in the following way:

- **Analog** systems for which the input and output signals are analog ($u : \mathbb{R} \mapsto \mathbb{R}^n$, $y : \mathbb{R} \mapsto \mathbb{R}^n$ or $u : \mathbb{R}^+ \mapsto \mathbb{R}^n$, $y : \mathbb{R}^+ \mapsto \mathbb{R}^n$). Example: analog filter, audio amplifier etc.

- **Discrete-time systems** for which the input and output signals are discrete in time but take analog values ($u : \mathbb{Z} \mapsto \mathbb{R}^n$, $y : \mathbb{Z} \mapsto \mathbb{R}^n$ or ($u : \mathbb{Z}^+ \mapsto \mathbb{R}^n$, $y : \mathbb{Z}^+ \mapsto \mathbb{R}^n$). Example: Switched-capacitor filter.

- **Digital** systems (discrete-time and discrete-value) for which the input and output signals are defined for discrete time moments and can take values from a finite set only. Example: digital filter.

Obviously there exist systems which are mixed analog/digital such as an A/D converter for which the input signals are analog and the outputs are digital. Also one can easily identify systems that do not fall into any of the above-mentioned categories like measuring instruments where outputs might be the read-outs in alpha-numeric form and many others.

The definition of a system given above, although admitting a wide variety of possible operations and signals, suffers from one major drawback. In many cases and especially in nonlinear systems it is not possible to define properly the operation linking the system outputs with its inputs. Either the mathematical description can be given only in the form of implicit equations which are impossible to solve explicitly for the outputs, or determination of the output signals requires knowledge of some additional internal variables referred to as **states** of the system. In this case we say that the system has **memory**. To be able to deal with systems with memory we have to augment our set of definitions.

Throughout this book, we will restrict ourselves to deterministic systems and we will consider in particular **deterministic dynamical systems**. The notion of **dynamics** is closely associated with time evolution of the system variables (internal states and outputs). A **deterministic** system is defined as one for which knowledge of its initial state at some initial time t_0, equations of evolution and input signals fully determines its state and outputs for any $t > t_0$.

In strict mathematical terms we can introduce the following definition:

Definition 1.3 (Dynamical system—abstract definition) *The triplet* (X, T, π), *where* X *is a metric space*[a] T *is a group* [b] *and* π *is a continuous*

[a]A metric space is a pair (X, ρ) where X is a set (later called the space, and its elements called points) and ρ is a mapping $\rho : X \times X \mapsto R^+ \cup 0$ called the distance, satisfying

$$\begin{aligned} \rho(x,y) &= 0 \Leftrightarrow x = y \\ \rho(x,y) &= \rho(y,x) \text{ for all } x,y \in X \\ \rho(x,y) &\leq \rho(x,z) + \rho(z,y) \text{ for all } x,y,z \in X \end{aligned}$$

[b]A nonempty set T is called a semi-group if an operation $*$ on its elements can be defined

operation from $X \times T$ into X is called a dynamical system if for all $x \in X$ and all $t_1, t_2 \in T$ the following conditions are satisfied:

$$\pi(x, 0) = x \tag{1.1}$$

$$\pi(\pi(x, t_1), t_2) = \pi(x, t_1 + t_2) \tag{1.2}$$

Again, if $T = \mathbb{Z}$ we will say that we consider a discrete dynamical system, and if $T = \mathbb{R}$ we will have a continuous-time (analog) system.
In some situations we consider $T = \mathbb{Z}^+$ or $T = \mathbb{R}^+$; in this case T is a semi-group. In these cases we have a so-called **semi-dynamical system**.
• **Note:** In this definition the input signals are not explicitly mentioned. To define dynamical system in mathematical terms we just describe the time evolution of the state and put some mild restriction on the properties on the operation describing this evolution.

In engineering applications we tend to consider a less abstract definition in which the sets of inputs, states and outputs and operations linking these signals are explicitly defined. This framework enables us to consider for example general problems of finding classes of output signals for given classes of inputs or addressing the common control theory questions (like controllability) for general classes of input signals.

Definition 1.4 (Dynamical System—control engineer's definition)
The quintuplet (U_s, X, Y_s, F_1, F_2), where $U_s = \{u : T \to U\}$ is called the input space (sometimes called also the space of controls), T is a linearly ordered set interpreted as time, U is the set of momentarily values of inputs , X is the state space (often called in the literature the phase space), $Y_s = \{y : T \to Y\}$ is the output space , Y is the set of values of the outputs.
• **Note:** *U_s and Y_s are sets of functions of time.*
F_1 is the state transition operator linking the values of the state vector at two moments of time:

$$F_1 : X \times U_s \times T \times T \to X \tag{1.3}$$

F_2 is the output function:

$$F_2 : X \times U \times T \to Y \tag{1.4}$$

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such that the following conditions are satisfied:
1. $T \times T \ni (a, b) \mapsto a * b \in T$
2. the operation is associative ie. $a * (b * c) = (a * b) * c$ for all $a, b, c \in T$,
If there exists so-called neutral element $e \in T$ satisfying $a * e = e * a = a$ for all $a \in T$ we call such a set a semi-group with a neutral element.
Further, if for every $a \in T$ there exists a $b \in T$ (called an inverse element) such that $a * b = b * a = e$ the set T is called a group.

The quintuplet is called a **dynamical system** *if the following conditions are satisfied:*

1. **Evolution property** *for the operation* F_1 *meaning that:*
   if for all $t_1, t_2, t_3 \in T$, $t_1 \leq t_2 \leq t_3$ *and for all* $u_1, u_2 \in U_s$ *such that* $u_1(t) = u_2(t)$ *for* $t \in [t_2, t_3]$ *we have:*

$$F_1(x(t_1), u_1, t_1, t_3) = F_1(F_1(x(t_1), u_1, t_1, t_2), u_2, t_2, t_3) \tag{1.5}$$

   and

$$F_1(x(t_1), u_1, t_1, t_1) = x(t_1) \tag{1.6}$$

2. **Completeness condition** *for* F_1 *ie.:*
   for all $t, s \in T$, $s < t$, *for all* $u_1, u_2 \in U_s$ *such that* $u_1(h) = u_2(h)$ *for* $h \in [s, t)$ *we have*

$$F_1(x(s), u_1, s, t) = F_1(x(s), u_2, s, t) \tag{1.7}$$

   Often instead of the completeness condition we require that F_1 *satisfies the* **causality** *condition which is easily obtained by fixing s in the completeness condition as a minimal element of the set* T *(sometimes referred to as "beginning of time" eg. for* $T = [t_0, \infty)$ *we have to put* $s = t_0$*).*

3. **Separability condition** *for the inputs meaning that:*
   if u_1, $u_2 \in U_s$ *then also* $u_3 \in U_s$, *where* $u_3(t) = u_1(t)$ *for* $t \in [t_1, t_2)$ *and* $u_3(t) = u_2(t)$ *for* $t \in [t_2, t_3)$ *where* $t_1 < t_2 < t_3$, $t_1, t_2, t_3 \in T$. *(Comment: This condition is clearly introduced to be able to define later various control problems and combine control signals).*

Interpretation of the functions F_1 and F_2 is straightforward—if we know the state of the system $x(t_0)$ at some time moment $t_0 \in T$ and we know the values of $u(t)$ for $t > t_0$, then the state of the system at $t > t_0$ is only a function of the initial state, the initial and actual time and the input function in the interval between the initial and actual times, and can be calculated using

$$x(t) = F_1(x(t_0), u, t_0, t) \tag{1.8}$$

The output of the system is a function of the values of the actual time and the state input functions at that moment and can be determined by

$$y(t) = F_2(x(t), u(t), t) \tag{1.9}$$

As can be seen from the above discussion and the three problem statements there is some freedom in choosing the definition of a system — depending on purpose such definition could serve.

• **Note:** There exist also definitions of system in which initial conditions or parameter values can be considered as inputs. The latter case is often considered in bifurcation studies where influence of parameter changes on dynamic behavior is the principal problem to be solved.

Traditionally the general class of systems is divided into **autonomous** and **non-autonomous** ones. A system is said to be autonomous if it has no external inputs.

In this work we will mainly consider dynamical systems described (in mathematical terms, "generated") by ordinary differential equations of the form:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(\mathbf{x}(t), \mathbf{u}(t), t), \ \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1.10}$$

with an output function defined as a linear combination of states and inputs

$$\mathbf{y}(t) = \mathbf{C}^T\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{1.11}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{F} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}^n$.

This kind of description is often shortened by hiding the functions of time $\mathbf{u}(t)$ in the general dependence of the right-hand side of the equation on time.

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(\mathbf{x}(t), t), \ \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1.12}$$

Where: $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{F} : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$.

Thus we obtain two equivalent forms corresponding to two general definitions of the dynamical system introduced before.

For such a description we will say that the system is autonomous if the right-hand side of equation (1.12) does not depend explicitly on time, ie.:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(x(t)), \ \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1.13}$$

The function $\mathbf{F} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is called a **vector field**. The solution of equation (1.13) with the initial condition $\mathbf{x}_0$ is called a **trajectory** and denoted by $\varphi(\mathbf{x}_0, t)$. We will also often denote such a solution simply by $\mathbf{x}(t)$. Trajectory represents a set of pairs "(point coordinates, time)".

The following theorem tells us exact conditions under which the description in terms of differential equations defines a dynamical system in the sense of definition (1.3):

Theorem 1.1 *Let us consider an autonomous system of differential equations (1.13) with a continuous function* $\mathbf{F} : \mathbb{R}^n \mapsto \mathbb{R}^n$. *Let us assume that for every* $\mathbf{x} \in \mathbb{R}^n$ *there exists exactly one solution* $\varphi(\mathbf{x}, t)$ *passing through* $\mathbf{x}$ *and defined on* $\mathbb{R}$ *satisfying the condition* $\varphi(\mathbf{x}, 0) = \mathbf{x}$. *Then the transformation* $\pi(\mathbf{x}, t) \triangleq \varphi(\mathbf{x}, t)$ *defines a continuous-time dynamical system on* $\mathbb{R}^n$.

Theorem 1.2 *If the transformation* $\mathbf{F}$ *satisfies a global Lipschitz condition, ie.:*

$$\exists K > 0 : \; ||\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})|| \leq K||\mathbf{x} - \mathbf{y}|| \; \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \tag{1.14}$$

then the condition on existence and uniqueness of solutions mentioned in the previous theorem is satisfied.

• **Note:** For many mathematical models of systems (eg. differential equations) the solutions exist only on subsets of $\mathbb{R}$; they do not exist for all times, eg. they do not exist for $t \to -\infty$ or can be defined only on an interval $t \in [t_0, t_f)$. From the engineer's point of view this can be viewed as a modeling error as real systems always possess solutions for all times.

In the case of discrete-time systems we have the following theorem which clarifies the properties of function f which are required to guarantee the generation of a dynamical system structure:

Theorem 1.3 *If* $f : X \to X$ *is a continuous bijective function and* $\varphi(x, n) \triangleq f^n(x)$ *then* $(X, \mathbf{Z}, \varphi)$ *is a discrete-time dynamical system. If* f *is not bijective then* (X, φ) *is a semi-dynamical system.*

There exists a special class of discrete-time dynamical systems which are often encountered when analyzing continuous dynamical systems. These are so-called Poincaré maps. Let us introduce the definition of this notion next.

Definition 1.5 (Poincaré map) *Let us denote by* $\varphi_t = \varphi(., t)$ *a family of solutions of the system (1.13). Let us define locally a hyper-plane* $\Sigma \subset \mathbb{R}^n$ *of dimension* $n - 1$, *transversal*[c] *to* φ_t *at some point* $\mathbf{x}_1$. *Let us assume further that there exists a point* $\mathbf{x}_2 = \varphi(\mathbf{x}_1, T) \in \Sigma$ *which belongs to the trajectory of* $\mathbf{x}_1$ *and at which the trajectory intersects* Σ *(ie.* Σ *is also transversal to flow at* $\mathbf{x}_2$*). We will assume that the trajectory for* $t \in (0, T)$ *does not pass through* Σ. *Let* $U \subset \Sigma$ *be some neighborhood of* $\mathbf{x}_1$. *Then the Poincaré map (return map)* $\Pi : U \to \Sigma$ *for point* $\mathbf{x}_u \in U$ *is defined by*

$$\Pi(\mathbf{x}_u) = \Pi_\Sigma(\mathbf{x}) \triangleq \varphi(\mathbf{x}_u, \tau) \tag{1.15}$$

where $\tau = \tau(\mathbf{x}_u)$ *is the time after which the trajectory* $\varphi(\mathbf{x}_u, t)$ *for the first time returns (and intersects) to* Σ.

[c]Transversal intersection is such that vector tangent to the flow at the intersection point together with the vectors spanning the hyper-plane span whole n-space.

In the definition given above, it is sufficient to consider a bundle of trajectories that correspond to recurrent motion i.e. after a finite time they return to a neighborhood of their initial states. In many systems, trajectories are not recurrent, at least in certain regions of their state space.

We have the following theorem which guarantees the existence of a continuous Poincaré map:

Theorem 1.4 *If the hyper-plane Σ is transversal to the flow φ_t at points $\mathbf{x}_1$ and $\mathbf{x}_2 = \varphi(\mathbf{x}_1, T)$, then there exists an open neighborhood U of $\mathbf{x}_1$ and exactly one mapping $\tau : U \mapsto \mathbb{R}$ which is C^1 and for every $\mathbf{x} \in U$, $\varphi_{\tau(\mathbf{x})}(\mathbf{x}) \in \Sigma$ and $\tau(\mathbf{x}_1) = T$.*

In this way instead of looking at the behavior of system trajectories we can analyze the properties of series of points being successive intersections of a trajectory with the Poincaré plane $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ (this is valid in regions of the phase space where the Poincaré map is well-defined by system trajectories). The series $\Pi^k(\mathbf{x}_0)$ is called the **trajectory of the Poincaré map** for the point $\mathbf{x}_0$. We will denote by

$$\Pi^n(\mathbf{x}_u) = \Pi(\Pi^{n-1}(\mathbf{x}_u)) \tag{1.16}$$

the n-times composition of the Poincaré map. Equation (1.13) has a *periodic solution* if and only if for the associated Poincaré map there exist a positive integer n and an initial point $\mathbf{x}_*$ such that $\Pi^n(\mathbf{x}_*) = \mathbf{x}_*$. The point $\mathbf{x}_*$ is a *fixed point* of the map Π and is the point of intersection of the periodic *orbit* γ of (1.13) with the plane Σ. The stability of this fixed point reflects the stability properties of the orbit γ. Looking at the trajectory one could say that the periodic trajectory makes n loops before closing itself in the space. If $n = 1$ we have so-called period-one orbit. In more general mathematical setting we have the following result concerning the stability of fixed points of the Poincaré map:

Theorem 1.5 *Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be any two points belonging to some periodic orbit γ. Let Σ_1 be a hyper-plane transversal to the flow at $\mathbf{x}_1$ and let Σ_2 be a hyper-plane transversal to the flow at $\mathbf{x}_2$. Then the Jacobian matrices $S_1 = D\Pi_{\Sigma_1}(\mathbf{x}_1)$ and $S_2 = D\Pi_{\Sigma_2}(\mathbf{x}_2)$ are* similar *i.e. there exists a non-singular matrix P such that $S_1 = P^{-1}S_2P$.*

A consequence of this theorem is that the eigenvalues of the Jacobian matrix of the Poincaré map generated by a flow (solutions of a continuous-time dynamical system) do not depend on the choice of the point on the periodic orbit or on the choice of the transversal plane.

1.2 Basic properties of solutions

Throughout this book we will be interested in asymptotic properties of solutions of dynamical systems and in particular of (1.13) i.e. we will analyze the behavior of system solutions for $t \to \infty$. [d] Asymptotic behaviors of systems can be very complicated and even bizzare [140,165,166,256]. To be able to distinguish between them we need to introduce some further notions and definitions. [39,171].

Definition 1.6 (Trajectory) *The set* $\gamma(x) = \{(\pi(x,t),t), t \in T\}$ *(set of all pairs* (x,t) *— "(point,time)") is called a trajectory of point* x. *The sets* $\gamma^+(x) = \{\pi(x,t) : t \geq 0, t \in T\}$ *and* $\gamma^-(x) = \{\pi(x,t) : t \leq 0,\ t \in T\}$ *are called a positive- and negative semi-trajectories of point* x *respectively. The transformation* $\pi_x : T \to X$ *by* $\pi_x(t) = \pi(x,t)$ *is called the* **motion** *through* x.

Definition 1.7 (Orbit) *The set all points* $\Gamma(x) = \{\pi(x,t) : t \in T\}$ *is called an orbit of point* x.

Orbit is a geometric object (a set of points).

Definition 1.8 (Invariant set) *The set* $S \subset X$ *(*$S \subset \mathbb{R}^n$*) such that*

$$\pi(x,t) \in S \ \forall x \in S \text{ and } \forall t \in T$$

is called an invariant set.

In terms of trajectories, the invariant set contains whole trajectories i.e. for all $x \in S$, $\gamma(x) \in S$.

Typical examples of limit sets are fixed points, closed (periodic) trajectories, integral manifolds etc. We will be interested in particular in bounded invariant sets.

Definition 1.9 (Non-decomposable set) *Set* $S \subset X$ *(*$S \subset \mathbb{R}^n$*) is called non-decomposable if it is closed and invariant and for any* $x, y \in S$ *and for any* $\varepsilon > 0$ *there exist* $x = x_0, x_1, \ldots, x_n = y$ *and* $t_1, \ldots t_n \geq 1$ *such that the distance between* $\pi(x_{i-1}, t_i)$ *and* x_i *is smaller then* ε.

Definition 1.10 (Minimal set) *Set* $S \subset X$ *(*$S \subset \mathbb{R}^n$*) is called minimal if it is non-empty, closed and invariant and no subset of it has these properties.*

Definition 1.11 (Fixed point) *The point* $x \in X$ *is called a fixed point if for every* $t \in T$ *we have* $\pi(x,t) = x$.

Definition 1.12 (Periodic point) *The point* $x \in X$ *is called a* **periodic point** *if* x *is not a fixed point and there exists* $\tau > 0$ *such that* $\pi(x,\tau) = x$. *The number* $\tau_0 = inf\{t > 0 : \pi(x,t) = x\}$ *is called the period of the periodic point* x. *If the point* x *is periodic then the trajectory* $\gamma(x)$ *is also called periodic.*

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[d]In engineering practice such a solution, if it exists, is called a steady state.

Definition 1.13 (Quasi-periodic motion) *The motion π_x is called* **quasi-periodic** *if $\forall \varepsilon > 0$ there exists a relatively dense set*[e] *of numbers τ_n, $n \in \mathbf{Z}$ such that*

$$\rho(\pi(x,t), \pi(x, t+\tau_n)) < \varepsilon \tag{1.17}$$

for every $t \in T$ and every integer n, where ρ denotes the distance between points.

Definition 1.14 (Limit sets) *The set $\omega(x)$ of all points p, such that*

$$\exists \{t_i\} : \lim_{t_i \to \infty} \pi(x, t_i) = p$$

is called the positive limit set of point x.

The set $\alpha(x)$ of all points p, such that

$$\exists \{t_i\} : \lim_{t_i \to -\infty} \pi(x, t_i) = p$$

is called the negative limit set of point x.

Definition 1.15 (Positive (negative) recursive set) *Set $S \subset X$ ($S \subset \mathbb{R}^n$) is called positively (resp. negatively) recursive with respect to the set $B \subset X$ if for every $x = x(t) \in B$, (for some $t \in T$) there exists $s > t$ (resp. $s < t$) such that $\pi(x,s) \in S$.*

1.3 Stability notions

We will use several notions of stability to help us establish the asymptotic properties of solutions.

Definition 1.16 (Stability in the sense of Lyapunov)
The trajectory $\gamma(x_0) = \{\pi(x_0,t) : t \in T\}$ is stable in the sense of Lyapunov if for every $\varepsilon > 0$ exists $\delta > 0$ such that:

$$||x - x_0|| < \delta \Rightarrow sup_{t \geq 0} ||\pi(x,t) - \pi(x_0,t)|| < \varepsilon \tag{1.18}$$

If, in addition, $\lim_{t\to\infty} ||\pi(x,t) - \pi(x_0,t)|| = 0$ then the solution $\pi(x_0,t)$ is called **asymptotically stable** *in the sense of Lyapunov.*

Stability in the sense of Lyapunov is a *local* property valid in the vicinity of a chosen solution.

Comments:

[e] The set D of real numbers is called *relatively dense* if there exists $t_0 > 0$ such that $D \cap (t - t_0, t + t_0) \neq \emptyset$ for all $t \in \mathbb{R}$, i.e. D has at least one *accumulation point.*

• For systems described by equations of the type (1.12) the analysis of the stability of a chosen solution $x_0(t) = x(x_0, t)$ can be transformed into analysis of the stability of the zero solution of an associated system. Substituting $x(t) = x_0(t) + z(t)$ we obtain for the new variable $z(t)$ the equation:

$$\frac{dz(t)}{dt} = G(z(t), t) \tag{1.19}$$

where $G(z,t) = F(x_0(t) + z(t), t) - F(x_0(t), t)$, $G(0,t) = 0$. Following the definition of stability in the Lyapunov sense, stability of $x_0(t)$ (trajectory through x_0) is equivalent to stability of the zero solution of (1.19).

• **Linearization**

Let us assume that the function $G(z,t)$ is C^1 with respect to z. Then there exists a matrix $A(t)$ such that for small $||z||$ we have:

$$G(z,t) = A(t)z + g(z,t) \text{ where } \lim_{z \to 0} \frac{||g(z,t)||}{||z||} = 0 \tag{1.20}$$

The matrix $A(t) = \frac{\partial F(x,t)}{\partial x}|_x = x_0(t)$ is called the Jacobian matrix and the equation

$$\frac{dz(t)}{dt} = A(t)z(t) \tag{1.21}$$

is called the linearized equation for the problem (1.19) .

• Let us assume that $G(z,t)$ is periodic in time or a constant matrix. Then the matrix $A(t)$ in (1.21) is also periodic or constant. We have then the following theorem due to Lyapunov:

Theorem 1.6 (Lyapunov) *If all the* characteristic exponents *of the linearized equation (1.21) have negative real parts (ie. if its zero-solution is asymptotically stable) then the zero solution of the equation (1.19) is also asymptotically stable. If at least one characteristic exponent of (1.21) has a positive real part then the zero solution of (1.19) is not stable in the Lyapunov sense.*

• In an autonomous system (1.13) the function F does not depend on time. Let x^0 denote a fixed point of (1.13) ie. $F(x^0) = 0$. The linearized equation around the fixed point is a differential equation with fixed coefficients;

$$\frac{dz(t)}{dt} = Az(t) \text{ where } A = \frac{\partial F(x)}{\partial x}|_{x=x^0} \tag{1.22}$$

In this case the stability properties of solutions are determined by the eigenvalues of the matrix A ie. the zeros of the characteristic polynomial $W(\lambda) = det(\lambda I - A)$. If all the zeros have negative real parts then the fixed point is asymptotically stable in the sense of Lyapunov. A polynomial $W(\lambda)$, all of whose roots have negative real parts, is called *Hurwitz*.

- *Eigenvalues and stability of equilibrium points*

Let x^* be an *equilibrium point* ie. a solution of the equation:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(x(t)) = 0 \tag{1.23}$$

A complete description of stability of this equilibrium is contained in the linearization of equation 1.13 about this equilibrium point. Eigenvalues of the matrix $\mathbf{A} = \frac{\partial F(x)}{\partial x}|_{x=x^*}$ determine the stability type of the equilibrium point.

If the real parts of all of the eigenvalues of $\mathbf{A}$ are strictly negative, then the equilibrium point x^* is *asymptotically stable* and is called a *sink* because all nearby trajectories converge towards it.

If any of the eigenvalues has a positive real part, the equilibrium point is *unstable*; if all eigenvalues have positive real parts, the equilibrium point is called a *source*. An equilibrium point which has both stable and unstable eigenvalues is called a *saddle*.

An equilibrium point is said to be *hyperbolic* if all of the eigenvalues of $\mathbf{A}$ have non-zero real parts. All hyperbolic equilibrium points are either unstable or asymptotically stable.

For discrete-time dynamical system:

$$\mathbf{x}_{k+1} = \mathbf{G}(\mathbf{x}_k)$$

stability of equilibria x^* is determined by the eigenvalues of the linearization $\mathbf{A} = \frac{\partial G(x)}{\partial x}|_{x=x^*}$ of the vector field $\mathbf{G}$, evaluated at x^*.

The equilibrium point is classified as *stable* if all of the eigenvalues of $\mathbf{A}$ are strictly less than unity in modulus, and *unstable* if any has modulus greater than unity.

- *Eigenvalues, eigenvectors, eigenspaces, stable and unstable manifolds*

Associated with each distinct eigenvalue λ of the Jacobian matrix $\mathbf{A}$ is an eigenvector $\mathbf{v}$ defined by

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{1.24}$$

A real eigenvalue γ has a real associated eigenvector $\vec{\eta}$. Complex eigenvalues of a real matrix occur in pairs of the form $\sigma \pm j\omega$. The real and imaginary parts of the associated eigenvectors $\vec{\eta_r} \pm j\vec{\eta_c}$ span a plane called a *complex eigenplane.*

For linear (linearized) systems the n_s-dimensional subspace of $\mathbb{R}^n$ associated with the stable eigenvalues of the Jacobian matrix is called the *stable* *eigenspace*, denoted $E^s(x^*)$. The n_u-dimensional subspace corresponding to the unstable eigenvalues is called the *unstable eigenspace*, denoted $E^u(x^*)$. Geometrically speaking these eigenspaces are lines, planes or hyper-planes.

The analogs of the stable and unstable eigenspaces for a general nonlinear system are called the local stable and unstable *manifolds*[f] $W^s(x^*)$ and $W^u(x^*)$.

The stable manifold $W^s(x^*)$ is defined as the set of all states from which trajectories remain in the manifold and converge under the flow to x^*. The unstable manifold $W^u(x^*)$ is defined as the set of all states from which trajectories remain in the manifold and diverge under the flow from x^*.

By definition, the stable and unstable manifolds are *invariant* under the flow (if $\mathbf{x} \in W^s$, then $\varphi(\mathbf{x}, t) \in W^s$). Furthermore, the n_s- and n_u-dimensional *tangent* spaces to W^s and W^u at x^* are E^s and E^u. In the special case of a linear or affine vector field $\mathbf{F}$, the stable and unstable manifolds are simply the eigenspaces E^s and E^u themselves[171].

• In systems with periodic driving (having a periodic input signal) the function $F(x,t)$ in (1.12) is a periodic function of t. Let $x_T(t)$ be a periodic solution of equation (1.12) with a period which is commensurate with (i.e. rationally related to) that of the driving signal. In such a case, the matrix $A(t) = \frac{\partial F(x,t)}{\partial x}|_{x=x_T(t)}$ is a periodic function of time. In this case, the stability analysis of solutions can be reduced the analysis of a system of differential equations with constant coefficients, due to the famous Floquet theorem:

Theorem 1.7 (Floquet) *Let $A(t)$ be a matrix of continuous periodic functions with period T_0. Let us consider a differential equation of the form:*

$$\frac{dx(t)}{dt} = A(t)x(t) \tag{1.25}$$

In this case there exists a nonsingular quadratic matrix $P(t)$ of continuous periodic functions of period T_0 such that the change of variables $x(t) = P(t)z(t)$ brings equation (1.25) into the form

$$\frac{dz(t)}{dt} = Qz(t) \tag{1.26}$$

with a constant Q matrix.

Unfortunately there is no simple way for finding $P(t)$ and Q. This theorem is however important from the qualitative point of view because the solutions will always have a form of a linear combination of solutions of the equation

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[f] An m-dimensional manifold is a geometrical object every small section of which looks like $\mathbf{R}^m$. For example, a limit cycle of a continuous-time dynamical system is a one-dimensional manifold.

with constant coefficients and continuous periodic functions. The eigenvalues of the matrix Q are called the characteristic multipliers of the equation (1.25).

• Let us consider next the stability of periodic solutions of the equation (1.13). Let $x_T(t) \triangleq x_T(x+T)$ be a nontrivial (non-constant) periodic solution of the autonomous system (1.13). Such a solution can be stable or unstable in the sense of Lyapunov but never asymptotically stable. To prove this it is enough to see that for any t_0 $x_T(t+t_0)$ is also a solution of the equation (1.13) and if t_0 is small then $x_T(0)$ is very near $x_T(t_0)$ and the difference $x_T(t)-x_T(t+t_0)$ is a periodic function not vanishing to 0 as $t \to \infty$. For periodic solutions of autonomous systems we need to introduce a special notion of stability, namely *orbital stability.*

Definition 1.17 (Orbital stability) *Let $x_T(t)$ be a periodic solution of equation (1.13) and let $\gamma_T = \{x_T(t) : t \in \mathbb{R}\}$ be the periodic orbit (trajectory) associated with this solution. In the state space this orbit represents a closed trajectory. Let $\rho(z, \gamma_T)$ be the distance between a point z and the periodic orbit γ_T.*

The periodic solution $x_T(t)$ is called orbitally stable if for every $\varepsilon > 0$ exists $\delta > 0$ such that

$$||x_0 - x_T(0)|| < \delta \textit{ implies } \rho_{t \geq 0}\{x(x_0,t), \gamma_T\} < \varepsilon \tag{1.27}$$

If, in addition,

$$\lim_{t\to\infty} \rho\{x(x_0,t), \gamma_T\} = 0 \tag{1.28}$$

we say that the periodic solution $x_T(t)$ is orbitally asymptotically stable.

Definition 1.18 (Stability in the sense of Poisson) *The point $x \in X$ is called* positively (negatively) stable in the sense of Poisson *if every neighborhood U of the point x is positively (resp. negatively) recursive in respect to the one-element set x ie. for every $t \in T$ there exists $s > t$ such that $\pi(x,s) \in U$. (An equivalent condition can be used for this definition, namely:*

point x is positively (negatively) stable in the Poisson sense if $x \in \omega(x)$ (resp. $x \in \alpha(x)$).

Definition 1.19 (Non-wandering point) *Point $x \in X$ is called non-wandering if every neighborhood U of x is positively recursive with respect to itself ie. $\forall t \in T$, $\forall y \in U$, $\exists s > t$ such that $\pi(y,s) \in U$.*

Definition 1.20 (Stability in the sense of Lagrange) *For any $x \in X$ the motion π_x is stable in the sense of Lagrange (positively or negatively) if the set $\overline{\gamma(x)}$ ($\overline{\gamma^+(x)}$ or $\overline{\gamma^-(x)}$) is compact.*

In typical situations where $X = \mathbb{R}^n$, stability in the sense of Lagrange is equivalent to boundedness of the sets $\gamma(x)$, $\gamma^+(x)$, $\gamma^-(x)$.

All these notions of stability are closely related to a concept which is widely used when studying nonlinear dynamical systems and chaos, namely **sensitive dependence on initial conditions** .

Definition 1.21 (Sensitive dependence on initial condition) *The mapping $\xi : X \mapsto X$ (defined by some discrete-time dynamical system or a Poincaré map associated with solutions of a continuous-time dynamical system) is said to have sensitive dependence on initial conditions*[102] *if there exists a number $\tau > 0$ such that for all $x \in X$ and for any neighborhood U of x, there exist $y \in U$ and $n > 0$ such that $\rho(\xi^n(x), \xi^n(y)) > \tau$ (where $\rho(x, y)$ denotes the distance between x and y).*

It means that there exists a **separation constant** τ such that in every infinitely small neighborhood of the point $x \in S$ there is always a point y which will eventually but not necessarily permanently move away from x to a distance of at least τ.

1.3.1 Attracting sets and attractors

Definition 1.22 (Attracting set) *An invariant set $A \subset X$ ($A \subset \mathbb{R}^n$) is called* attracting *if there exists some neighborhood U of A, such that*

$$\forall x \in U : \varphi(x, t) \in U \text{ for } t \geq 0 \text{ and } \lim_{t\to\infty} \varphi(x, t) = A$$

Definition 1.23 (Attractor) *An attracting, bounded invariant set $A \subset X$ ($A \subset \mathbb{R}^n$) which contains a dense trajectory*[9] *is called an attractor.*

This means that an attractor has a neighborhood $U(A)$ such that for any $x_0 \in U(A)$ the positive semi-trajectory starting at x_0 remains in $U(A)$ for $t \in [0, \infty)$ and tends to A as $t \to \infty$.

Definition 1.24 (Domain of attraction) *The set $A_\Omega = \bigcup_{t\leq 0} \varphi(U, t)$ is called the domain of attraction of the attractor A (It is a union of pre-images of points contained in the neighborhood U mentioned in the previous definition).*

For systems described by differential equations (1.12) this definition can be reformulated for domain of attraction of an asymptotically stable solution $x^*(t)$. Namely:

Definition 1.25 (Domain of attraction for solutions of ODE) *The set of all initial points x_0 for which the solutions $x(x_0, t)$ satisfy:*

$$\lim_{t\to\infty} ||x(x_0, t) - x^*(t)|| = 0 \qquad (1.29)$$

[9]The condition on existence of a dense trajectory means that the attractor is a minimal set in the sense that it does not contain any other attractor, i.e. it is non-decomposable.

is called the domain of attraction of the solution $x^*(t)$.

1.4 Classification of attracting limit sets

Birkhoff[30] proposed a basic classification of trajectories and their corresponding limit sets which was further refined by Andronov. They have proposed the following classification of asymptotic behavior of system trajectories and their associated attracting limit sets:

- trajectories constant in time (limit set – a point),
- Periodic trajectories (limit set – closed curve),
- Quasi-periodic trajectories (limit set - torus),
- Chaotic trajectories. Recurrent, bounded in space trajectories not belonging to any of the above-mentioned classes, (stable in the Poisson sense) ; limit set of the Cantor type often referred to as "strange attractor"[h]

Trajectories belonging to this last class will be our principal interest throughout this book. We will try to refine their description to be able to understand the underlying mechanisms and analyze them.

• **Notes:**

There exist also *special trajectories* which are doubly (in positive and negative time) to equilibrium points or periodic trajectrories. Such trajectories are called *homoclinic* if they constitute a self-link between an equilibrium or periodic orbit and *heteroclinic* if they link two different equilibria or periodic orbits. Homoclinic and heteroclinic trajectories will be discussed also in Chapter 6. From the electronic engineer's point of view limit sets correspond to so-called steady state behavior ie. after all the transients die out.

1.5 Structural stability and bifurcations

To discuss the notion of *structural stability* we have to consider that the system behavior depends on a set of parameters μ. For example the dynamics of the system is described by a parameterized differential equation of the form:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, \mu), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{1.30}$$

[h]A Cantor set is a closed, non-empty set not possessing either internal points or isolated points ie. in every neighborhood of any point belonging to this set there are always points belonging and not belonging to it.

Structural stability refers to sensitivity of the dynamic behavior observed in the system to changes in the parameters. The vector field **F** will be called structurally stable if sufficiently close vector fields **F′** have equivalent dynamics (in terms of existence of a continuous invertible function which transforms **F** into **F′**. The term *conjugate* is sometimes used as a synonym for *equivalent.*

The dynamic behavior observed in the system may vary qualitatively when changing its parameters. For example tuning an oscillator one can clearly identify parameter values for which we observe onset of oscillations - thus we can observe stable equilibrium (no oscillations) or limit cycle (oscillations) depending on the chosen parameter value in the same circuit.

Qualitative changes in the observed dynamic behavior when changing system parameters are called *bifurcations.* The chosen variable parameter for the variation of which we observe changes in dynamic behavior is called the *bifurcation parameter.*

The value of bifurcation parameter at which a qualitative change in observed dynamic behavior is called a *bifurcation point.*

1.5.1 Basic types of bifurcations

In this section we will describe the most common types of bifurcations only. For high-dimensional systems very complex types of bifurcations not considered here can be encountered[2,14,15]. For understanding the phenomena analyzed later in this book we will consider only local bifurcations: the *Hopf bifurcation*, the *saddle-node* or *fold* bifurcation, the *symmetry-breaking* and the *period-doubling* or *flip* bifurcation. These bifurcations are called *local* because they may be understood by linearizing the system close to an equilibrium point or limit cycle.

• **Hopf bifurcation**

The Hopf bifurcation occurs when an equilibrium point changes stability from stable to unstable and a stable limit cycle is born. Looking at the eigenvalues of the linearized system we find that Hopf bifurcation occurs when a pair of complex eigenvalues moves out from the left-hand half of the complex plane to the right-hand half-plane passing the imaginary axis (ie. the sign of the real parts of a pair of complex eigenvalues changes from negative to positive). In electronic circuits this type of bifurcation is easily observable in all generators — at the onset of oscillations the Hopf bifurcation occurs. Hopf bifurcation also occurs for periodic orbits — in such a case a torus is born.

• **Saddle-node bifurcation (fold)**

The simplest case of saddle-node bifurcation is the case in which in the system we have two orbits — a stable one and an unstable one which when changing the bifurcation parameter, approach each-other and disappear at the bifurcation point.

This type of bifurcation is also typical in systems having multiple attractors. In a saddle-node bifurcation, one of two attractors loses its stability and "jumps" to the other. In the simplest case we observe jumps between two equilibrium points. Switching of states in a flip-flop or low-high transition in a memory cell corresponds in terms of dynamics to a saddle-node bifurcation. A common example of this in electronic circuits is also a Schmitt trigger. At the threshold for switching, a stable equilibrium point corresponding to the "high" state merges with the high-gain region's unstable saddle-type equilibrium point and disappears. After a switching transient, the trajectory settles to the other stable equilibrium point, which corresponds to the "low" state.

A saddle-node bifurcation may also manifest itself as a switch between periodic attractors of different size, between a periodic attractor and a chaotic attractor, or between a limit cycle at one frequency and a limit cycle at another frequency.

- **Symmetry-breaking bifurcation**

This is a specific type of bifurcation of periodic orbits (equilibria for maps) in which when changing the value of the bifurcation parameter a single stable periodic orbit of period T splits into two stable orbits of the same period co-existing at the same time. This kind of bifurcation often occurs in systems with symmetric nonlinearities (an example of such behavior will be shown when analyzing bifurcations in the RC-ladder generator).

- **Period-doubling bifurcation (flip)**

A period-doubling bifurcation occurs only with periodic solutions. At the bifurcation point, a periodic trajectory with period T changes smoothly into one with period $2T$. An interesting fact is that the period-1 orbit daos not disappear – it just changes stability becoming unstable. The unstable periodic orbits created apart from stable orbits with doubled period in this kind of bifurcations will manifest themselves when analyzing chaotic attractors. As a matter of fact the infinite number of unstable periodic orbits embedded in a chaotic attractor were born in period doubling bifurcations leading to chaos.

- **Crisis**

The notion of crisis is associated with specific types of bifurcation phenomena in which orbits (attractors) co-existing in the state space collide with each-other or with an unstable orbit resulting in a qualitative change of behavior. This is the case fro example for so-called *blue sky catastrophe* in which the attractor disappears "in the blue sky" after collision with a saddle-type (unstable) periodic orbit. This is a global bifurcation.

1.6 Routes to chaos

Each of the local bifurcations may give rise to a distinct route to chaos if the bifurcations appear repeatedly when changing the bifurcation parameter. These routes are important because it is often difficult to conclude from experimental data alone whether irregular behavior is due to measurement noise or to underlying chaotic dynamics. Recognition of one of the typical routes to chaos in experiments is a good indication that the dynamics may be chaotic. In experiments we typically construct so-called *bifurcation trees* in which for changing parameter values we show corresponding behavior by plotting maxima of chosen variables, coordinates of points on a Poincaré section etc.. In discrete systems, one simply plots successive values of a state variable.

• **Period-doubling route to chaos**

When a cascade of successive period-doubling bifurcations occurs when changing the value of the bifurcation parameter it is often the case that finally the system will reach chaos. For this an infinite sequence of period-doubling is necessary - each of the successive bifurcations occurring at a smaller step of parameter variation. An infinite cascade of such doublings results in a chaotic trajectory of infinite period and a broad frequency spectrum. over a finite range of the bifurcation parameter because of a geometric relationship between the intervals over which the control parameter must be moved to cause successive bifurcations. Period-doubling is the most common type of routes to chaos and often is governed by a universal scaling law which holds in the vicinity of the bifurcation point to chaos μ_∞.

The ratio δ_k of successive intervals μ,

$$\delta_k = \frac{\mu_{2^k} - \mu_{2^{k-1}}}{\mu_{2^{k+1}} - \mu_{2^k}},$$

where μ_{2^k} is the bifurcation point for the period from 2^kT to $2^{k+1}T$. In the limit as $k \to \infty$, a universal constant called the *Feigenbaum number* δ is obtained:

$$\lim_{k\to\infty} \delta_k = \sigma = 4.6692\ldots.$$

The period-doubling route to chaos can be identified from a state-space plot (qualitative changes of orbits observed eg. on an oscilloscope), time series, power spectrum (successive appearance of spikes half-way between the already existing ones), or a Poincarémap (doubling number points on the section) when changing the bifurcation parameter value.

• **Intermittency route to chaos**

The route to chaos caused by saddle-node bifurcations comes in different forms, the common feature of which is a direct transition from regular motion to chaos. The most common type is the intermittency route and results from a single saddle-node bifurcation. Just after the bifurcation, the trajectory is characterized by long intervals of almost regular motion (called *laminar phases*) and short bursts of irregular motion. The period of the oscillations is approximately equal to that of the system just before the bifurcation. At the parameter passes through the critical value μ_∞ at the bifurcation point into the chaotic region, the laminar phases become shorter and the bursts become more frequent, until the regular intervals disappear altoghter. The scaling law for the average interval of the laminar phases depends on $|\mu - \mu_\infty|$ so chaos is not fully developed until some distance from the bifurcation point

Intermittency is best characterized in the time domain since its scaling law governs on the length of laminar phases.

• **Torus breakdown route to chaos**

The quasiperiodic route to chaos results from a sequence of Hopf bifurcations. In the first one a periodic orbit is born. In the second one this orbit bifurcates into a two-torus. The three-torus generated after the third Hopf bifurcation is not stable in the sense that it is destroyed by an arbitrarily small perturbation of the system (in terms of parameters) for which it disappears giving way to chaos.

1.7 Asymptotic behavior, attractors, limit sets—what can an electronic engineer see in practice ?

In this section we will try to clarify the introduced notions from an electronic engineer's point of view.

We should first note that engineers easily distinguish between linear and nonlinear circuits and systems, and in the everyday practice it is assumed in most cases that everything is linear (ie. in simple words a response of a system to a sum of signals is a sum of the responses to each of the signals separately and a response to a scaled copy of the input (multiplied by a constant) will be a scaled copy of the output (with the same scaling constant)). Linear systems are also "nicely behaved" – there is not much freedom in the classification of the possible responses[69,70,180,182,360,361]:

- all are almost all[i] solutions converge to a fixed point (unique!)

[i] Except the initial conditions in a set of measure 0

- all solutions are periodic (with an amplitude depending on the initial conditions,
- for systems with external periodic or quasi-periodic driving all (or almost all) solutions will converge to a periodic or quasi-periodic solution.
- all or almost all solutions are divergent (in reality this situation is not possible due to the finite energy supplied to the circuits and existing dissipation)

Typically such circuits as filters or amplifiers should belong to this class – they must have a unique operating point (stable fixed point in terms of dynamics). In all the design procedures one tries to maintain as close as possible the linearity of the circuitry with respect to the signals.

However there are many applications which require by definition of their functionality the existence of different types of behavior. From the above given classification it is clear that a linear circuit can not serve as a periodic signal generator (supplying the same waveform independently on the initial condition). Neither a linear circuit could serve as a bistable cell which clearly requires existence of two stable operating points (fixed points).

In fact there is a large variety of different circuit behaviors in the case of circuits with nonlinearities. Even considering autonomous nonlinear circuits we can encounter in practice for example the following types of behavior:

- as in the case of linear circuits all solutions converge to a unique operating point (fixed point) (this is the mode of operation of real analog RLC filters, amplifiers etc.)
- all solutions converge to one out of many equilibrium points (this is the mode of operation of bistable circuits, memory cells, threshold detectors, Schmitt triggers, sample-and-hold circuits etc.)
- All solutions converge to a unique periodic or quasi-periodic solution (this is the mode of operation of the oscillators, periodic signal generators etc.)

These are "normal" modes of operation every electronic engineer is accustomed with.

There are some more rarely met situations that are also known in practice: eg. depending on circuit parameters we can observe sub-harmonic solutions in some power circuits (eg. ferro-resonant circuits) (ie. various kinds of stable periodic solutions), one can observe so-called false synchronizations in the PLL circuits or (which means again existence of various stable periodic solutions depending on some parameter, initial condition or input signal choices).

1.7.1 How to recognize the behavior

In the simplest case in the time domain, an equilibrium point of an electronic circuit is simply a DC solution or operating point. This is a typical case when using an oscilloscope we can see the waveforms converging towards a constant level or in the XY mode towards a point on the screen.

Periodic solutions are more difficult to confirm. The time waveform may look periodic but we need better tools to confirm this fact. First we could look at phase plots (XY mode) - periodic solutions form closed curves in space and projections of such curves are easily visible on the scope screen. Further we can consider that every periodic signal $x(t)$ may be decomposed into a Fourier series—a weighted sum of sinusoids at integer multiples of a fundamental frequency Thus, a periodic signal appears in the frequency domain as a set of spikes at integer multiples *harmonics* of the *fundamental* frequency. The amplitudes of these spikes correspond to the coefficients in the Fourier series expansion of $X(t)$. The Fourier transform is an extension of these ideas to aperiodic signals; one considers the distribution of the signal's power over a continuum of frequencies rather than on a discrete set of harmonics[352,358]. The distribution of power in a signal $x(t)$ is most commonly quantified by means of the *power density spectrum*, often simply called the *power spectrum*. Using a spectrum analyzer we can readily make a read-out of at least a part of the signal spectrum. Processing the signals digitally we have to note the following. The simplest estimator of the power spectrum is the periodogram which, given N uniformly spaced samples $X(k/f_s), k = 0, 1, \ldots, N-1$ of $x(t)$, yields $N/2+1$ numbers $P(nf_s/N), n = 0, 1, \ldots, N/2$, where f_s is the sampling frequency. $P(nf_s/N)$ is an estimate of the power in the component at frequency nf_s/N. By Parseval's theorem, the sum of the power in each of these components equals the mean squared amplitude of the N samples of $x(t)$. If $x(t)$ is periodic with period T, then its power will be concentrated in a DC component, a fundamental frequency component $1/T$, and harmonics. In practice, the discrete nature of the sampling process causes power to "leak" between adjacent frequency components; this leakage may be reduced by "windowing" the measured data before calculating the periodogram

Even more difficult situation occurs when observing quasi-periodic trajectories in experiments. Quasiperiodic behavior occurs in systems where two (or more) incommensurate frequencies are present. A periodically-forced or discrete-time dynamical system has a frequency associated with the period T of the forcing or sampling interval T of the system; if a second frequency is introduced which is not rationally related to T, then quasi-periodicity may occur.

A quasiperiodic function may typically be expressed as a countable sum of periodic functions with incommensurate frequencies (for example, $x(t) = \sin(t) + \sin(2\pi t)$ is a quasiperiodic signal). Time waveforms are very difficult to recognize from measurements. In the time domain, a quasiperiodic signal may look like an amplitude- or phase-modulated waveform. XY projection in some cases might help if we can identify a "donut-like" shape in the state space projections. This however is very difficult to visualize in higher dimensions. While the Fourier spectrum of a periodic signal consists of a discrete set of spikes at integer multiples of a fundamental frequency, that of a quasiperiodic solution comprises a discrete set of spikes at *incommensurate* frequencies. In principle, a quasiperiodic signal may be distinguished from a periodic one by determining whether or not the frequency spikes in the Fourier spectrum are harmonically related. In practice, it is impossible to determine whether a measured number is rational or irrational; therefore, any spectrum which appears to be quasiperiodic may simply be periodic with an extremely long period.

DC equilibrium, periodic, and quasiperiodic steady-state behaviors have been correctly identified and classified since the pioneering days of electronics in the 1920s. By contrast, the existence of more exotic steady-state behaviors in electronic circuits has only been acknowledged in the past twenty years. While the notion of chaotic behavior in dynamical systems has existed in the mathematics literature since the turn of the century, unusual behaviors in the physical sciences as recently as the 1960s were described as "strange". Probably most of the practicing engineers have also encountered "wild" types of behaviors – suddenly observing on the oscilloscope a "cloud" of waveforms or being not able to synchronize the scope at all. These bizzare behaviors in most practical cases are immediately judged as useless and usually the circuitry has to be redesigned without looking in much detail into what really happened in our experiment. Often the phenomena are accounted to be a result of noise or "strange couplings". I would dare to say that in most of these bizzare cases of "wrong design" the engineers observe in reality these strange waveforms that are neither constant nor periodic or quasi-periodic but are chaotic! I would even risk a statement that almost all circuits under specific circumstances (choices of parameters, initial conditions, input signals etc) can become chaotic! Today, we classify as *chaos* low-dimensional recurrent[j] motion in deterministic dynamical systems which exhibits both "randomness" and "order"[432].

[j] **Because a chaotic steady-state does not settle down onto a single well-defined trajectory, the definition of *recurrence* must be used to identify post-transient behavior.**

From a experimentalist's point of view, chaos may be defined as *bounded steady-state behavior which is not an equilibrium point, not periodic, and not quasiperiodic*. In the time domain waveforms look "random". Looking at an XY plot on the screen of an oscilloscope the trajectory winds around in a strange way filling the fragments of the space.

Repeated experiments show usually different waveforms as two trajectories started from almost identical initial conditions diverge exponentially and soon become uncorrelated (sensitive dependence on initial conditions).

"Randomness" manifests itself in the frequency domain as a broad "noise-like" Fourier spectrum.

Whether or not such a kind of behavior is interesting in practice we will try to analyze and give some answers in the next chapters.

In fact there are no general methods enabling the forecasting of asymptotic behavior of nonlinear circuits. It is very difficult to find design procedures which take into account system nonlinearities, In many cases the design is linear and the influence of real nonlinearities is tested either by bread-boarding or massive simulations.

At this moment the fundamental questions in analysis of possibly chaotic circuits are: understanding of the underlying phenomena, elaboration of new design methods to avoid unwanted chaotic/complex behavior, elaboration of new techniques in which chaos might prove useful.

To be able to analyze chaotic behavior three types of approaches are being combined:

- modeling of the circuitry and phenomena. Building an abstract mathematical model (dynamical system!) is a first step to understanding the behavior. To a certain extent analytical analysis of the model is also possible (mathematical proofs).

- simulation experiments using the developed model(s) (into this category we could put calculation of system responses for various sets of parameters, calculation of Lyapunov exponents, dimensions, entropy, construction of bifurcation diagrams etc.)

- laboratory experiments and analysis of experimental data.

Only a combination of these three approaches gives convincing results concerning dynamic behavior of the given nonlinear circuit.