Chapter 1

Superfluidity

As their name indicates, superfluids flow without dissipation. This remarkable property, shared by systems as different as superconductors where the flowing particles are electrons, and superfluid Helium where there are atoms, is due to the macroscopic occupation of a quantum state. It is a manifestation of macroscopic quantum coherence, a notion fundamental to superfluidity.

1.1 The Landau critical velocity

Let us consider first the case of liquid Helium, and assume that its special properties below a critical temperature called the λ point are due to the fact that a macroscopic fraction of He atoms condense in a state of zero momentum, not subject to thermal agitation. Let us further assume that this macroscopic occupation is not immediately destroyed when we set the liquid in motion. All atoms initially at rest have now acquired a small velocity V. This is the velocity of the *condensate.* As we increase the velocity, we guess that at some point the condensate will start to be depleted by the creation of *excitations.* When will this happen?

An excitation is characterized by its momentum p, with respect to the frame of reference of the condensate, and by its energy $\epsilon(p)$. In the energetically most favorable case, the momentum of the excitation will be opposite to the velocity of the condensate, reducing the total kinetic energy. The change in energy of the system will then be, to first order in p,

$$\Delta E = \epsilon(p) - pV \tag{1.1}$$

This energy is positive as long as V is smaller than a critical velocity

$$V_c = \left[\frac{\epsilon(p)}{p}\right]_{\min} \tag{1.2}$$

This is the critical velocity introduced by Landau for the case of superfluid Helium. But it is, in fact, a very general expression, valid as well for superconductors.

In liquid Helium, the low lying excitations are phonons, or density waves, with an energy linear in p. The ratio $\left(\frac{\epsilon(p)}{p}\right)$ is then a constant. At higher momenta, there are other excitations, called *rotons*, giving a local minimum in $\epsilon(p)$, and therefore also a minimum in $\left(\frac{\epsilon(p)}{p}\right)$ (Fig. 1.1).

The dispersion curve $\epsilon(p)$ has been determined by neutron scattering experiments, so the measured value of V_c can be compared to that calculated from the Landau criterion. In fact, the experimental value is much smaller than the calculated one, presumably because of interactions between the superfluid and the walls of the channel in which it flows. However, a value very close to the theoretical one is found when ions are accelerated within the superfluid.

In superconductors, there are no low lying excitations near the Fermi level. The first excited states have an energy equal to the *energy gap* Δ (at least for the standard BCS superconductors) (Fig. 1.2). From Landau's expression, one then obtains for the critical velocity in superconductors:

$$V_c = \frac{\Delta}{p_F} \tag{1.3}$$

where p_F is the Fermi momentum. This is exactly the result given by the BCS theory. Note, however, that the existence of an energy gap is not a pre-requisite for the existence of superconductivity. All that is really needed is a macroscopic condensate. For instance, a finite critical velocity is still observed in *gapless superconductors,* obtained for instance by the introduction of magnetic impurities, or

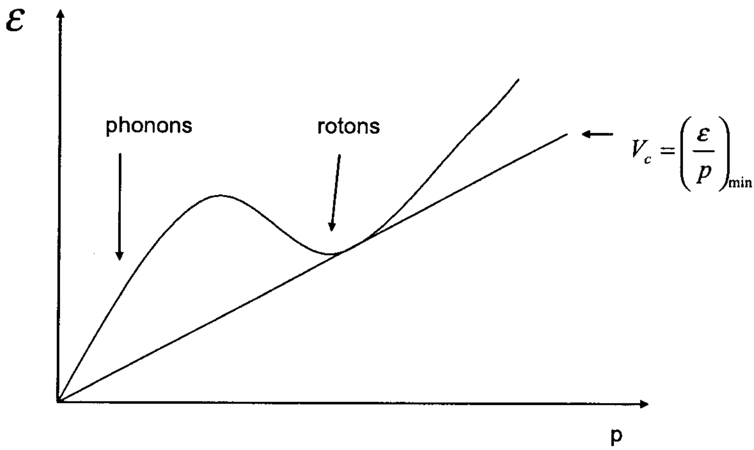

Figure 1.1: Excitation spectrum of superfluid Helium. Superfluidity does not require an energy gap, only a minimum value for (ϵ/p).

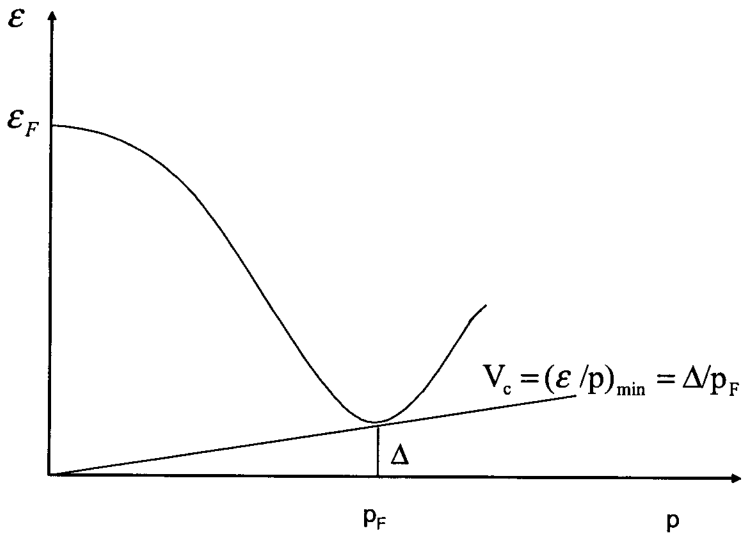

Figure 1.2: Electronic excitations in a superconductor. Landau's critical velocity is equal to the energy gap divided by the Fermi momentum. The excitation energy at zero momentum is the energy required to bring an electron from the bottom of the conduction band up to the Fermi level.

by the application of strong magnetic fields. A more recent, and very relevant example for these lecture notes, is that of the high temperature superconductors, which in most cases are also gapless, but do have — fortunately for applications — a large critical velocity, and associated critical current density.

To obtain an order of magnitude of the critical current density, let us consider a BCS superconductor having an energy gap of 1 meV.

The critical current density is given by:

$$j_c = neV_c \tag{1.4}$$

where n is the carrier density. For ordinary metals, which is the case of BCS superconductors in their normal state, the carrier density is of the order of $10^{22}/\text{cm}^3$. The Fermi wave vector is of the order of 1Å^{-1}, hence the Fermi momentum $hk_F \cong 1 \cdot 10^{-19}$erg·sec/cm. We obtain:

$$j_c \cong 1 \cdot 10^7 \text{A/cm}^2 \tag{1.5}$$

This theoretical value is in good agreement with experiment. It gives the basis for all power applications of superconductors. In a film having a thickness of 1 μm, this translates into a critical current per unit width of $1{\cdot}10^3$ A/cm-width. Such values have now been reached in High T_c coated conductors, as we shall see in the last chapter.

1.2 Origin of the condensate

The notion of macroscopic occupation of a quantum state is counter-intuitive for the case of electrons in metals: these particles are Fermions, and according to the Pauli principle, only two electrons of opposite spins can share the same energy state. So we leave aside, for the moment, superconducting metals, and consider the easier case of liquid Helium. Here, the particles are bosons and there is in principle no limit to the occupation number of a given state.

We assume, for simplicity, that there are no interactions between the Helium atoms. This idealized description is that of the *Bose Gas*.

We treat the Helium atoms as particle-waves having the de Broglie wave length:

$$\lambda = \frac{h}{p} \tag{1.6}$$

where p is determined by the thermal energy:

$$\frac{p^2}{2m} = k_B T \tag{1.7}$$

$$p = \sqrt{2mk_B T} \tag{1.8}$$

At low enough temperatures, the de Broglie wave length will eventually become larger than the inter-particle distance, hence a quantum description of the gas must be used. Treating the Helium atoms as free particle-waves in a box of size L, the density of states of this gas is given by:

$$N(E) = L^{-3} 2^5 \sqrt{2} \pi^4 \left(\frac{m}{h^2}\right) E^{\frac{1}{2}} \tag{1.9}$$

and the occupation probability is:

$$f(E) = \frac{1}{\exp(\frac{E-\mu}{k_B T}) - 1} \tag{1.10}$$

where the value of the chemical potential μ is determined by the condition that the particle density n be kept constant:

$$n = \int_0^\infty N(E) f(E) dE \tag{1.11}$$

This equation sets the value of the chemical potential. Let us first notice that it must be negative, since otherwise, for small values of E, the occupation probability would be negative (Fig. 1.3). Since it cannot cross the zero value, there must be, below that temperature T_c, an increase in the low energy states occupation probability. The picture then is that below T_c, a macroscopic occupation of the state

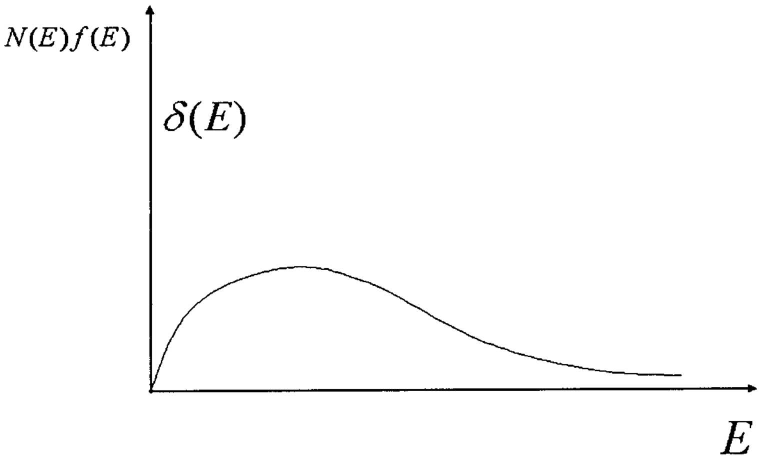

Figure 1.3: Below T_c, the number of particles in the zero energy state becomes macroscopic.

E=0 will start with all particles being eventually in that state at T=0.

We calculate the value of T_c as the temperature at which μ=0:

$$n = A\left(\frac{m}{h^2}\right)^{\frac{3}{2}} \int_0^\infty \frac{1}{\exp\left(\frac{E}{k_B T_c}\right) - 1} E^{\frac{1}{2}} dE \tag{1.12}$$

where A is a numerical constant. This gives:

$$T_c = 3.31 \left(\frac{h^2}{m}\right) n^{\frac{2}{3}} \tag{1.13}$$

This expression has a simple physical interpretation: it is the temperature at which the de Broglie wave length of the particles becomes of the order of the inter-particle distance $a = n^{-\frac{1}{3}}$:

$$a \cong \frac{h}{\sqrt{2mk_B T_c}} \tag{1.14}$$

As soon as we cross T_c, the occupation of the zero energy state becomes macroscopic, and superfluidity appears. The superfluid fraction can move without dissipation with a finite velocity V, while the rest of the fluid (the "normal" fraction) stays at rest.

The remarkable fact is that the Bose–Einstein (BE) result for T_c has no adjustable parameter. If we put in the density of liquid Helium, we obtain a value of 3K, quite close to the experimental value of 2.14K. There is undoubtedly quite a bit of luck here, since the BE result neglects all interactions between the particles, an unreasonable assumption since the liquid is actually quite dense. Yet, the agreement is remarkable. One effect of the particle-particle interaction is that not all He atoms are condensed at T=0, but only approximately 10%. But this does not matter: what matters is that the condensed fraction is macroscopic.

1.3 Phase of the condensate

The quantum macroscopic condensate can be described by a wave function, having an amplitude and a phase:

$$\Psi = \mid \Psi \mid e^{i\varphi} \tag{1.15}$$

The amplitude of this wave function squared is the density of the condensed particles. In its ground state, i.e. when it is at rest, the phase is uniform and arbitrary. The importance of the phase appears in two kinds of situations: when we set the condensate in motion, and when we change the number of particles in the condensate.

When we set the condensate in motion, the current density of particles is proportional to the gradient of the phase:

$$\mathbf{j} = \frac{\hbar}{2mi}(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \tag{1.16}$$

which we can interpret as the product of the density $\mid \Psi \mid^2$ by the velocity:

$$\mathbf{V} = \frac{\hbar \nabla \varphi}{m} \tag{1.17}$$

To Landau's critical velocity corresponds a critical phase gradient:

$$(\nabla \varphi)_c = \left(\frac{m}{\hbar}\right)\left(\frac{\epsilon(p)}{p}\right)_{\min} \tag{1.18}$$

Let us calculate the critical phase gradient in the case of a BCS superconductor having an energy gap Δ at $p = p_F$. We obtain:

$$(\nabla\varphi)_c = \frac{\Delta}{\hbar v_F} \tag{1.19}$$

where v_F is the Fermi velocity. The critical phase gradient is determined by a length scale:

$$(\nabla\varphi)_c = \pi\xi^{-1} \tag{1.20}$$

where:

$$\xi = \frac{\hbar v_F}{\pi\Delta} \tag{1.21}$$

This length scale is called the *coherence length.* For BCS superconductors, the critical current density is reached when the phase gradient is of the order of π divided by the coherence length. For a typical low temperature superconductor, having an energy gap of 1 meV and a Fermi velocity of $1{\cdot}10^8$cm/sec, it is of the order of 1000Å, several orders of magnitude larger than the interatomic distance.

The other case where the phase is important is that where we change the number of particles in the condensate. Suppose for instance that we introduce a weak link (a small orifice of atomic size) between two condensates, say two recipients filled with superfluid Helium. If there is no particle flow through this weak link, the phases of the two condensates must be identical. But suppose that we now set the superfluid in motion through this weak link. Then, a phase difference $\Delta\varphi$ will appear between the two condensates. By analogy with the case of a bulk current flow, we write:

$$\Delta\varphi = l\frac{m}{h}V \tag{1.22}$$

where l is the size of the orifice. In this experiment, the velocity V is equal to the flow of particles dN/dt, multiplied by the inter-particle distance, which we have assumed is of the same order as l. We finally find for the phase gradient:

$$\Delta\varphi = l^2\frac{m}{h}\frac{dN}{dt} \tag{1.23}$$

Since the phase difference cannot exceed π, without the system going back effectively to an equivalent state, this relation puts a limit to the flow of particles between the two condensates that can take place *without dissipation.* It is one aspect of the Josephson effect. Of course, this *gedanken* experiment is highly impractical: we assumed that the orifice was of atomic size, otherwise the opening would be a regular channel and dissipation-less flow would just be identical to bulk superfluidity. Not only is such a small orifice difficult to make in a controlled way, but the outcome of the experiment becomes in this limit very sensitive to the boundary condition for the wave function at the walls, on the atomic scale. We shall have the occasion to return later to this boundary condition problem.

Josephson effects are in fact much better studied in superconductors. In that case, the weak link can be a well controlled tunnel junction, or a constriction of size ξ, which as we have shown is much larger than the interatomic distance. What is important to note at this stage, is that dissipation-free currents can flow through weak links between superfluid condensates.

1.4 Two-Fermion superfluids

Now let us go back to the case of electrons in solids. As already noted, electrons will not condense as Helium atoms do, because they are Fermions. But let us imagine that there exists a strong attractive potential between electrons, so that electrons can form pairs. This might happen for instance if there is a strong electron-phonon interaction, in which case electrons can dig their own potential well in the lattice to form polarons, and under certain circumstances, bipolarons. We assume that this potential is strong enough to produce pair formation at high temperature, so that at low temperatures the pairs are very stable and can be considered as bosons. In such a case, Bose–Einstein condensation of these pairs will occur as we have calculated: to obtain the transition temperature, we only need to replace the Helium atom mass by twice the electron mass:

$$T_{co} = 3.31 \frac{h^2}{2m_e^*} \left(\frac{n}{2}\right)^{\frac{2}{3}} \tag{1.24}$$

where n is the electron density, and m_e^* is the effective mass of the electrons.

Because the electron mass is very much smaller than that of He atoms, this expression predicts for our hypothetical two-Fermion superfluid a very high condensation temperature.

How realistic is such a picture? Several stringent conditions must apply for such a pair formation, and pair condensation, to occur:

(i) the binding energy ϵ_0 of the pair must be of the order of, or larger than the typical energy that free (not bound) electrons would have, namely the Fermi energy:

$$\frac{\epsilon_0}{2} > E_F \tag{1.25}$$

(ii) the formed pairs must be mobile in order for them to condense.

The first condition certainly cannot be met in ordinary metals, where the Fermi energy is large (on the order of several eV), and screening is strong, preventing the formation of strong attractive potentials between electrons. On the other hand, if screening is weak and interactions are strong, we may be dealing with an insulator rather than with a metal, and pairs — if they are indeed formed — will tend to be localized, and the second condition will not be met.

Our conclusion is that, if at all, these two conditions can only be met in a material that is very close to a metal-insulator transition. Then, the free carrier density is small, the Fermi energy is reduced compared to that of an ordinary metal, interactions are stronger, but electrons are not yet strongly localized. Keeping these caveat in mind, we see that the value of the BE T_c is not quite as high as we might naively have expected from the ratio of the masses: instead of a typical metallic density of the order of $1{\cdot}10^{22}/\text{cm}^3$, we must rather use a value typical of a degenerate semiconductor, say $1{\cdot}10^{21}/\text{cm}^3$; and the effective mass of the electron may be substantially larger than that of the free particle, because we are in a regime of strong interactions. Taking both effects into account, the BE T_c would only be one to two orders of magnitude larger than that of liquid He, instead of four. But this is still a very high condensation temperature, in the 100K range.

Superconductivity is indeed observed in a number of systems that are close to the metal-insulator transition: granular superconductors have often critical temperatures higher than that of the parent bulk superconductor; organic superconductors and last but not least, the high-temperature superconductors cuprates. These are natural candidates as Bose–Einstein superconductors.

Assuming that such pair formation and condensation are possible, the normal state is characterized by the amplitude of the potential, V_m and by its range a. For pairs to form at all, we must have:

$$V_m > \frac{h^2}{2ma^2} = V_a \tag{1.26}$$

We wish to calculate the pair binding energy ϵ_0 and its extension in the potential well, a_0. In the strong coupling limit $V_m \gg V_a$, the pair is strongly bound at the bottom of the potential well, and its binding energy is close to the depth of the well

$$\epsilon_0 \approx V_m \tag{1.27}$$

For a deep potential well we can use the harmonic approximation. The pair extension is then given by the harmonic oscillator solution, it varies as the inverse fourth power of the potential amplitude, i.e. of the pair binding energy

$$a_0 \propto \epsilon_0^{-1/4} \tag{1.28}$$

In the normal state, there is a gap:

$$\Delta_p = \frac{\epsilon_0}{2} \tag{1.29}$$

Δ_p is the *single particle excitation energy* of the system. If, for instance, we try to inject a single electron in our material from a normal metal through a tunnel junction, we shall have to apply a potential difference across the junction of at least Δ_p/e.

Electrons are paired over the length a_0. This is in a sense the equivalent of the *coherence length* that we introduced earlier for the case of a BCS superconductor. But there is of course a major difference: in the present case, the gap and the pairing length are properties of the *normal state*. When the Bose–Einstein condensation of the pairs occurs at some lower temperature, there will be no change in this gap. In the condensed state, a new length scale appears, which characterizes the scale over which the phase of the condensate (and not its amplitude) can vary. This is really the coherence length of the Bose–Einstein condensate. Neither the pairing length, nor the coherence length are anymore related to the single particle energy gap in the way they are in a BCS superconductor.

We conclude that, if Bose–Einstein superconductivity does exist, it will be characterized by a large tunneling gap, only weakly temperature dependent on the scale of the condensation temperature, by a small pairing length, of the order of interatomic distances, also temperature independent and distinct from the length scale that characterizes changes in the phase of the condensate. In principle, these properties are easily distinguishable from those of BCS superconductors in which the gap is a thermodynamical gap, proportional to T_c at low temperatures, and going to zero at T_c, and the pairing (in that case, also coherence) length is large compared to interatomic distances, temperature dependent (it diverges at T_c) and given by the Fermi velocity divided by the gap at T=0.

1.4.1 The Meissner effect

In a two-Fermion superfluid, the expression for the current that we have used for non-charged superfluids, Eq. (1.16), must be modified to take into account the charge 2e of our new "two-Fermion bosons" . By analogy with the expression for the current for particles of charge 2e and wave function $\Psi(r)$, we write:

$$\mathbf{j} = \frac{e\hbar}{im^*}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) - \frac{4e^2}{m^*c} \mid \Psi \mid^2 \mathbf{A} \tag{1.30}$$

where we have taken the particles to be of mass m^*, and $\mathbf{A}$ is the vector potential.

The vector potential will be non-zero in the presence of applied magnetic fields or currents, or both. This covers a wide variety of situations. In general, Ψ and $\mathbf{A}$ will be position dependent, and both dependencies must be known in order to calculate the current $\mathbf{j}(\mathbf{r})$.

The situation is much simpler when the applied fields and currents are small: we can then assume that $|\Psi|$ retains its zero-field value, $|\Psi_0|$. Taking the **curl** of both sides, and making use of:

$$\text{curl}\,\mathbf{h} = \frac{4\pi\mathbf{j}}{c} \tag{1.31}$$

we obtain:

$$\text{curlcurl}\,\mathbf{h} = \frac{16\pi e^2}{m^* c^2} \mid \Psi_0 \mid^2 \mathbf{h} \tag{1.32}$$

This relation defines a length scale λ such that:

$$\lambda^{-2} = \frac{16\pi e^2}{m^* c^2} \mid \Psi_0 \mid^2 \tag{1.33}$$

To take a specific example, assume that we have applied an external magnetic field H parallel to the surface of a very thick sample. The equation for the current then tells us that it will decay exponentially inside the sample, over the length scale λ. So will the internal magnetic field h. If we orient the x axis normal to the surface of the sample, this internal field will obey the relation:

$$h(x) = H \exp -\left(\frac{x}{\lambda}\right) \tag{1.34}$$

since the parallel component of the field must be continuous across the surface of the sample.

The assumption of a rigid, macroscopic wave function to describe the condensate was originally proposed by London. λ is called the London penetration depth. The exact nature of the condensate — whether it is a Bose–Einstein condensate as we discuss here, or a more classical BCS condensate of which we recall the main properties in a later chapter — does not affect the validity of the London equation.

1.4.2 Flux quantization

Let us consider a hollow cylinder whose wall is much thicker than the penetration depth λ. Inside the wall we can find a path around the cylinder along which the local magnetic field and the current is zero, according to Eq. (1.34). Integrating Eq. (1.30) along such a path, and using Eq. (1.15), we obtain:

$$\oint \nabla\varphi \cdot d\mathbf{l} = \frac{2e}{\hbar c} \oint \mathbf{A}.d\mathbf{l} \tag{1.35}$$

Since the l.h.s must be a multiple of 2π, the flux enclosed within the selected path is given by:

$$\Phi = n\Phi_0 \tag{1.36}$$

where n is an integer and:

$$\Phi_0 = \frac{hc}{2e} \tag{1.37}$$

is the flux quantum.

To go back to our Bose–Einstein notation, we replace $\mid \Psi_0 \mid^2$ by the superfluid density $n_s{=}n/2$, and m^* by $2m$. Assuming that all particles have condensed at $T{=}0$, we have:

$$\lambda(0)^{-2} = \frac{4\pi e^2}{c^2}\frac{n}{m} \tag{1.38}$$

Now, we remember that the Bose–Einstein condensation temperature is a power law of the particle density (Eq. 1.13), here $(\frac{n}{2})$. Therefore, in our two-Fermion superfluid, the condensation temperature is a power law of the zero-temperature penetration depth:

$$k_B T_c \propto 3.31 \left(\frac{c^2}{4\pi e^2}\right)^{\frac{2}{3}} \frac{h^2}{2m^{\frac{1}{3}}}\lambda(0)^{-\frac{4}{3}} \tag{1.39}$$

This relation can in principle be checked quantitatively. Note however that the effective mass of the individual electrons must be determined independently. A power law relation between T_c and $\lambda(0)$ ($T_c \propto \lambda(0)^{-2}$) has indeed been discovered by Uemura, and interpreted as an indication that the superfluid condensation in these materials is of the Bose–Einstein type. This will be discussed in more detail in the next chapter.

1.5 BCS superconducting metals

The case of the two-Fermion Bose–Einstein superfluid is a very special one. As we have seen, it can only occur in a material that is close to a metal-insulator transition. In fact, it is a well established fact that most superconductors are well behaved metals, with strong electronic screening and relatively weak interactions, so that this description cannot be applied to them.

Yet, the principle that the condensed state, characterized by a macroscopic occupation of a quantum state, is composed of pairs of electrons of opposite spins must apply.

1.5.1 Condensation energy

The microscopic description of metal-superconductors starts from the existence of a large Fermi surface, and correspondingly large Fermi energy, namely much larger than the energy scale — the energy gap Δ — that characterizes the superconducting state. The difference between the electronic structure of the normal and superconducting state is then seen only close to the Fermi level. Only electrons within an energy Δ from the Fermi level, have their energy substantially lowered in the condensed state; hence, the condensation energy per unit volume is of the order of $N(0)\Delta^2$ (the exact expression is $\frac{1}{2}N(0)\Delta^2$) where $N(0)$ is the normal state density of states, and the condensation energy per particle is of the order of $(\frac{\Delta^2}{E_F})$. Because this energy is so small, we might think that superconductivity in metals is going to be very fragile. In fact, this is not the case: perturbations of the superconducting state (due for instance to thermodynamical fluctuations) will take place over the scale of the coherence length, because this is the length scale over which the Ψ function can vary (as we have seen in the case of its gradient, Eq. (1.19)). Because the coherence length in a BCS superconductor is very large as we have seen, not just one pair will be affected, but many. The relevant energy scale for these perturbations is not the condensation energy per particle, *but the condensation energy per coherence volume:*

$$U \cong N(0)\Delta^2\xi^3 \tag{1.40}$$

Using Eq. (1.21) for ξ and the free electron expression for the normal state density of states, we get:

$$U = \frac{2}{\pi^5}\frac{E_F^2}{\Delta} \tag{1.41}$$

This is a very large energy — much larger even than the Fermi energy in the considered situation $E_F \gg \Delta$. The condensate is in fact very robust. We say that coherence in metal-superconductors is very strong, a useful property that has important practical consequences for their applications.

In a BCS superconductor, it is the destruction of pairs by the thermal energy that determines the critical temperature. Vice versa, in metal-superconductors, pairs form and condense at the same temperature T_c. In the weak coupling limit, $\Delta \ll E_F$, the critical temperature is proportional to the gap, $2\Delta = 3.5 k_B T_c$.

1.5.2 The BCS wave function

BCS introduced the following wave function to describe the condensed state:

$$\tilde{\Phi} = \prod_k (u_k + v_k a^+_{k\uparrow} a^+_{-k\downarrow})\Phi_0 \tag{1.42}$$

where Φ_0 describes the vacuum, the operator $a^+_{k\uparrow}$ creates an electron of spin up and wave vector k and the operator $a^+_{-k\downarrow}$ an electron of spin down and wave vector $-k$. The normalization of $\tilde{\Phi}$ is ensured by the condition $u_k^2 + v_k^2 = 1$. The probability to find a condensed electron at wave vector k and spin α is given by:

$$v_k^2 = \langle \tilde{\Phi} \mid a^+_{k\alpha} a_{k\alpha} \mid \tilde{\Phi} \rangle \tag{1.43}$$

where the operator $a_{k\alpha}$ destroys a condensed electron in the state $k\alpha$.

For an isotropic gap Δ the energy of an electron of wave vector k in the condensed state is given by:

$$\epsilon_k = \sqrt{\xi_k^2 + \Delta^2} \tag{1.44}$$

where ξ_k is the energy in the normal state of an electron of wave vector k, measured from the Fermi level, and:

$$v_k^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{\sqrt{\xi_k^2 + \Delta^2}}\right) \tag{1.45}$$

Electrons in the condensed state are spread over an energy of the order of Δ around the Fermi surface. They lie both above and below the Fermi energy, while in the normal state at zero temperature there are no electrons above the Fermi wave vector. Hence the average kinetic energy of the electrons in the condensed state is *larger* than that in the normal state. The energy gain in the condensed state comes about because the lowering of the potential energy is larger than the increase of the kinetic energy. This is a fundamental difference with Bose–Einstein condensation, where in the absence of interaction there is no change in the potential energy, and the gain in energy in the condensed state comes about entirely from a diminution of the kinetic energy. Since the kinetic energy of a boson is of the order of k_BT_c in the normal state at the transition, and zero in the condensed state, this energy gain is of the order of k_BT_c per boson, or nk_BT_c per unit volume, where n is the boson density.

There are thus two important differences between the two modes of condensation:

(i) In the BCS condensation mode, the kinetic energy increases, while it decreases in the BE condensation. A measurement of the *sign* of the change in the kinetic energy upon condensation is of particular interest when there are indications that pairing may occur at higher temperatures. We come back later to this point in Chapter 7 when we discuss experimental ways of determining the actual mode of condensation in the cuprates.

(ii) The condensation energy per boson is much larger in BE condensation than in BCS condensation. The former is of the order of k_BT_c, and the later of the order of $\frac{\Delta^2}{E_F}$ or within numerical factors $k_BT_c\left(\frac{T_c}{T_F}\right)$, where T_F is the Fermi temperature, $T_F \gg T_c$. One may be tempted to conclude that the BE condensed state is "stronger" (for a given transition temperature) than the BCS state. This is in fact wrong: the BCS state is more rigid against fluctuations than the BE state. The reason is that the later has a much shorter coherence length (of the order of the inter-boson distance) than the former, and what actually determines fluctuation amplitudes is the *condensation energy per coherence volume,* rather than the condensation energy per boson. For the BE condensate, the condensation energy per coherence volume is then of the order of $k_B T_c$, while it is as we have seen of the order of $\left(E_F^2/k_B T_c\right)$ in a BCS superconductor.

These considerations are not purely academic. Fluctuations can have a strong impact on some important practical properties, such as the ability to retain a zero resistance state in the presence of strong magnetic fields and strong currents. There is a close connection between the mode of condensation and applications. It will be discussed in some details when we review properties of the vortex state in the last two chapters.

While the two kinds of condensation that we have described are physically very different, Eagles, Leggett and Nozieres and Schmitt-Rink (see further reading for references) have shown that as the interaction strength is increased, the transition between the two modes is in fact continuous as a function of the ratio of the pair breaking energy to the chemical potential.

1.6 Summary

Superfluids can be divided broadly into two categories:

Short coherence length superfluids

They include superfluid Helium, and possibly electronic systems near a metal-insulator transition called two-fermion superconductors, in which electron pairs would form above the condensation temperature. Their main properties are:

- a coherence length of the order of some microscopic scale, independent of the condensation temperature T_c.

- a condensation energy per particle, hence also per coherence volume, of the order of $k_B T_c$.

- a temperature independent gap for the creation of single particle excitations, but no thermodynamical gap.
- a penetration depth related to the condensation temperature by a power law.

Long coherence length superconductors

They include superconducting metals and alloys. Their main properties are:
- a coherence length much larger than interatomic distances, inversely proportional to T_c.
- a condensation energy per particle much smaller than $k_B T_c$, but a condensation energy per coherence volume much larger than $k_B T_c$.
- a thermodynamical gap in the excitation spectrum.
- a penetration depth unrelated to T_c.

1.7 Further reading

For a description of the BCS condensation, see P.G. de Gennes, "Superconductivity of Metals and Alloys", Benjamin Inc., New York 1966.

For one of the original articles on the cross over between the Bose–Einstein and the BCS condensation modes, see P. Nozieres and S. Schmitt-Rink, J. Low Temp. Phys. **59**, 195 (1985). For a recent review on this topic, see for instance Q. Chen *et al.*, Phys. Reports **412**, 1 (2005) and references therein.