were used.

If we put $\mu = 0$ in Eq. (1.29), it is possible to obtain the total power of instantaneous radiation

$$W = \frac{d\mathcal{E}}{dt} = \frac{2}{3}\frac{e^2}{c^3}\, w_\mu w^\mu .$$

Hence it is obvious that this value is invariant. In other notation

$$W = \frac{2}{3}\frac{e^2}{c^3}\,\gamma^6\left(a^2 - [\boldsymbol{\beta a}]^2\right). \tag{1.31}$$

When $\boldsymbol{\beta} \perp \boldsymbol{a}$ we have synchrotron radiation power

$$W_{\mathrm{SR}} = \frac{2}{3}\frac{e^2}{c^3}\, a^2\gamma^4 . \tag{1.32}$$

This formula was obtained by A.A. Liénard [6] as far back as 1898. In the nonrelativistic approximation $\gamma \to 1$ it comes to the Larmor formula[19].

1.3. Angular Distribution of Instantaneous Radiation Power

In a general case, the angular distribution of instantaneous radiation power can be obtained by Eq. (1.27) at $\mu = 0$

$$dW = -\frac{e^2}{4\pi\gamma}\, d\Omega \left[\frac{w_\alpha w^\alpha}{(n_\rho v^\rho)^3} - c^2\frac{n_\alpha w^\alpha}{(n_\rho v^\rho)^5}\right]. \tag{1.33}$$

The angular distribution in terms of fields can be obtained after a substitution of $\overset{\circ}{\tilde{A}}_\lambda \overset{\circ}{\tilde{A}}{}^\lambda$ from Eq. (1.21) into Eq. (1.27)[e]

$$dW = \frac{e}{4\pi}\,\tilde{r}^2\, d\Omega \tilde{E}^2[1 - (\boldsymbol{n\beta})].$$

If $\tilde{E}^2$ is substituted here from Eq. (1.7) then the angular distribution of radiation will take the form

$$\frac{dW}{d\Omega} = \frac{e^2}{4\pi c^3}\,\frac{a^2[1-(\boldsymbol{n\beta})]^2 - (\boldsymbol{na})^2(1-\beta^2) + 2(\boldsymbol{\beta a})(\boldsymbol{na})[1-(\boldsymbol{n\beta})]}{[1-(\boldsymbol{n\beta})]^5}. \tag{1.35}$$

Of course, Eq. (1.33) gives the same result.

---

[e] We note that the reference application of the Poynting vector $\boldsymbol{S} = c/4\pi[\boldsymbol{EH}]$ for calculation of the angular distribution does not give the power but the radiation intensity

$$dI = (\boldsymbol{S}d\boldsymbol{\Omega}) = \frac{c}{4\pi}\, d\Omega \tilde{E}^2, \tag{1.34}$$

which manifests itself here in absence of factor of $1 - (\boldsymbol{n\beta})$. The radiation intensity is a noninvariant value and depends on the reference frame. Failure to respond in this problem can lead to serious misunderstandings (see discussion in Refs[28−31] and also Ref.[32])

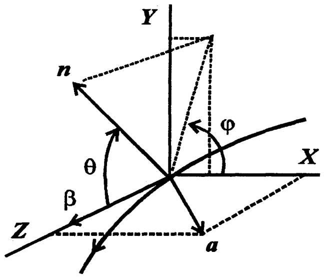


Fig. 1.3. Coordinate system for analysis of the angular distribution of the total power of instantaneous radiation. Plane ZX coincides with the plane of velocity and acceleration vectors. The components of the vectors $\boldsymbol{n}$, $\boldsymbol{\beta}$, and $\boldsymbol{a}$ are equal to $\boldsymbol{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, $\boldsymbol{\beta} = \beta(0,0,1)$, $\boldsymbol{a} = a(\sin\alpha, 0, \cos\alpha)$.

Space anisotropy of angular distribution essentially depends on the value of $1 - (\boldsymbol{n\beta})$ in the denominator of this expression. In the ultrarelativistic case, when the angle ψ included between the vectors $\boldsymbol{n}$ and $\boldsymbol{\beta}$ is very small

$$1 - (\boldsymbol{n\beta}) \approx \frac{1}{2}\left(\frac{1}{\gamma^2} + \psi^2\right)$$

this value is close to zero and the angular distribution is highly extended along the direction of the particle velocity (projector effect).

For the analysis of the angular distribution it is more convenient to use a coordinate system shown in Fig. 1.3. In the general case of arbitrary orientation of the velocity and acceleration vectors, the angular distribution of the total power of instantaneous radiation has the form[26]

$$\frac{dW}{d\Omega} = \frac{e^2a^2}{4\pi c^3}\rho(\theta,\varphi;\alpha),$$

$$\rho(\theta,\varphi;\alpha) = \frac{\sin^2\varphi\sin^2\alpha}{(1-\beta\cos\theta)^3} + \frac{[(\beta-\cos\theta)\cos\varphi\sin\alpha + \sin\theta\cos\alpha]^2}{(1-\beta\cos\theta)^5}. \tag{1.36}$$

If ρ is interpreted as a radius-vector, then it will describe a surface in the spherical coordinate system which can be called an indicatrix of radiation. It follows from Eq. (1.36) that even in the general case, this surface is symmetrical with respect to $\boldsymbol{\beta a}$ plane and nonsymmetrical with respect to $\boldsymbol{H\beta}$ plane (at the expense of α). All singularities of radiation are in the $\boldsymbol{\beta a}$ plane. There are always two such directions exactly in this plane where the radiation is absent completely. According to Eq. (1.36), these directions are determined by the condition[1,10]

$$\cos\theta_{\min} = \beta\sin^2\alpha \pm \cos\alpha\sqrt{1-\beta^2\sin^2\alpha}.$$

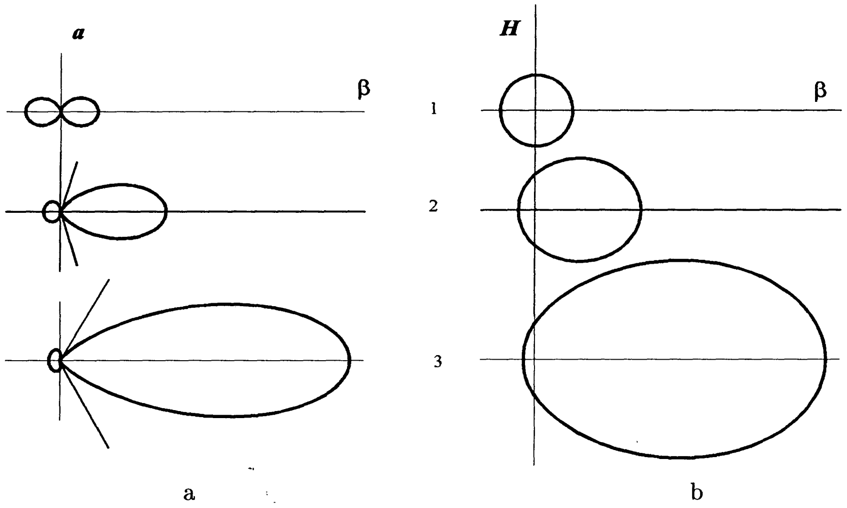


Fig. 1.4. $\boldsymbol{\beta a}$ (a) and $\boldsymbol{H\beta}$ (b) plane sections of the indicatrix of SR at $\beta = 0.01$ (1), $\beta = 0.3$ (2), and $\beta = 0.5$ (3).

We note that in these directions the wave zone tends to infinity.

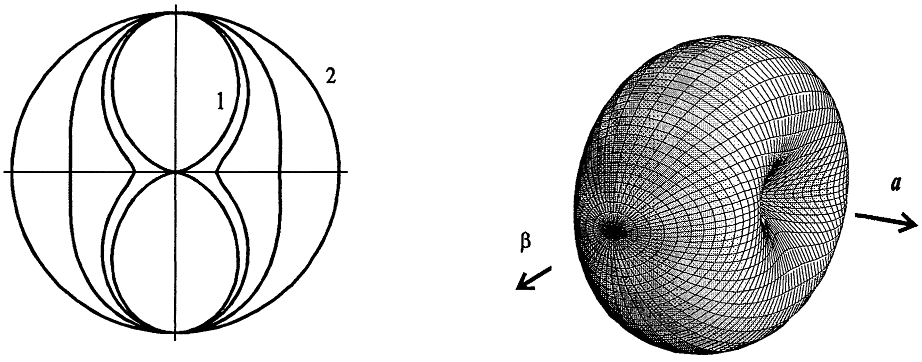
Let us consider some particular cases. SR is of the greatest interest. In this case, the velocity of the particle is orthogonal to acceleration ($\alpha = \pi/2$). The indicatrix of SR has the form

$$\rho(\theta, \varphi) = \frac{(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\varphi}{(1-\beta\cos\theta)^5}.$$

One can see that this function is symmetrical with respect to both $\boldsymbol{\beta a}$ and $\boldsymbol{H\beta}$ planes (see Fig. 1.4). We obtain sufficiently complete information on the indicatrix by investigation of its sections in these planes

$$\rho(\theta, \varphi = 0, \pi) = \frac{(\beta - \cos\theta)^2}{(1-\beta\cos\theta)^5}, \quad \rho(\theta, \varphi = \pm\frac{\pi}{2}) = \frac{1}{(1-\beta\cos\theta)^5}.$$

The angle $\theta = 0$ for SR corresponds to the principal maximum of the radiation power. The directions where the radiation is absent are determined by the condition of $\cos\theta_{\min} = \beta$. In the nonrelativistic case, they almost coincide with the forward or opposite direction of acceleration ($\theta_{\min} \approx \pi/2$). The angle $\theta_{\min}$ decreases with increasing velocity and at $0.63 \le \beta < 1$ the secondary maximums appear. It is of



a

b

Fig. 1.5. Evolution of $\boldsymbol{aH}$ plane section of SR indicatrix as a function of β from $\beta = 0.01$ (1) to $\beta = 1$ (2) and the space indicatrix of SR at $\beta = 0.01$.

interest to estimate the relation "forward-backward"

$$\frac{\rho(\theta = 0, \varphi)}{\rho(\theta = \pi, \varphi)} = \left(\frac{1+\beta}{1-\beta}\right)^3 \Bigg|_{\beta \to 1} \approx 2^6 \gamma^6$$

and the relation of the principal maximum to the secondary one

$$\frac{\rho(\theta = 0, \varphi)}{\rho(\theta_{\text{max}}', \varphi = 0)} = \frac{5^5}{216} \beta^2 (1+\beta)^3 \,|_{\beta \to 1} \approx 231.48.$$

These sections for various quantities of β are shown in Fig. 1.4a,b.

The $\boldsymbol{aH}$ plane section is given by the curves

$$\rho(\theta = \pi/2, \varphi) = \sin^2 \varphi + \beta^2 \cos^2 \varphi.$$

The evolution of this section as a function of β is shown in Fig. 1.5a.

Space indicatrixes for various quantities of β are shown in Fig. 1.5b and Fig. 1.6.

Now let us consider another extreme case when the velocity and acceleration are parallel. The corresponding indicatrix has the form

$$\rho(\theta, \varphi) = \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}.$$

This radiation possesses azimuthal symmetry in the plane orthogonal to velocity. Thereat in the direction of $\beta = 0$ the radiation is absent at all. It is interesting that

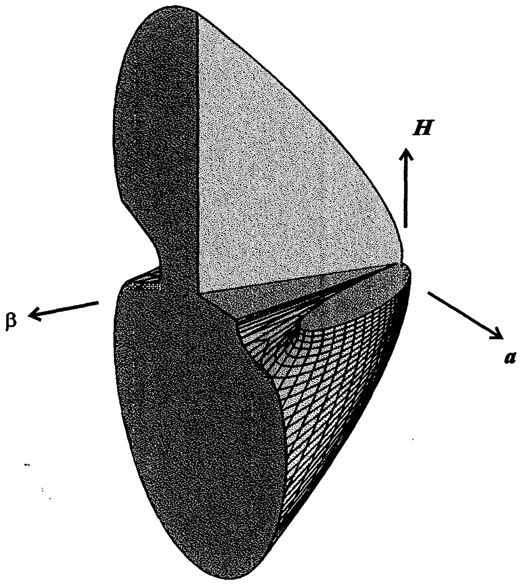


Fig. 1.6. Space indicatrix of SR for $\beta = 0.9$.

at $\theta = \pi/2$ the cross section does not depend on the velocity. The evolution of the longitudinal section as the velocity increases is shown in Fig. 1.7a. The angles related to the radiation maximum are determined by the formula

$$\cos\theta_{\max} = \frac{5\beta}{1+\sqrt{1+15\beta^2}}\,.$$

Space distribution of this radiation is shown in Fig. 1.8. The evolution of the maximum of radiation is shown in Fig. 1.7b.

In the general case, in presence of both longitudinal and orthogonal acceleration the evolution of a ZX plane section of the angular distribution is shown in Fig. 1.9 and corresponding indicatrix is shown in Fig. 1.10.

Of course, further integration over the angles leads to the total power of instantaneous radiation which coincides with Eq. (1.31).

In conclusion of this section we note that the angular distribution of the radiation power can be obtained in another way by means of Lorentz transformations[1,32]. Thereat it is necessary to bear in mind that $d\widetilde{\mathcal{P}}^\mu$ is a four-vector which, according to Eq. (1.27), satisfies the condition

$$n_\mu d\widetilde{\mathcal{P}}^\mu = 0,$$

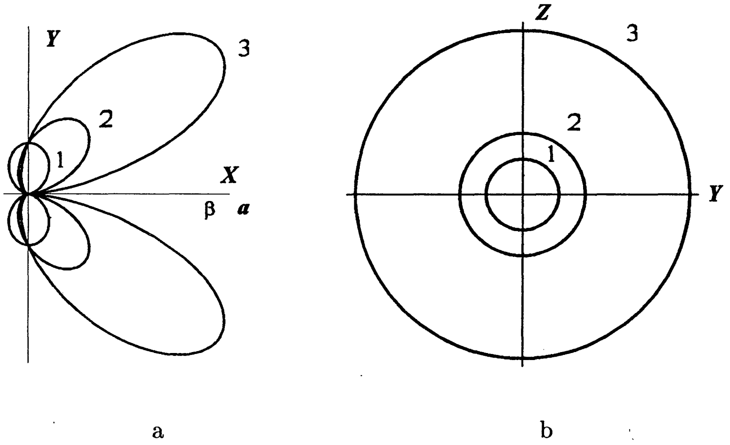


Fig. 1.7. Evolution of the longitudinal section of the indicatrix as the velocity increases when $\boldsymbol{\beta} \parallel \boldsymbol{a}$ (a); the evolution of the maximum of radiation in this case (b); $\beta = 0.01$ (1), $\beta = 0.3$ (2), $\beta = 0.5$ (3).

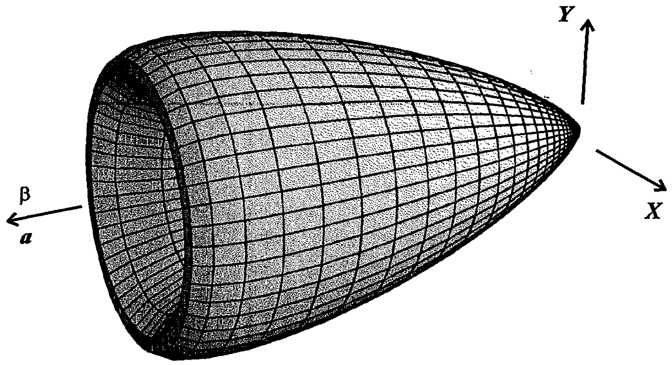


Fig. 1.8. Space distribution of radiation when $\boldsymbol{\beta} \parallel \boldsymbol{a}$.

Fig. 1.9. A section of the indicatrix of the instantaneous radiation of an arbitraty moving charge by ZX plane containing the vectors $\boldsymbol{\beta}$ and $\boldsymbol{a}$ (see the next Fig.).

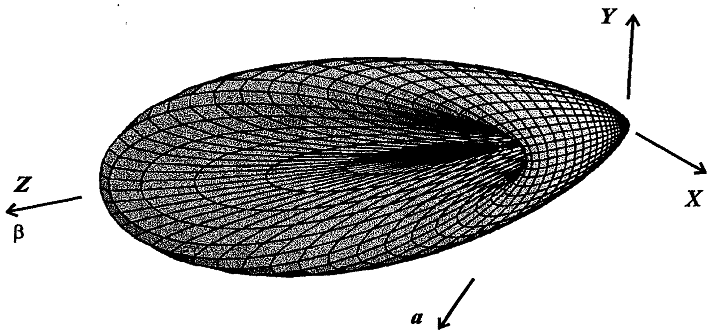


Fig. 1.10. Space indicatrix of radiation when $\alpha = \pi/4$.

as $n_\mu n^\mu = 0$. Hence it follows that the radiation energy is related to the momentum by the simple formula

$$d\mathcal{E} = c d\widetilde{\mathcal{P}}.$$

Now it is not difficult to write the Lorentz transformations for $d\mathcal{E}$. Supposing that the charge rests at the origin of a primed coordinate system (see Fig. 1.3) we have

$$d\mathcal{E}' = \frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}} d\mathcal{E}.$$

Taking into account that $dt/d\tau = \sqrt{1-\beta^2}$ we obtain

$$dW = \frac{1-\beta^2}{1-\beta\cos\theta} dW'.$$

In the system of rest the angular distribution of radiation is determined by Eq. (1.35) where we it should be put $\beta = 0$. Then we obtain

$$dW' = \frac{e^2}{4\pi c^3}[a'^2 - (\boldsymbol{n}'\boldsymbol{a}')^2]d\Omega'.$$

The expression in the square brackets is an invariant functional

$$I_a(\theta,\varphi) = a'^2 - (\boldsymbol{n}'\boldsymbol{a}')^2 = w_\mu w^\mu - (e_\mu w^\mu)^2. \tag{1.37}$$

Here we have taken into a consideration that

$$w^\mu|_{\beta=0} = (0, \boldsymbol{a}'), \quad e^\mu|_{\beta=0} = (0, \boldsymbol{n}').$$

One can prove that the direct calculation of the invariant I_a on the basis of Eqs (1.3), (1.6), and (1.26) gives the same result.

The solid angle is transformed by the rule[1]

$$d\Omega' = \frac{d\Omega}{\gamma^2(1-\beta\cos\theta)^2}.$$

As $(1-\beta\cos\theta) = -n_\rho v^\rho/\gamma$ we finally have

$$dW = -\frac{e^2}{4\pi\gamma}\frac{d\Omega}{(n_\rho v^\rho)^2}\left[w_\alpha w^\alpha - c^2\frac{(n_\alpha w^\alpha)^2}{(n_\rho v^\rho)^2}\right],$$

which coincides with Eq. (1.33).

1.4. Linearly Polarized SR and Its Angular Distribution

As well as the angular distribution of instantaneous radiation power, the linearly polarized radiation and its angular distribution can be analyzed in the most general