

## Chapter 2

# Brownian Motion

### 2.1 Historical background

The fact that each degree of freedom of a particle which is in equilibrium with gas molecules has an average kinetic energy equal to that of a degree of freedom of the gas molecules, i.e.  $\frac{1}{2}kT$ , has far-reaching consequences. The possibility of explaining the motion of small specks, hovering in a gas or in a liquid, within the framework of the kinetic theory was, from a historical point of view, the most important of these consequences, due to its contribution to the resolution of the ideological struggle for and against the atomic structure of matter. This titanic struggle went on during the second half of the nineteenth century and involved the prominent physicists of that era. We can learn of its acuteness from the fact that even in the beginning of this century one of the pillars of physics — Ernst Mach — could be heard to say:

“If the belief in the existence of atoms is so crucial in your eyes, I hereby withdraw from the physicist’s way of thought. . . ”

An even stronger evidence for the heat of the debate is the fact that the sharp criticism that was aimed at Ludwig Boltzmann seems to have contributed to his suicide in 1906.

The experiment by the botanist Robert Brown, concerning the drifting of specks with radii on the order of micrometers ( $1\mu\text{m} = 10^{-6}\text{ m}$ ) in liquids and in gases, had been known since 1827. However, only in 1905 did Einstein explain the phenomenon. Within his explanation, which was based on the kinetic theory, Einstein connected in a quantitative manner the Brownian motion and quantities that appear in the kinetic theory — such as the coefficients of mobility and viscosity — and he brought the debate to a conclusion in a very short time.

But the importance of the subject far transcends the resolution of the heated debate about atomism. This subject is also of great practical

importance: understanding the effect of the thermal motions of atoms on sensitive instruments that are not too heavy and are in equilibrium with their surroundings, is very important for understanding the limits on the sensitivity of very accurate measuring instruments. Among these are the galvanometers, which are based on the deviations of a small mirror hanging from a thin thread, and the voltage variations in sensitive electric circuitry containing resistors. The random motion, which is due to the equipartition of thermal energy, is a source of “noise” in many systems. The theoretical treatment of Brownian motion is a workshop for the understanding of such phenomena. Moreover, the treatment of this subject begins to shed some light on the deep and important problem of the connection between thermal fluctuations and the “erosion of energy” (dissipation), or frictional phenomena.

**energy  
dissipation**

## 2.2 Characteristic scales of Brownian motion

The sort of phenomenon we are about to treat is schematically as follows: a large *body* hovers in a crowd of tiny particles — as a giant bear floats in a huge crowd of bees (see *Winnie the Pooh*, Chapter 1).

The tiny particles are moving to and fro. In this process they make many fast knocks against the large body. The knocks are random and each has a very small effect on the body, since the ratio of the mass of the body to that of the particle is very large.

The hovering particles that Robert Brown was able to see under his microscope have a diameter of a few micrometers. To acquaint ourselves with the data of the problem described in Fig. 1.2.1, we suppose that the body is hovering in a gas under standard conditions. The density of such a gas is  $n \approx 10^{26} \text{ m}^{-3}$ . The thermal energy of a particle at room temperature is

$$\epsilon \approx kT \approx 5 \times 10^{-21} \text{ J}$$

( $k$  is the Boltzmann constant). We are interested only in orders of magnitude, so the factor  $3/2$  was omitted. We will assume that the density of the body is a tenth of a gram per  $\text{cm}^3$ . Its mass will therefore be  $M \approx 10^{-16} \text{ kg}$ .

From the assumption that the body is in thermal equilibrium with the gas, we can deduce that its typical velocity is  $10^{-2} \text{ m s}^{-1}$  (check this). Similarly, if we take a typical value of  $10^{-26} \text{ kg}$  for the mass of the surrounding molecules, the molecular velocities will be  $10^3 \text{ m s}^{-1}$ .

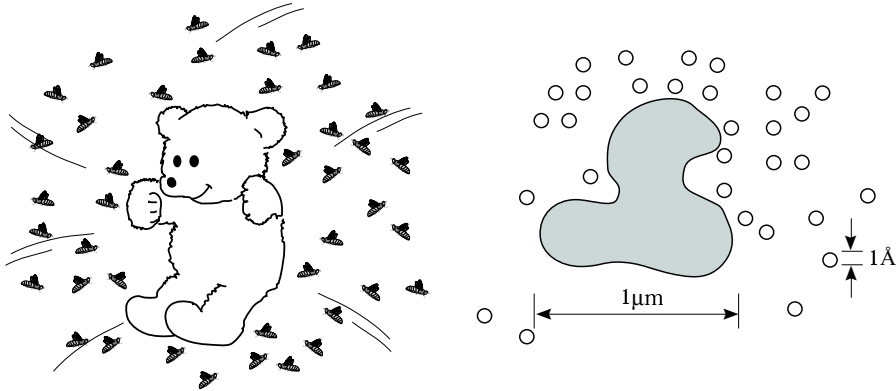


Fig. 1.2.1 A body executing Brownian motion.

**Exercise 2.1**

Calculate, for the above data, the velocity which a molecule of typical velocity can impart to the body in a head-on elastic collision.

**Solution on page 88**

**mean free path**

Another scale that characterizes the situation is the distance that the body can traverse between two collisions. This distance is called the *mean free path*, or simply the *free path* (see also Chap. 3). In order to find the orders of magnitude of the free path, we apply the following consideration: if the body were to move, parallel to itself, a distance  $L$  (Fig. 1.2.2), it would bump into  $N = n \cdot (S \cdot L)$  molecules, where  $n$  is the gas density and  $S$  is the cross section area of the body. In order to bump into one molecule the body must traverse, typically, a distance  $L$  such that  $N = 1$ . Namely

$$L = \frac{1}{nS}. \quad (1.2.1)$$

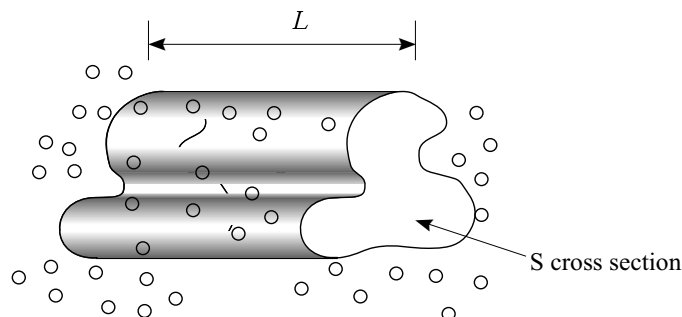


Fig. 1.2.2 Motion of a body in gas.

Under the conditions of our problem  $L \approx 10^{-14}$  m (do check), so that even using a very advanced microscope we are unable to discern isolated events: the motion will seem continuous and smooth, but lacking a preferred direction.

Along with the free path,  $L$ , it is possible to define the corresponding time scale, *mean free time*, which, as indicated by its name, is the time it takes for a particle to traverse a distance  $L$ . In our case  $\tau \approx 10^{-12}$  s, which means that the body gets hit by  $10^{12}$  molecules per second!

**mean free  
time**

## 2.3 Random walk

If we follow the motions of different bodies from an initial time  $t = 0$ , we see that each body follows a different trajectory, reflecting the randomness of the collisions. The average over many trajectories of the displacement,  $\mathbf{R}$ , of each body from its initial position gives zero. However, the average square distance is not zero. Experiments show that it grows linearly with time. Namely

$$\langle \mathbf{R}^2 \rangle = \alpha t. \quad (1.2.2)$$

Thus if, for example, at time  $t = 0$  all the bodies were at the same location, they would gradually move from their initial location so that most of them would be found near a sphere of radius proportional to  $t^{1/2}$ . The major achievement in Einstein's work, as already mentioned, was that the atomic assumption and the use of the kinetic theory enabled him to express  $\alpha$  in terms of other quantities of the kinetic theory. How this can be done will be shown in the following paragraphs.

First we inquire whether Eq. (1.2.2) fits a simple intuitive picture of the nature of the process. We assume, for the sake of simplicity, that the collisions of the body with the molecules cause it to move as in a random walk. A two-dimensional example of such a walk is shown in Fig. 1.2.3. The sketched walk consists of 36 steps. The characteristic of such a walk, a drunk's walk from a bar, is a series of steps of more or less constant length, that are randomly directed, in the corresponding space — two-dimensional for the drunk, three-dimensional for the hovering body. Obviously, every drunk will reach a specific point, in a given series of steps. We cannot say where every particular walk will arrive; however, we will have something to say about the average location of a group of drunks who leave the bar together. You may be surprised to find that we do not need to compute the probability of reaching every location. We can get along with a few very general considerations.

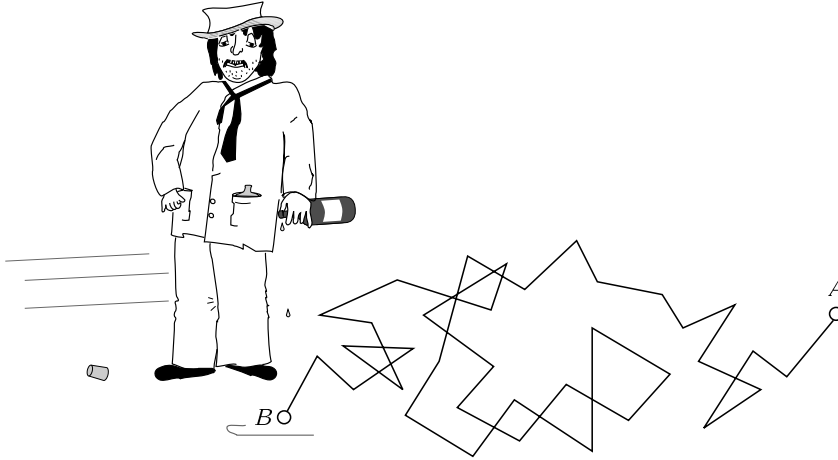


Fig. 1.2.3 Thirty-six steps in the walk of a drunk.

We will assume that the length of a step is  $L$ , that after  $N$  steps the drunk has reached the point  $\mathbf{R}_N$ , and that his next step will be in the direction  $\mathbf{n}$  ( $\mathbf{n}$  is the unit vector, which determines the direction of the next step). After  $N + 1$  steps the position of the drunk will be  $\mathbf{R}_{N+1}$ :

$$\mathbf{R}_{N+1} = \mathbf{R}_N + L\mathbf{n}. \quad (1.2.3a)$$

First, clearly, the average position of the drunks remains at the bar, since each one of them moves in a different direction,  $\langle \mathbf{R}_N \rangle = 0$  (see also Exercise 2.2). We can learn about the extent of the scattering of the drunks from the square distance from the initial position, which is given by

$$\mathbf{R}_{N+1}^2 = (\mathbf{R}_N + L\mathbf{n})^2 = \mathbf{R}_N^2 + L^2 + 2L\mathbf{n} \cdot \mathbf{R}_N. \quad (1.2.3b)$$

(Note that we used the fact that  $\mathbf{n}^2 = 1$ .)

We apply the randomness argument to Eq. (1.2.3b) in the following manner: we average both sides of the equation. The fact that the direction of  $\mathbf{n}$  is random with respect to  $\mathbf{R}_N$  leads to a zero average for the last term on the right hand side [cf. (1.1.9)]. The conclusion is that

$$\langle \mathbf{R}_{N+1}^2 \rangle = \langle \mathbf{R}_N^2 \rangle + L^2, \quad (1.2.4)$$

from which it is immediately inferred that

$$\langle \mathbf{R}_N^2 \rangle = NL^2. \quad (1.2.5)$$

$L^2$  is constant and  $N$  is the number of steps, which is proportional to the duration of the walk, so that the result of Eq. (1.2.5), is not different from that of Eq. (1.2.2).

Finally, we note that exactly the same results can be obtained if we interpret the averages in Eqs. (1.2.4) and (1.2.5) as averages over time for a single drunk, in the following manner:

We observe a certain drunk over a very long period of time — a large number of steps. Every point along the drunk's path can be considered as an initial point (assuming that the effect of the liquor does not diminish). Every such point can be considered as the origin of a coordinate system. For every such choice it is possible to find the point that is reached by the drunk after  $N$  additional steps. In a coordinate system whose origin is at that initial point the drunk will be at  $\mathbf{R}_N$ .

After reaching  $\mathbf{R}_N$  the drunk will take his next step in a random fashion (for if it were not so he would not be drunk); this means that for every choice of an initial point  $L\mathbf{n}$  will point in a different direction. The averaging over all the choices of the initial point is the averaging from the second point of view.

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### Exercise 2.2

Prove that for both points of view  $\langle \mathbf{R}_N \rangle = 0$ .

Solution on page 89

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## 2.4 Brownian motion, random force and friction: the Langevin equation

The treatment we use below is not exactly the same as that formulated by Einstein in 1905, but is similar to the formulation by his close friend, Pierre Langevin, a short while later (1908).

We describe the center of mass motion of a body in a gas as it evolves under an external force  $\mathbf{F}_e$  and friction, i.e. a restraining force. The equation of motion is

$$M\ddot{\mathbf{r}} + \mu\dot{\mathbf{r}} = \mathbf{F}_e. \quad (1.2.6)$$

■ A simple example of the appearance of a restraining force that is proportional to the velocity can be found in Self-Assessment Exercise 6 of this part.

$\mu\dot{\mathbf{r}}$  is the restraining force, proportional to the velocity — this is a typical description of a frictional force.  $\mu$  is a *friction coefficient* and is connected, as will be mentioned later, to the viscosity. First, let us elucidate the role of the friction term in Eq. (1.2.6). We do this by way of the following exercises:

**Exercise 2.3**

- (a) Calculate the dimensions of  $\mu$ .
- (b) Show that in the absence of an external force the velocity of the body tends to zero for long times, even if its initial value is different from zero, i.e. the friction restores the system to equilibrium.

**Solution on page 89**

We will not enter here into an involved discussion of the connection between  $\mu$  and the viscosity. We will limit ourselves to the following description: The viscosity describes an internal friction between different layers of the fluid (see also Sec. 3.6 below). A body in motion through a liquid drags along nearby liquid layers, giving rise to friction between successive liquid layers. This friction is expressed as a restraining force that acts on the body and is proportional to its velocity. If the body is a ball of radius  $a$  then the proportionality coefficient is especially simple, as was found by Stokes:  $F = 6\pi a\eta v$ .  $\eta$  is called the viscosity coefficient. Its dimensions are  $[M][L]^{-1}[T]^{-1}$ , i.e. mass divided by length and time, and its units in cgs are poise, P. (1P =  $10^{-1}$  kg m $^{-1}$  s $^{-1}$ .) For a gas,  $\eta$  is about  $10^{-4}$  P; it is  $10^{-2}$  P for water and 8.5 P for glycerine.

viscosity

**Exercise 2.4**

The one-dimensional system (1.2.6) with  $F_e = mg$  describes a sky diver in a gravitational field in air. Solve the equation, and verify that for long times the velocity of the sky diver is constant. What are long times?

**Solution on page 90**

The solution to Exercise 2.4 indicates a way of measuring  $\mu$  directly. The method is the same as that used for measuring the viscosity. Namely, bodies are dropped in a gravitational field in a certain medium and their final speed is measured. Its absolute value is  $Mg/\mu$ .

From the fact that the frictional force causes the sky diver of Exercise 2.4 to accelerate at a slower rate than in a free fall, it follows that its total energy decreases with time.

**Exercise 2.5**

Show that if  $\mathbf{F}_e$  is a force derived from a potential, then the rate of change of the total energy of a body whose motion is described by Eq. (1.2.6) is

$$\frac{dE}{dt} = -\mu\dot{\mathbf{r}}^2.$$

This is the rate of energy dissipation.

■ Reminder: the connection between the force and the potential is given by  $\mathbf{F} = -\nabla U$ , where  $U$  is the potential energy.

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**Solution on page 91**

We have seen, therefore, that due to the friction  $\mu\dot{\mathbf{r}}$  the particle can lose energy with time. Where does the friction come from and where does the energy go to?

The friction that restores the body to equilibrium originates of course from the numerous fast collisions with the gas molecules. These collisions also give rise to the Brownian motion of the body. The connection between  $\mu$  and microscopic factors will be considered in the next chapter. Here we note that  $\mu\dot{\mathbf{r}}$  is part of the effect of the momentum exchange between the gas molecules and the body, due to collisions. In these momentum exchanges the body transfers more energy to the gas molecules than it receives from them, due to the fact that in the direction of motion the body makes more frequent and harder collisions than it makes in the opposite direction. In other words, the energy of the body dissipates.

Beyond the damping effect of the collisions with the molecules, these fast momentum exchanges contribute a sort of random force,  $\mathbf{F}_e$ , that acts on the body in the absence of any external force.

We make here two remarks:

- (a) The randomness of the force  $\mathbf{F}_e$  is expressed by the fact that, if we average over many particles (averaging over an *ensemble*), or over different initial times, we get

$$\langle \mathbf{F}_e \rangle = 0 \quad (1.2.7)$$

as well as

$$\langle \mathbf{r} \cdot \mathbf{F}_e \rangle = 0. \quad (1.2.8)$$

Compare this with the discussion in Sec. 2.3. Figure 1.2.4 shows a series of graphs that describes the behavior with time of a random force. The series can be read as an ensemble of different systems, or as different time intervals in the behavior of the same system along the time axis.

The effect of a random force acting on one system can be substituted by random forces acting on many systems. The assumption that these two approaches lead to the same result, an assumption that seems so natural, is called the ergodic hypothesis. The generality of its validity is still the subject of active research.

- (b) It is especially important to note that, if we were to assume that the entire effect of the collisions amounts to the appearance of

a random force, i.e.  $\mu = 0$  in Eq. (1.2.6), the result would have been that the body could maintain a constant average velocity, without re-equilibrating with its surroundings. The fact that the *average* velocity is conserved, in the absence of friction, is inferred from the observation that the average of a random force is zero — Eq. (1.2.7). Hence, if we substitute  $\mu = 0$  in Eq. (1.2.6) and average both sides we obtain

$$\frac{d}{dt}\langle \dot{\mathbf{r}} \rangle = 0.$$

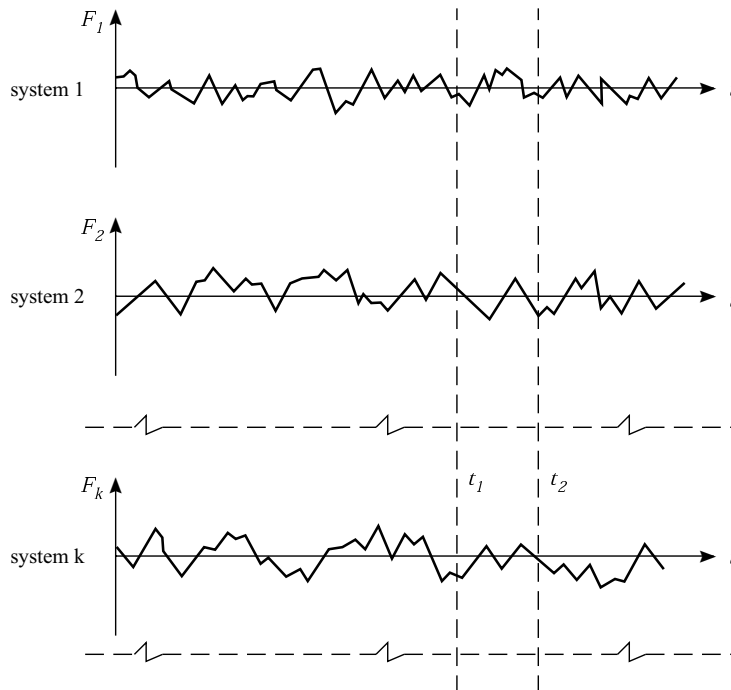


Fig. 1.2.4 Random forces.

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### Exercise 2.6

Show that the above result is an immediate consequence of Eq. (1.2.6).

**Solution on page 92**

The practical summary of the discussion above is, therefore, that the effect of the collisions can be written (following Langevin) as a sum of two contributing forces: one gives rise to the friction term and is proportional to the velocity, while the other is the random force that we called  $\mathbf{F}_e$ . Equation (1.2.6) with a random force  $\mathbf{F}_e$  is the Langevin equation.

## 2.5 Solving the Langevin equation: approximations and orders of magnitude

Let us suppose that the force  $\mathbf{F}_e$  in the Langevin equation (1.2.6) is the random force due to collisions with the gas molecules only, and attempt to deduce the time variation of the average square displacement of the body,  $\langle r^2 \rangle$ . Our aim is to show that this system, described by Eq. (1.2.6), has a solution that describes Brownian motion. We expect, therefore,  $\langle r^2 \rangle$  to be proportional to the time.

Since the solution involves analytic arguments along with statistical arguments, we shall discuss it in detail. As we are interested in the change of the magnitude  $\langle r^2 \rangle$ , we first obtain an equation for  $r^2$ :

$$\frac{1}{2}M \frac{d^2 r^2}{dt^2} + \frac{1}{2}\mu \frac{dr^2}{dt} - M\dot{\mathbf{r}}^2 = \mathbf{r} \cdot \mathbf{F}_e. \quad (1.2.9)$$

Note that Eq. (1.2.9) goes beyond the familiar context of differential equations. Beside functions of  $t$  and their derivatives, it contains a random element corresponding to some temporal sequence of the force,  $\mathbf{F}_e$ , as those exemplified in Fig. 1.2.4. It is a *stochastic differential equation*. Here we will restrict ourselves to a few comments concerning such equations:

- To any given sequence,  $\mathbf{F}_e$ , corresponds a particular solution;
- The solution corresponding to any particular sequence of the random force is of little interest;
- A quantity can be significant only if it is not strongly dependent on the particular sequence;
- Such a quantity can be calculated by averaging over all “acceptable” sequences, just because it is insensitive.
- The “acceptable” set of sequences, the *ensemble*, has to be specified.

The transition from Eq. (1.2.6) to Eq. (1.2.9) is obtained by taking the scalar product of Eq. (1.2.6) with  $\mathbf{r}$  and using the identities

$$\begin{aligned} \frac{d}{dt}r^2 &= 2\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}, \\ \frac{d^2}{dt^2}r^2 &= 2\mathbf{r} \cdot \frac{d^2\mathbf{r}}{dt^2} + 2\dot{\mathbf{r}}^2. \end{aligned} \quad (1.2.10)$$

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### Exercise 2.7

Complete the deduction of Eq. (1.2.9).

**Solution on page 92**

The next step is to average both sides of Eq. (1.2.9) over the ensemble, simplifying the right hand side with the help of Eq. (1.2.8). In this manner

**stochastic  
differential  
equation**

we obtain a differential equation for  $\langle r^2 \rangle$ :

$$\frac{1}{2}M \frac{d^2}{dt^2} \langle r^2 \rangle + \frac{1}{2}\mu \frac{d}{dt} \langle r^2 \rangle - 2 \left\langle \frac{1}{2}Mv^2 \right\rangle = 0,$$

where  $\mathbf{v}$  was substituted for  $\dot{\mathbf{r}}$ .

The last average on the left hand side can be evaluated, at equilibrium, by the equipartition principle. Every degree of freedom, for each dimension of space, is assigned an energy of  $\frac{1}{2}kT$ . The last term is therefore  $DkT$ , where  $D$  denotes the number of dimensions of space.

Thus the equation becomes

$$M\ddot{u} + \mu\dot{u} = 2DkT, \quad (1.2.11)$$

where we denoted  $\langle r^2 \rangle$  by  $u$ . This equation can be fully solved. The initial conditions are chosen to be  $r(t=0) = 0$ , namely the origin of the coordinate system of each body in the ensemble is chosen as its position at  $t = 0$ . In this case  $u(t=0) = \dot{u}(t=0) = 0$ , and

$$u(t) = \frac{2DkT}{\mu} \left[ t + \theta(e^{-t/\theta} - 1) \right], \quad (1.2.12)$$

where

$$\theta = \frac{M}{\mu}. \quad (1.2.13)$$

### Exercise 2.8

Check that Eq. (1.2.12) is in fact the solution to Eq. (1.2.11), satisfying the initial conditions, and that the parameter  $\theta$  has the dimensions of time.

### Solution on page 93

Let us inquire what happens to the body a very short and a very long time after the initiation of its motion, and compare to our physical intuition. Short and long times must be measured with respect to a characteristic time appearing in the problem. In our case this characteristic time is  $\theta$ . That is, at short times  $t \ll \theta$ , it is possible to expand the exponential in Eq. (1.2.12):

$$u(t) = \langle r^2 \rangle \approx \frac{DkT}{M} t^2. \quad (1.2.14)$$

■  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Putting it simply, at very short times, relative to times between collisions, the body moves as a free particle with constant velocity. The

constant velocity is the thermal velocity as determined at equilibrium, i.e.  $v_T = (DkT/M)^{1/2}$ . Notice that although we considered short times we have not ignored the body's previous collisions (before  $t = 0$ ), which allowed it to acquire thermal velocity. Otherwise our entire discussion of short times is invalid, since without thermal velocity we cannot use Eq. (1.2.11).

For long times  $t \gg \theta$ , the exponential decays away, and  $\theta$  can be neglected, in this limit, compared to  $t$ . So

$$u(t) = \langle r^2 \rangle \sim \frac{2DkT}{\mu} t. \quad (1.2.15)$$

This result is the same as Eq. (1.2.2), with a bonus of a relation between the coefficient  $\alpha$  and the macroscopic characteristics of the problem:

$$\alpha = \frac{2DkT}{\mu}. \quad (1.2.16)$$

To obtain an idea of the orders of magnitude of the times for which the two approximations are valid, we compute the magnitude of the time  $\theta$ . To this end, let us assume that our Brownian particles are spherical. This will allow us to use Stokes' law for a sphere:  $\mu = 6\pi\eta a$ . We further assume that the small sphere is floating in water, whose viscosity is  $\eta = 10^{-2}$  P. The particle's mass, whose radius is about one micrometer and whose specific weight is close to that of water, will be about  $5 \times 10^{-12}$  g, so that

$$\frac{M}{\mu} \approx 2 \times 10^{-7} \text{ s}.$$

This means that in any reasonable experiment, lasting more than  $10^{-3}$  s, we will not notice the region  $t \ll \theta$  but only the region  $t \gg \theta$ . Hence, the average square distance of the body from its initial position, will be linear with time.

We further remark on the role of  $D$  — the dimensionality of space — in Eq. (1.2.16). In experiment  $\langle r^2 \rangle$  is usually measured in a space whose dimensionality is less than that of the space in which the body actually moves. This happens, for example, when we measure the projection of the position of the body on the focal plane of the microscope lens. In this case  $D = 2$ , though the body's real motion is in three-dimensional space.

As already mentioned,  $\mu$  and  $\eta$  can be measured directly. With the help of the Brownian motion, i.e. with the help of Eq. (1.2.16) it is possible to measure the Boltzmann constant  $k$ . Combining this with the gas equation which gives  $R$ , it is possible to obtain Avogadro's number. Indeed, this is how J. Perrin obtained the first precise determination of Avogadro's number in 1908.

## 2.6 Applications and implications

The Brownian behavior — the fluctuations that are induced in the motion of a fine system as a result of the thermal agitation of the surroundings with which the system is in equilibrium — appears in different contexts. We shall here elaborate on the two cases that were mentioned in Sec. 2.1.

But before doing that let us broaden the discussion in Secs. 2.4 and 2.5, even if in a somewhat artificial manner. Let us assume that the floating body is attached to a spring connected at the origin. In this case Eq. (1.2.6) will take the form

$$M\ddot{\mathbf{r}} + \mu\dot{\mathbf{r}} + C\mathbf{r} = \mathbf{F}_e. \quad (1.2.17)$$

The last term on the left hand side originates, of course, from the work required in order to stretch the spring ( $\frac{1}{2}Cr^2$ ). Since it is quadratic in  $r$ , the thermal average of  $\frac{1}{2}Cr^2$  is given by the equipartition principle, and is identical to that of the kinetic energy, as in Eq. (1.1.60) of Sec. 1.6. That is, at thermal equilibrium (and only then)

$$\left\langle \frac{1}{2}Cr^2 \right\rangle = \left\langle \frac{1}{2}M\dot{\mathbf{r}}^2 \right\rangle = \frac{1}{2}DkT. \quad (1.2.18)$$

If we perform on (1.2.17) the same operations that brought us from Eq. (1.2.6) to Eq. (1.2.9) and to Eq. (1.2.11), we obtain for  $u = \langle r^2 \rangle$  the equation

$$M\ddot{u} + \mu\dot{u} + 2Cu = 2DkT. \quad (1.2.19)$$

### Exercise 2.9

Derive the above equation.

### Solution on page 93

Equation (1.2.19) appears exactly like the one-dimensional version of Eq. (1.2.17), except that instead of the random external force a constant force is acting. Alternatively, it is possible to transfer the “force”  $2DkT$  to the left hand side, and to imagine that  $u$  describes the displacement of the spring not from a loose state but from a state in which it is stretched on average according to  $2Cu = 2DkT$ . Thus, if we define a new variable  $v = u - DkT/C$  we obtain an equation that is identical to the equation of a damped harmonic oscillator:

$$M\ddot{v} + \mu\dot{v} + 2Cv = 0. \quad (1.2.20)$$

The solution to Eq. (1.2.20) includes an exponentially decaying factor, as for the case  $C = 0$ , in addition to a restoring force ( $2Cv$ ) producing

vibrations around  $v = 0$ . Thus, even without investigating the solution in detail we can reach the conclusion that at long times  $v$  tends to its equilibrium value, i.e. zero, and that  $u$  tends to its equilibrium value,  $DkT/C$ , as inferred from Eq. (1.2.18).

If we search for a solution of the form  $e^{-\gamma t}$ , we find that the substitution in Eq. (1.2.19) gives two possible values (both positive) for  $\gamma$ :

$$\gamma_{1,2} = \frac{1}{2\theta} \left( 1 \pm \sqrt{1 - \frac{8CM}{\mu^2}} \right), \quad \gamma_2 > \gamma_1. \quad (1.2.21)$$

And the solution corresponding to the initial conditions  $u(t = 0) = \dot{u}(t = 0) = 0$  is

$$\langle r^2 \rangle = u(t) = \frac{DkT}{C} \left( 1 - \frac{\gamma_2 e^{-\gamma_1 t} - \gamma_1 e^{-\gamma_2 t}}{\gamma_2 - \gamma_1} \right). \quad (1.2.22)$$

Indeed, for long times  $\langle r^2 \rangle$  tends to its equilibrium value. (See also Fig. 1.2.5.)

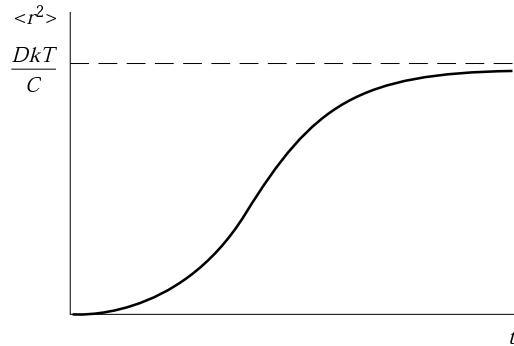


Fig. 1.2.5 A graphic representation of Eq. (1.2.22).

Finally, we note that when the square root in Eq. (1.2.21) becomes imaginary, i.e. when the damping is small, the exponential solution becomes a solution of damped oscillations. This solution also tends to  $DkT/C$  at long times.

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### Exercise 2.10

Obtain Eq. (1.2.22), and check its short time behavior. When is  $u(t)$  a linear function of time?

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**Solution on page 93**

Notice that the behavior at times which are not too long resembles that of ordinary Brownian motion without an elastic force. At first  $u$

grows quadratically with time, when the effect of the collisions and of the elastic force are still negligible. Later, when the effect of the collisions starts to be significant, but the elastic force is still negligible, the body does indeed move as in Brownian motion. In contrast, at long times the elastic force dominates and does not allow further separation beyond the limit set by the temperature. This situation does not describe a group of drunk people who move freely from the moment they leave the bar, but rather a group of drunk horses that are tied by flexible straps at the entrance to the bar.

A more physical example for this state of affairs is the galvanometer. This instrument is used to measure very small currents, by means of the very small angles of rotation of a quartz whisker that these currents induce. We shall not halt here to explain how the currents induce the rotations of the whisker, but rather concentrate on the way these rotations are measured. A tiny mirror is connected to the whisker. Light is projected on the mirror and the angle of rotation  $\phi$  of the whisker is measured by registering the angle into which light is reflected from the mirror, on a scale (see Fig. 1.2.6).

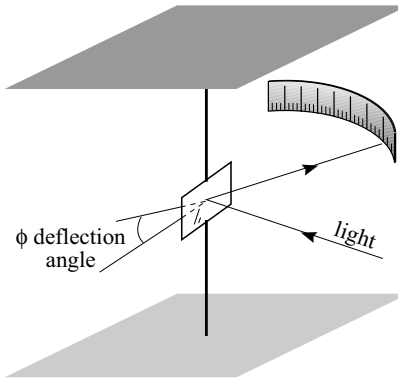


Fig. 1.2.6 Measurement of the angle of reflection in a galvanometer.

Here as well, the mirror and the quartz whisker are in thermal equilibrium with the surrounding gas. The rotations of the whisker have a kinetic energy that depends on the moment of inertia of the system, i.e.  $\frac{1}{2}I\dot{\phi}^2$ , and a potential energy  $U = \frac{1}{2}a\phi^2$ , where  $-a\phi$  is the restoring force of the whisker. In this case the temporal behavior of  $\phi(t)$  is determined by Eq. (1.2.17), with the following substitutions:

$$\text{dimensionality } D = 1, C \rightarrow a, M \rightarrow I, \mathbf{r} \rightarrow \phi.$$

Here, too, there exists a random force or torque, originating from the collisions of the whisker and the mirror with the gas molecules, that gives

a friction term  $\mu\dot{\phi}$ , which restores equilibrium, as well as the force  $\mathbf{F}_e$ , whose average vanishes.

The result is that even in the absence of a current, deviations will appear due to thermal fluctuations. The average square angular deviations will be given by Eq. (1.2.22) in terms of the parameters of the galvanometer. For long times one finds

$$\langle\phi^2\rangle = \frac{kT}{a},$$

which is appropriately dimensionless. This is a noise that limits the precision of the instrument. In order to reduce it, the instrument has to be cooled. But where? The answer can be found in Sec. 2.4. The fluctuations that give rise to  $\langle\phi^2\rangle$  are the ones that produce the friction, and therefore the part in which friction is created has to be cooled. This can be, for example, the mirror which suffers collisions with the gas molecules. In this case the gas has to be cooled.

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### Exercise 2.11

How can we measure the restoring force constant  $a$  of the whisker?

**Solution on page 94**

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In Sec. 2.5 we saw that the characteristic time for Brownian motion  $\theta = M/\mu$  is very short —  $10^{-7}$  s. Thus, it is impossible to observe in an experiment on Brownian motion the exponential decay to the linear region. Here, on the other hand,  $\theta = I/\mu$ , but  $\mu$  can be reduced by reducing the pressure of the gas, making it possible to increase  $\theta$  up to measurable magnitudes and to check the predictions of the theory in great detail, as was done by Kappler in 1931.

Finally, we mention another analogous instance — the Johnson noise, caused by the fluctuations in a resistor, of a resonating circuit with a high  $Q$  factor (see Fig. 1.2.7). The thermal fluctuations, caused by the collisions of electrons, which form the current  $I$ , with the atoms in the resistor, produce also here the double effect — resistance  $R$ , which damps the system and drags it towards equilibrium at zero current, and a random

**Johnson  
noise  
quality  
factor**

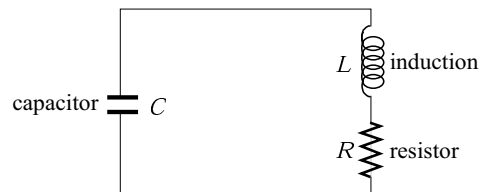


Fig. 1.2.7 A resonating circuit. A high  $Q$  value means that the circuit amplifies significantly only in a very narrow band around the resonant frequency  $\omega = 1/\sqrt{LC}$ .

(electromotive) force. The equation of the circuit determines the charge on the capacitor  $q$ :

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = V_e. \quad (1.2.23)$$

Here  $C$  is the capacitance,  $L$  the induction and  $V_e$  the random electromotive force originating from the fluctuations in the resistor. This is the Langevin equation, analogous to Eq. (1.2.17), with the substitutions

$$D = 1, \quad \mathbf{F}_e \rightarrow V_e, \quad C \rightarrow \frac{1}{C}, \quad \mu \rightarrow R, \quad M \rightarrow L, \quad \mathbf{r} \rightarrow q.$$

We are of course interested not in the solution to Eq. (1.2.23), but in the solution to the equation that determines the time dependence of  $\langle q^2 \rangle$  or of the directly measurable quantity  $\langle V^2 \rangle$ , where  $V = \frac{q}{C}$  is the voltage across the capacitor. The equation obtained for  $\langle V^2 \rangle$  has the same form as Eq. (1.2.19), which was solved in detail.

But even without solving the equation we can reach the following conclusions: the first term of Eq. (1.2.23) comes about from the “kinetic energy” term,  $\frac{1}{2}LI^2$ , while the last term originates in a “potential energy” term,  $\frac{q^2}{2C}$ . Therefore, at equilibrium we obtain an equation that resembles Eq. (1.2.18):

$$\left\langle \frac{q^2}{2C} \right\rangle = \left\langle \frac{1}{2}LI^2 \right\rangle = \frac{1}{2}kT. \quad (1.2.24)$$

Since there are charge fluctuations, there will also be voltage fluctuations (Johnson noise), which are

$$\langle V^2 \rangle = \frac{kT}{C} \quad (1.2.25)$$

or

$$\langle V^2 \rangle = \omega^2 LkT, \quad (1.2.26)$$

where  $\omega$  is the resonant frequency of the circuit.