Chapter 1

CHUA'S CIRCUIT AND THE QUALITATIVE THEORY OF DYNAMICAL SYSTEMS

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Simple electronic oscillators were at the origin of many studies related to the qualitative theory of dynamical systems. Chua's circuit ([Chua, 1992; Madan, 1993; Chua, 1993; Chua & Pivka, 1995; Wu & Chua, 1996; Pivka *et al.*, 1996]) is now playing an equivalent role for the generation and understanding of complex dynamics.

In honour of my friend Leon Chua on his 60th birthday.

1. Oscillating Circuits and the Origin of the Qualitative Theory

In the 19th century, Joseph Fourier wrote: "*The study of Nature is the most productive source of mathematical discoveries. By offering a specific objective, it provides the advantage of excluding vague problems and unwieldy calculations. It is also a means to form the Mathematical Analysis, and isolate the most important aspects to know and to conserve. These fundamental elements are those which appear in all natural effects*".

The important development of the theory of dynamic systems during this century essentially has its origins in the study of the "natural effects" encountered in systems of mechanical, electrical, or electronic engineering, and the rejection of non-essential generalizations. Most of the results obtained in the abstract dynamic systems field have been possible on the foundations of results of the concrete dynamic systems field. It is also worth noting that the majority of scientists (including mathematicians) were not led to their discoveries by a process of deduction from general postulates or general principles, but rather by a thorough examination of properly chosen cases and observation of concrete processes. The generalizations have come later because it is far easier to generalize an established result than to discover a new line of argument.

Since Andronov (1932), three different approaches have traditionally been used for the study of dynamical systems: *Qualitative methods, analytical methods and numerical methods.* To define the "strategy" of *qualitative methods*, one has to note that the solutions of equations of nonlinear dynamic systems are in general nonclassical, transcendental functions of Mathematical Analysis, which are very complex. This "strategy" is of the same type as the one used for the characterization of a function of the complex variable by its singularities: Zeros, poles, essential singularities. Here, the complex transcendental functions are defined by the singularities of continuous (or discrete) dynamic systems such as:

— stationary states which are equilibrium points (fixed points), or periodical solutions (cycles), which can be stable, or unstable;
— trajectories (invariant curves,) passing through saddle singularities of two-dimensional systems;
— stable and unstable manifolds for dimensions greater than two;
— boundary, or separatrix, of the influence domain (domain of attraction, or basin) of a stable (attractive) stationary state;
— homoclinic, and heteroclinic singularities;
— or more complex singularities of fractal, or nonfractal type.

The qualitative methods consider the nature of these singularities in the phase (or state) space, and their evolution when parameters of the system vary, or in the presence of a continuous structure modification of the system (study of the *bifurcation* sets in the parameter space, or in a function space) [Andronov *et al.*, 1966, 1966a, 1967].

In fact, at the beginning qualitative methods developed from the fundamental studies of circuits of radio-engineering. Indeed in 1927, Andronov, the most famous student of Mandelstham, defends his thesis with the topic formulated by Mandelstham *The Poincaré limit cycles and the theory of oscillations*. This thesis is a first-rank contribution for the evolution of the theory of nonlinear oscillations because it opens up new possibilities for application of Poincaré's qualitative theory of differential equations, with great practical significance. With this work, Andronov was the first to see that phenomena of free (or self) oscillations, for example that generated by the Van der Pol oscillator, correspond to limit cycles. It is from the study of oscillators that afterwards, Andronov amplifies his activity with a precise purpose: The development of a *theory of nonlinear oscillations*, in order to make use of mathematical tools common to different scientific disciplines [Andronov *et al.*, 1966].

Andronov and Pontrjagin formulated in 1937 the necessary and sufficient conditions for structural stability of *autonomous two-dimensional systems.* These conditions are: The system only has a finite number of equilibrium points and limit cycles, which are not in a critical case in the Liapunov's sense; no separatrix joins the same, or two distinct saddle points. In this case, it is possible to define, in the parameter space of the system, a set of cells inside of which the same qualitative behavior is preserved [Andronov *et al.*, 1966].

The knowledge of such cells is of first importance for the analysis, and the synthesis of dynamic systems in physics or engineering. On the boundary of a cell, the dynamic system is structurally unstable, and for *autonomous two-dimensional systems* (*two-dimensional vector fields*), *structurally stable systems are dense* in the function space. Until 1966, the conjecture of the extension of this result for higher dimensional systems was generally believed to be true.

Andronov also extended the notion of structural stability for dynamic systems described by:

$$dx/dt = f(x, y)\,, \quad \boldsymbol{\mu} dy/dt = g(x, y)\,, \quad \boldsymbol{\mu} > 0\,, \tag{1}$$

where x, y, are vectors, $\boldsymbol{\mu}$ is a "small" parameter vector representing the parasitic elements of the system, $f(x, y)$, and $g(x, y)$ are bounded and continuous in the domain of interest of the phase space. If $\boldsymbol{\mu} = 0$, (1) reduces to a system of lower dimension,

$$dx/dt = f(x, y), \quad g(x, y) = 0. \tag{2}$$

For theoretical, as well as practical purposes, a fundamental problem consists in determining when the "small" terms $\boldsymbol{\mu} dy/dt$, representing the effects of the parasitic elements (small capacitances and inductances in an electrical system, small damping and inertia in a mechanical one) are negligible. In other words, when is the motion described by (1) sufficiently close to the motion described by (2), so that it can be represented by the solution of (2) defined for a lower dimension?

It is interesting to note that the formulation of this important problem has its origin in a discussion (1929) between Andronov and Mandelstham, related to the *one time-constant electronic multivibrator.* Without considering the parasitic elements, such as parasitic capacitances, and inductances, the multivibrator is nominally described by a first-order (one-dimensional) autonomous differential equation, such as (2) where x is now a scalar (voltage). If it is required that $y(t)$ be a continuous function of time, then it was shown by Andronov that (2) does not admit any non-constant periodic solution. Such a mathematical result is contrary to physical evidence, because the one time-constant multivibrator is known to oscillate with a periodic waveform. In the Mandelstham—Andronov's discussion of this paradox, the following alternative was formulated: (a) either the nominal model (2) is not appropriate to describe the practical multivibrator, or (b) it is not being interpreted in a physically significant way.

Andronov has shown that either term of the alternative may be used to resolve the paradox, provided the space of the admissible solutions is properly defined. In fact, specifying that the solutions must be continuous and continuously differentiable leads to the conclusion that (2) is inappropriate on physical grounds, because the real multivibrator possesses several small parasitic elements. Then this leads to a model in the form (1), the vector $\boldsymbol{\mu}$ being related to the parasitic elements. However (1) appears as rather unsatisfactory from a practical point of view. Indeed the existence and the stability of the required periodic solution depends not only on the *presence* of parasitic parameters, which are difficult to measure in practice, but also on their *relative magnitudes*. Andronov has shown that the strong dependence on parasitic elements can be alleviated by means of the second term of the alternative. This is made by generalizing the set of admissible solutions, defined now as consisting of piecewise continuous and piecewise differentiable functions. Then the first-order differential equation (2) is supplemented by some ⟪*jump*⟫ conditions (called *Mandelshtam conditions*) permitting the joining of the various pieces of the solution, which can now be periodic. The theory of models having the form (1) associated with the problem of dimension reduction, and that of *relaxation oscillators* began with this study.

2. Chua's Circuit and the Contemporary Qualitative Theory

One of the reasons for the popularity of the Chua's circuit is due to the fact that it can generate a large variety of complex dynamics, and convoluted bifurcations, from a simple model in the form of a *three-dimensional autonomous piecewise linear ordinary differential equation* (*flow*). It concerns a concrete realization (with discrete electronic components, or implemented in a single monolitic chip) while the well-known *Lorenz equation*, which is also a three-dimensional flow, is related to a very rough low-dimensional model of atmospheric phenomena, far from the real complexity of the ⟪*nature*⟫.

As mentioned above, until 1966, an extension of two-dimensional structural stability conditions, for dimensions higher than two, was conjectured. But Smale [1966, 1967] showed that this conjecture is false in general. So, it appears that, with an increase of the system dimension, one has an