

shown by Andronov that (2) does not admit any non-constant periodic solution. Such a mathematical result is contrary to physical evidence, because the one time-constant multivibrator is known to oscillate with a periodic waveform. In the Mandelstham—Andronov's discussion of this paradox, the following alternative was formulated: (a) either the nominal model (2) is not appropriate to describe the practical multivibrator, or (b) it is not being interpreted in a physically significant way.

Andronov has shown that either term of the alternative may be used to resolve the paradox, provided the space of the admissible solutions is properly defined. In fact, specifying that the solutions must be continuous and continuously differentiable leads to the conclusion that (2) is inappropriate on physical grounds, because the real multivibrator possesses several small parasitic elements. Then this leads to a model in the form (1), the vector μ being related to the parasitic elements. However (1) appears as rather unsatisfactory from a practical point of view. Indeed the existence and the stability of the required periodic solution depends not only on the *presence* of parasitic parameters, which are difficult to measure in practice, but also on their *relative magnitudes*. Andronov has shown that the strong dependence on parasitic elements can be alleviated by means of the second term of the alternative. This is made by generalizing the set of admissible solutions, defined now as consisting of piecewise continuous and piecewise differentiable functions. Then the first-order differential equation (2) is supplemented by some *«jump»* conditions (called *Mandelstham conditions*) permitting the joining of the various pieces of the solution, which can now be periodic. The theory of models having the form (1) associated with the problem of dimension reduction, and that of *relaxation oscillators* began with this study.

2. Chua's Circuit and the Contemporary Qualitative Theory

One of the reasons for the popularity of the Chua's circuit is due to the fact that it can generate a large variety of complex dynamics, and convoluted bifurcations, from a simple model in the form of a *three-dimensional autonomous piecewise linear ordinary differential equation (flow)*. It concerns a concrete realization (with discrete electronic components, or implemented in a single monolithic chip) while the well-known *Lorenz equation*, which is also a three-dimensional flow, is related to a very rough low-dimensional model of atmospheric phenomena, far from the real complexity of the *«nature»*.

As mentioned above, until 1966, an extension of two-dimensional structural stability conditions, for dimensions higher than two, was conjectured. But Smale [1966, 1967] showed that this conjecture is false in general. So, it appears that, with an increase of the system dimension, one has an

increase of complexity of the parameter (or function) space. The boundaries of the cells defined in the phase space, as well as in the parameter space, have in general a complex structure, which may be a fractal (self-similarity properties) for n -dimensional vector fields, $n > 2$.

Sufficient conditions of structural stability were formulated by Smale in 1963. A system is structurally stable when the fixed (equilibrium) points and periodic solutions (orbits) are structurally stable and in finite number, when the set of nonwandering points consists of these stationary states only, when all the stable and unstable manifolds intersect transversally. Such systems are now known as *Morse–Smale systems*.

The analysis of *bifurcations, which transform a Morse–Smale system into a system having an enumerable set of periodic orbits*, has been a favorite choice of research topic since 1965. There certainly exists a lot of such bifurcations of different types. Gavrilov, Afraimovitch and Shilnikov have studied some of them which were related to the presence of structurally unstable homoclinic, or heteroclinic curves associated with an equilibrium point, or a periodic orbit for a dimension $m > 3$. Their results have contributed to the study of the popular *Lorenz differential equation* ($m = 3$) by Afraimovitch and Shilnikov [Afraimovitch *et al.*, 1983]. *Chua's circuit* belongs to the class of three-dimensional “continuous” dynamical systems (*flows* with $m = 3$). With respect to other studies it has the advantage of exhibiting “physical” bifurcations which transform a Morse–Smale system into a system having an enumerable set of periodic orbits.

Let us consider this class of three-dimensional “continuous” dynamical systems (*flows*), and two-dimensional diffeomorphisms associated with them from a Poincaré section. Newhouse [1979] formulated a very important theorem stating that in any neighborhood of a C^r -smooth ($r \geq 2$) dynamical system, in the space of discrete dynamical systems (diffeomorphisms), there exist regions for which systems with homoclinic tangencies (then with structurally unstable, or nonrough homoclinic orbits) are dense. Domains having this property are called *Newhouse regions*. This result is completed in [Gonchenko *et al.*, 1993] which asserts that systems with infinitely many homoclinic orbits of any order of tangency, and with infinitely many arbitrarily degenerate periodic orbits, are dense in the Newhouse regions of the space of dynamical systems. This has a considerable consequence:

Systems belonging to a Newhouse region are such that a complete study of their dynamics and bifurcations is impossible.

Then only particular characteristics of such systems can be studied, such as the presence of nontrivial hyperbolic subsets (infinite number of saddle cycles). Let us restrict to a one-parameter family of three-dimensional dynamical systems leading to *Newhouse intervals*, and the associated family

of two-dimensional diffeomorphisms (differentiable invertible maps). Then in such intervals there are dense systems with an infinite number of stable cycles (periodic orbits) if the modulus of the product of their multipliers (eigenvalues) is less than one, and with infinitely many totally unstable cycles if this modulus is higher than one [Shilnikov, 1994]. This last result furnishes a theoretical foundation to the fact that many of the attractors studied contain a “large” hyperbolic subset in the presence of a finite or infinite number of stable cycles. Generally such stable cycles have large periods, and narrow “oscillating” tangled basins, which are difficult to put in evidence numerically.

Systems having infinitely many unstable periodic orbits (they are not of Morse–Smale type) give rise either to *strange attractors*, or to *strange repellers*. Strange repellers are at the origin of two phenomena: Either that of *chaotic transient* toward only one attractor for small changes of initial conditions, or that of *fuzzy* (or *fractal*) *boundaries* [Grebogi *et al.*, 1983] separating the basins of several attractors. In fact, a fractal basin boundary also gives rise to chaotic transients, but toward at least two attractors in the presence of very small variations of initial conditions. The structure identification of strange attractors and repellers, and the bifurcations giving rise to such a complex dynamics, constitute one of the most important problem of this time.

Strange attractors are presently distinguished into three principal classes: *Hyperbolic*, *Lorenz-type*, and *quasi-attractors* [Shilnikov, 1994].

Hyperbolic attractors are the limit sets for which Smale’s Axiom A is satisfied, and are structurally stable. Periodic orbits and homoclinic orbits are dense and are of the same saddle type, that is the stable (resp. unstable) manifold of all the trajectories have the same dimension. In particular, this is the case of Anosov systems, and the Smale–Williams solenoid. Till now such attractors have not been found in concrete applications.

Lorenz attractors are not structurally stable, though their homoclinic and heteroclinic orbits are structurally stable (hyperbolic). They are everywhere, and no stable periodic orbits appear under small parameter variations [Afraimovitch *et al.*, 1983] (for more references cf. also [Shilnikov, 1994]).

Both hyperbolic and Lorenz attractors are stochastic, and thus can be characterized from the ergodic theory.

Quasi-attractors (abbreviation, of “quasistochastic attractors” [Afraimovitch & Shilnikov, 1983], for more references cf. also [Shilnikov, 1994]) are not stochastic, and are more complex than the above two attractors. A quasi-attractor is a limit set enclosing periodic orbits of different topological types (for example stable and saddle periodic orbits), structurally unstable orbits. Such a limit set may not be transitive. *Attractors generated by Chua’s circuits* [Chua, 1992, 1993], associated with *saddle-focus homoclinic loops*

are quasi-attractors. For three-dimensional systems, mathematically such attractors should contain infinitely stable periodic orbits, a finite number of which can only appear numerically due to the finite precision of computer experiments. They coexist with nontrivial hyperbolic sets. Such attractors are encountered in a lot of models, such as the Lorenz system, the *spiral-type* and the *double-scroll* attractor generated by a Chua's circuit, the Henon map, this for certain domains of the parameter space.

The complexity of quasi-attractor is essentially due to the existence of structurally unstable homoclinic orbits (homoclinic tangencies) not only in the system itself, but also in any system close to it. It results in a sensitivity of the attractor structure with respect to small variations of the parameters of the generating dynamical equation, i.e. *quasi attractors are structurally unstable*. Then such systems belong to Newhouse regions with the consequences given above.

In the n -dimensional case, $n > 3$, the situation becomes more complex and the first results (in particular a theorem showing that a system can be studied in a manifold of lower dimension) can be found in [Gonchenko *et al.*, 1993b, 1993c].

In addition to its interest in engineering applications, *Chua's circuit* generates a large number of complex fundamental dynamical phenomena. Indeed it is the source of different bifurcations giving rise to chaotic behaviors (period doubling cascade, breakdown of an invariant torus, etc.). The corresponding attractors are related to complex homoclinic heteroclinic structures. One of these attractors, the *double scroll*, characterized by the presence of three equilibrium points of saddle-focus type, arises from two nonsymmetric spiral attractors. It is different from other known attractors of autonomous three-dimensional systems in the sense that it is multistructural.

3. Conclusion

The important book by Madan [1993] collects many contributions devoted to applied and theoretical questions related to this circuit, which since this publication has given rise to many new developments. So the *synchronization of chaotic signals* generated by Chua's circuit leads to an increasing number of publications, with applications to secure communications [Lozi & Chua, 1993]. Moreover, a wide field of research is beginning to be opened through the use of a two- and three-dimensional grid of resistively coupled Chua's circuits. From such networks, waves and spatiotemporal chaos can be put in evidence with *travelling, spiral, target, scroll waves* [Chua & Pivka, 1995]. Here Chua's circuit is used as the basic cell in a discrete *cellular neural network (CNN)*.