1.2 The Heisenberg picture

In the Schrödinger picture the basis vectors are visualized as a fixed set of vectors and the state vector $|\Psi_S(t)\rangle$ as changing in time. Vice versa, the same system can be described equally well by time-dependent observable operators and by a fixed state vector. This picture is physically equivalent to the Schrödinger picture and is called the Heisenberg picture. Thus, in the Heisenberg picture the state vector is

$$|\Psi_H(t)\rangle = |\Psi_S(t_0)\rangle \tag{1.20}$$

here the subscript H designates the Heisenberg picture. It is clear that the state vector $|\Psi_H(t)\rangle$ coincides with the state vector at time $t = t_0$ in the Schrödinger picture. Therefore, the state vectors in the two pictures are connected by equation (1.6)

$$|\Psi_S(t)\rangle = U(t,t_0)|\Psi_H\rangle \tag{1.21}$$

Since the expectation values of the physical observables correspond to the experimental results and they are actually independent of the pictures. That is to say,

$$\begin{aligned}\langle A\rangle &= \langle\Psi_S(t)|A_S|\Psi_S(t)\rangle = \langle\Psi(t_0)|U^\dagger(t,t_0)A_S U(t,t_0)|\Psi(t_0)\rangle \\ &= \langle\Psi_H|U^\dagger(t,t_0)A_S U(t,t_0)|\Psi_H\rangle = \langle\Psi_H|A_H|\Psi_H\rangle\end{aligned} \tag{1.22}$$

It is easy to see from the above equation that the operators in the two pictures have the following transformation law

$$A_H(t) = U^\dagger(t,t_0)A_S U(t,t_0) \tag{1.23}$$

From (1.23) we see that the operators, which are stationary in the Schrödinger picture, become time-dependent ones in the Heisenberg picture according to this transformation law.

Now we examine how the eigenvectors of the operator $A_H(t)$ evolve in the Heisenberg picture. In the Schrödinger picture, the eigenvalue equation of the operator A_S is

$$A_S|u_n^S\rangle = \lambda_n|u_n^S\rangle \tag{1.24}$$

By means of (1.23) we obtain

$$U(t,t_0)A_H(t)U^\dagger(t,t_0)|u_n^S\rangle = \lambda_n|u_n^S\rangle$$

Noticing that $U^\dagger U = 1$, the above equation becomes

$$A_H(t)|u_n^H(t)\rangle = \lambda_n|u_n^H(t)\rangle \tag{1.25}$$

where $|u_n^H(t)\rangle$ is defined as

$$|u_n^H(t)\rangle = U^\dagger(t,t_0)|u_n^S\rangle \tag{1.26}$$

This is just the transformation law of the eigenvectors of the operator A in the two pictures. It shows that the time-independent eigenvectors $|u_n^S\rangle$ in the Schrödinger picture become time-dependent eigenvectors $|u_n^H(t)\rangle$ in the Heisenberg picture. Comparing (1.21) with (1.26) we see that in the Schrödinger picture the state vector $|\Psi_S(t)\rangle$ varies along a certain direction with the time development, but in the Heisenberg picture, the eigenvectors of the operator A_H vary along the opposite direction.

In the Heisenberg picture the time evolution of the system can be obtained by solving the Heisenberg equation of the operator $A_H(t)$. Differentiating both sides of (1.25) with respect to t, we obtain

$$\begin{aligned} i\hbar\frac{d}{dt}A_H &= U^\dagger A_S H_S U - U^\dagger H_S A_S U + i\hbar U^\dagger\frac{\partial}{\partial t}A_S U \\ &= U^\dagger A_S U U^\dagger H_S U - U^\dagger H_S U U^\dagger A_S U + i\hbar U^\dagger\frac{\partial}{\partial t}A_S U \\ &= [A_H, H_H] + i\hbar U^\dagger\frac{\partial}{\partial t}A_S U \end{aligned} \tag{1.27}$$

where we have used eq.(1.25) and we have defined

$$H_H(t) = U^\dagger(t,t_0)H_S U(t,t_0) \tag{1.28}$$

which is the Hamiltonian in the Heisenberg picture. Eq.(1.27) is interpreted as the Heisenberg equation of motion for the operator A_H. Like the Schrödinger equation in the Schrödinger picture, (1.27) is the fundamental equation describing the time evolution of the system in the Heisenberg picture. When the expression of $A_H(t)$ is explicitly obtained, we can obtain the expectation value of the operator A_H.

If $\frac{d}{dt}A_H = 0$, then A_H is a constant of the motion. When A_S has no explicit time dependence, (1.27) reduces to

$$\frac{d}{dt}A_H = \frac{1}{i\hbar}[A_H, H_H] \tag{1.29}$$

In fact, for a conservative system, the following relation in the Schrödinger picture is valid: $\frac{d}{dt}H_S$ =0. As a special case, we let $A_S = H_S$ and note $U(t,t_0)$ is defined by (1.10), then $[H_S, U] = 0$. In this case, eq.(1.23) gives that $H_S = H_H$. This means that for a conservative system, the Hamiltonians both in the Schrödinger picture and in the Heisenberg picture are identical. According to (1.29), we then have

$$\frac{d}{dt}H_H = 0 \tag{1.30}$$

which shows that H is a constant of the motion in the two pictures. In other words, H has the same form in the two pictures.

Another important property we have to point out is that the commutation relations in the two pictures should be the same form, because the commutation relations represent the correlation of physical quantities which must be independent of the different description (or picture) . As an example, we take A_S, B_S, and C_S to be three operators in the Schrödinger picture and satisfy the commutation relation

$$[A_S, B_S] = iC_S \tag{1.31}$$

If we multiply both sides from the left by $U^\dagger$ and from the right by U, we have

$$U^\dagger A_S B_S U - U^\dagger B_S A_S U = iU^\dagger C_S U$$

Moreover we insert $UU^\dagger = 1$ between A_S and B_S , which yields

$$U^\dagger A_S U U^\dagger B_S U - U^\dagger B_S U U^\dagger A_S U = iU^\dagger C_S U$$

Considering eq.(1.23), we obtain the commutation relation in the Heisenberg picture as

$$[A_H(t), B_H(t)] = iC_H(t) \tag{1.32}$$

Equation (1.32) has the same form as (1.31). Thus, the simultaneous commutation relations of the operators both in the Heisenberg picture and in the Schrödinger picture must have the same form. Since these commutation relations are independent of the pictures, then the Heisenberg uncertainty relations of the operators in the Heisenberg picture, taken at the same time, are also the same as those in the Schrödinger picture.

So far we have discussed two pictures for describing the system in quantum mechanics. It is worth to point out that for the two kinds of descriptions, the Schrödinger picture is more appropriate to describe a conservation system, because the state vector may be solved in a conservation system. But for an open system (such as the atom-field coupling system in a bad cavity), because the Hamiltonian of the system has more complicated form due to the effect of the surroundings, therefore, it is not easy to solve the state vector from equation (1.5). However, it is possible to solve the Heisenberg equations (1.29) , furthermore, the time evolution of the physical observables and their expectation values can be obtained explicitly. Thus, to deal with the problems of a quantum system, one must choose an appropriate picture in which the physical properties of the system can be easily revealed and the calculating processes are relatively simple in mathematics.

1.3 The Interaction Picture

1.3.1 Equation of motion in the interaction picture

Another picture besides the two pictures discussed in the previous sections, is the interaction picture. This picture is frequently used in quantum optics, when the Hamiltonian of the system can be written as a sum form of two terms

$$H_S = H_0^S + V_S \tag{1.33}$$

where H_0^S is independent of the time and its eigenvectors are easy to solve by the following equation

$$i\hbar\frac{\partial}{\partial t}|\psi_n\rangle = H_0^S|\psi_n\rangle \tag{1.34}$$

The term V_S can be regarded as the interaction energy operator of the system, and may depend explicitly on time although it may need not. In fact, V_S usually induces a strong influence on the time behavior of the system. The aim of introducing the interaction picture is to investigate the effects of the interaction Hamiltonian of the system on the time behavior of the system.

The method of transforming the Schrödinger picture into the interaction picture is performed by a unitary operator $U_0(t, t_0)$ such that

$$|\Psi_S(t)\rangle = U_0(t, t_0)|\Psi_I(t)\rangle \tag{1.35}$$

here the subscript I refers to the interaction picture and $U_0(t, t_0)$ satisfies

$$U_0(t, t_0) = \exp\left[-\frac{i}{\hbar}H_0^S(t - t_0)\right] \tag{1.36}$$