the Schrödinger picture is more appropriate to describe a conservation system, because the state vector may be solved in a conservation system. But for an open system (such as the atom-field coupling system in a bad cavity), because the Hamiltonian of the system has more complicated form due to the effect of the surroundings, therefore, it is not easy to solve the state vector from equation (1.5). However, it is possible to solve the Heisenberg equations (1.29) , furthermore, the time evolution of the physical observables and their expectation values can be obtained explicitly. Thus, to deal with the problems of a quantum system, one must choose an appropriate picture in which the physical properties of the system can be easily revealed and the calculating processes are relatively simple in mathematics.

1.3 The Interaction Picture

1.3.1 Equation of motion in the interaction picture

Another picture besides the two pictures discussed in the previous sections, is the interaction picture. This picture is frequently used in quantum optics, when the Hamiltonian of the system can be written as a sum form of two terms

$$H_S = H_0^S + V_S \tag{1.33}$$

where H_0^S is independent of the time and its eigenvectors are easy to solve by the following equation

$$i\hbar\frac{\partial}{\partial t}|\psi_n\rangle = H_0^S|\psi_n\rangle \tag{1.34}$$

The term V_S can be regarded as the interaction energy operator of the system, and may depend explicitly on time although it may need not. In fact, V_S usually induces a strong influence on the time behavior of the system. The aim of introducing the interaction picture is to investigate the effects of the interaction Hamiltonian of the system on the time behavior of the system.

The method of transforming the Schrödinger picture into the interaction picture is performed by a unitary operator $U_0(t,t_0)$ such that

$$|\Psi_S(t)\rangle = U_0(t,t_0)|\Psi_I(t)\rangle \tag{1.35}$$

here the subscript I refers to the interaction picture and $U_0(t,t_0)$ satisfies

$$U_0(t,t_0) = \exp\left[-\frac{i}{\hbar}H_0^S(t-t_0)\right] \tag{1.36}$$

Evidently

$$U_0^\dagger = U_0^{-1}$$
$$U_0(t_0, t_0) = 1 \tag{1.37}$$

Differentiating (1.36) with respect to time t, we find that $U_0(t, t_0)$ obeys

$$i\hbar\frac{\partial}{\partial t}U_0 = H_0^S U_0 \tag{1.38}$$

Next we deduce the equation of evolution for the state vector $|\Psi_I(t)\rangle$ in the interaction picture. For the Hamiltonian (1.33), the equation of evolution for the state vector $|\Psi_S(t)\rangle$ in the Schrödinger picture is

$$i\hbar\frac{\partial}{\partial t}|\Psi_S(t)\rangle = (H_0^S + V_S)|\Psi_S(t)\rangle \tag{1.39}$$

By using (1.35), this gives

$$i\hbar\frac{\partial U_0}{\partial t}|\Psi_I(t)\rangle + i\hbar U_0\frac{\partial}{\partial t}|\Psi_I(t)\rangle = [H_0^S + V_S]|\Psi_I(t)\rangle$$

If we use (1.38) and multiply both sides from the left by $U_0^\dagger$, then we obtain the Schrödinger equation for the state vector $|\Psi_I(t)\rangle$ in the interaction picture to be

$$i\hbar\frac{\partial}{\partial t}|\Psi_I(t)\rangle = V_I(t)|\Psi_I(t)\rangle \tag{1.40}$$

where

$$V_I(t) = U_0^\dagger V_S U_0 \tag{1.41}$$

Eq.(1.40) shows that the time evolution of the state vector $|\Psi_I(t)\rangle$ is determined by the interaction Hamiltonian, which emphasizes the effects of the interaction energy.

Since the expectation values of operators are independent of the pictures we adopt, it must have

$$\begin{aligned}\langle A\rangle &= \langle\Psi_S(t)|A_S|\Psi_S(t)\rangle = \langle\Psi_I(t)|U_0^\dagger A_S U_0|\Psi_I(t)\rangle \\ &= \langle\Psi_I(t)|A_I(t)|\Psi_I(t)\rangle\end{aligned} \tag{1.42}$$

From the above equation, we find the transformation law for the operators between the Schrödinger picture and the interaction picture as

$$A_I(t) = U_0^\dagger(t, t_0) A_S U_0(t, t_0) \tag{1.43}$$

Since not only the state vector $|\Psi_I(t)\rangle$ but also the operators depend on time in the interaction picture, we also have to discuss the equation of motion for operators in the interaction picture. Differentiating both sides of (1.43) with respect to t, and using (1.38) and its adjoint, we obtain

$$\begin{aligned} i\hbar\frac{d}{dt}A_I =& i\hbar U_0^\dagger A_S \frac{\partial}{\partial t}U_0 + i\hbar\frac{\partial}{\partial t}U_0^\dagger A_S U_0 + U_0^\dagger i\hbar\frac{\partial}{\partial t}A_S U_0 \\ =& U_0^\dagger A_S H_0^S U_0 - U_0^\dagger H_0^S A_S U_0 + U_0^\dagger i\hbar\frac{\partial}{\partial t}A_S U_0 \end{aligned} \tag{1.44}$$

As H_0^S is time-independent, and

$$[H_0^S, U_0] = 0 \tag{1.45}$$

then

$$H_0^S = H_0^I \tag{1.46}$$

Thus eq.(1.44) becomes

$$i\hbar\frac{d}{dt}A_I = [A_I, H_0^I] + i\hbar U_0^\dagger \frac{\partial}{\partial t}A_S U_0 = [A_I, H_0^S] + i\hbar U_0^\dagger \frac{\partial}{\partial t}A_S U_0 \tag{1.47}$$

This is the equation of motion for the operator $A_I(t)$ in the interaction picture. In accordance with (1.47) and the initial condition, we can in principle obtain the time evolution of the operator $A_I(t)$.

1.3.2 A formal solution of the state vector $|\Psi_I(t)\rangle$ by the perturbation theory

We return to discuss equation (1.40) and look for an expression of the state vector $|\Psi_I(t)\rangle$. From (1.35) we know $|\Psi_I(t_0)\rangle = |\Psi_S(t_0)\rangle$ at time $t = t_0$. Introducing a unitary transformation operator $U(t, t_0)$, such that

$$|\Psi_I(t)\rangle = U(t, t_0)|\Psi_I(t_0)\rangle \tag{1.48}$$

This means that the state vector evolves into $|\Psi_I(t)\rangle$ from $|\Psi_I(t_0)\rangle$ with the time development. Substituting (1.48) into (1.40) and noticing that $|\Psi_I(t_0)\rangle$ may be chosen arbitrarily, then U satisfies

$$i\hbar\frac{d}{dt}U(t, t_0) = V_I(t)U(t, t_0) \tag{1.49}$$

Solving equation (1.49) and considering $V_I(t)$ and the initial condition

$$U(t_0, t_0) = 1 \tag{1.50}$$

we can obtain the exact or perturbation solution of $U(t,t_0)$ in principle. Thus, we can get the state vector $|\Psi_I(t)\rangle$, and furthermore the state vector $|\Psi_S(t)\rangle$ in the Schrödinger picture

$$|\Psi_S(t)\rangle = U_0^\dagger(t,t_0)|\Psi_I(t)\rangle = U_0^\dagger(t,t_0)U(t,t_0)|\Psi_S(t_0)\rangle \tag{1.51}$$

Now we may obtain an approximate solution of the operator $U(t,t_0)$ from (1.49) by the perturbation theory. Integrating both sides of (1.49) and using the initial condition (1.50) we have

$$U(t,t_0) = 1 + \frac{1}{i\hbar}\int_{t_0}^{t} V_I(t_1)U(t_1,t_0)dt_1 \tag{1.52}$$

If we let $t_1 = t_2$ and the dummy integration variable $t_1 \longrightarrow t_2$, we may rewrite $U(t,t_0)$ as

$$U(t_1,t_0) = 1 + \frac{1}{i\hbar}\int_{t_0}^{t_1} V_I(t_1)U(t_2,t_0)dt_2 \tag{1.53}$$

Substituting (1.53) into the integrand in (1.52), then obtain

$$\begin{aligned} U(t,t_0) =& 1 + \frac{1}{i\hbar}\int_{t_0}^{t} V_I(t_1)U(t_1,t_0)dt_1 \\ &+ \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^{t} dt_1 V_I(t_1)\int_{t_0}^{t_1} dt_2 V_I(t_2)U(t_2,t_0) \end{aligned} \tag{1.54}$$

We may iterate this procedure indefinitely and obtain the series expansion of $U(t,t_0)$ as

$$U(t,t_0) = 1+\sum_{n=1}^{\infty}\left(\frac{1}{i\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n V_I(t_1)V_I(t_2)\cdots V_I(t_n) \tag{1.55}$$

If the interaction energy $V_I(t)$ is much smaller than H_0, this series is likely to converge rapidly. Inserting (1.55) into (1.48) we may obtain the perturbation expansion of the state vector at time t in the interaction picture.

As an example, we assume that the system is initially in one eigenket $|i\rangle$ of the unperturbed Hamiltonian H_0, so that

$$H_0|i\rangle = E_i|i\rangle \tag{1.56}$$

Due to the influence of the interaction energy $V_I(t)$, the system evolves into the state vector $|\Psi_I(t)\rangle$ with the time development. Now the question arises about the probability of finding the system in another eigenket $|k\rangle$ of H_0 different from $|i\rangle$ at time t. Clearly, the probability amplitude gives

$$\langle k|\Psi_I(t)\rangle = \langle k|U(t,t_0)|i\rangle \tag{1.57}$$

Inserting (1.55) into the above equation, we obtain

$$\begin{aligned}\langle k|\Psi_I(t)\rangle = {}& \frac{1}{i\hbar}\int_{t_0}^{t}\langle k|V_I(t_1)|i\rangle dt_1 \\ &+ \left(\frac{1}{i\hbar}\right)^2\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\langle k|V_I(t_1)V_I(t_2)|i\rangle \\ &+\cdots\end{aligned} \tag{1.58}$$

Substituting equation (1.41) into (1.58) and retaining the term in the first-order terms in $V_I(t)$, we have

$$\langle k|\Psi_I(t)\rangle \approx \frac{1}{i\hbar}\int_{t_0}^{t}\langle k|V_S|i\rangle\exp(i\omega_{ki}t)dt_1 \tag{1.59}$$

where

$$\omega_{ki} = (E_k - E_i)/\hbar$$

which is the transition frequency. Similarly, we can take into account the probability amplitude expression with high order of V_I, then obtain the probability amplitude of the system from initial state $|i\rangle$ to final state $|k\rangle$ at time t by means of equation (1.58).

1.4 The density operator

From the previous discussion we have learned that if the state vector of the system at time t is known, the time behavior of the system can be determined