Section 1 Linear Algebra

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Let V be a real vector space of dimension at least 3 and let $T \in \mathrm{End}_{\mathbb{R}}(V)$. Prove that there is a non-zero subspace W of V, $W \neq V$, such that $T(W) \subseteq W$.

(*Indiana*)

Solution.

Make V into an $\mathbb{R}[\lambda]$-module by defining $\lambda \cdot v = T(v)$ for all $v \in V$. Thus for $\sum_{i=0}^{n} a_i \lambda^i \in \mathbb{R}[\lambda]$ and $v \in V$

$$\left(\sum_{i=0}^{n} a_i \lambda^i\right) \cdot v = \sum_{i=0}^{n} a_i T^i(v).$$

It is clear that a subspace W of V is an $\mathbb{R}[\lambda]$-submodule of V if and only if $T(W) \subseteq W$.

Now suppose V is a simple $\mathbb{R}[\lambda]$-module. Then $V \simeq \mathbb{R}[\lambda]/I$ for some maximal ideal of $\mathbb{R}[\lambda]$. Since $\mathbb{R}[\lambda]$ is a P.I.D., there exists an irreducible polynomial $f(\lambda)$ of $\mathbb{R}[\lambda]$ such that $I = (f(\lambda))$. So

$$3 \leq \dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}} \mathbb{R}[\lambda]/(f(\lambda)) = \deg f(\lambda).$$

This implies that we have an irreducible polynomial $f(\lambda)$ with degree ≥ 3 in $\mathbb{R}[\lambda]$. This is a contradiction. Hence V is not a simple $\mathbb{R}(\lambda)$-module, that is, there is a non-zero subspace W of V, $W \neq V$, such that $T(W) \subseteq W$.

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Let V be a finite dimensional vector space over a field K.

Let S be a linear transformation of V into itself. Let W be an invariant subspace of V (that is, $SW \subseteq W$). Let $m(t)$, $m_1(t)$, and $m_2(t)$ be the minimal polynomial of S as linear transformation of V, W and V/W respectively.

(a) Prove that $m(t)$ divides $m_1(t) \cdot m_2(t)$.