polynomial of the linear transformation $T : V \to V$. Then $m(\lambda) \in \mathrm{Ann}_{\mathbb{R}[\lambda]}(v)$. Since $m(\lambda)$ is irreducible, we have

$$\mathbb{R}[\lambda]/(m(\lambda)) \simeq \mathbb{R}[\lambda] \cdot v = V$$

(we may assume that $V \neq 0$). So V is an irreducible $\mathbb{R}[\lambda]$-module. Thus, V does not have proper T-invariant subspace.

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Let A be an $n \times n$ matrix with entries in $\mathbb{C}$. Show that A has n distinct eigenvalues in $\mathbb{C}$ if and only if A commutes with no nonzero nilpotent matrix.

(*Indiana*)

Solution.

Necessity. Suppose that A has n distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ in $\mathbb{C}$. Then there exists an invertible $n \times n$ matrix P such that

$$PAP^{-1} = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}.$$

If A commutes with some nilpotent matrix B, we have to show $B = 0$. Since $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct and

$$PAP^{-1} = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}$$

commutes with $P^{-1}BP$, $P^{-1}BP$ is a diagonal matrix. But the nilpontency of B implies that $P^{-1}BP$ is nilpotent. Hence we have $P^{-1}BP = 0$. So $B = 0$.

Sufficiency. Suppose that the characteristic polynomial of A has multiple roots. We have to show that A commutes with some nonzero nilpotent matrix. Let $\mathrm{diag}(J_1, J_2, \cdots, J_t)$ be the Jordan canonical form of A and

$$PAP^{-1} = \mathrm{diag}(J_1, J_2, \cdots, J_t),$$

where P is an $n \times n$ invertible matrix and J_i is a Jordan block of order e_i. Without loss of generality, we may assume that $e_1 > 1$ (If all the $e_i = 1$, then it is easy to see that A commutes with some nonzero nilpotent matrix).

Let B_1 be the Jordan block of order e_1, with 0 on the diagonal. Then $J_1B_1 = B_1J_1$ and $B_1(\neq 0)$ is nilpotent. Let

$$B' = \mathrm{diag}(B_1, B_2, \cdots, B_t)$$

where $B_i(i \geq 2) = 0 \in M_{e_i}(\mathbb{C})$. Then

$$B' \cdot PAP^{-1} = PAP^{-1} \cdot B'.$$

Taking $B = P^{-1}B'P$, we have $B \neq 0$, which is nilpotent, and $AB = BA$.