1105

Suppose V is a finite dimensional vector space over a field K, $T : V \to V$ a linear map such that the minimal polynomial of T coincides with the characteristic polynomial, which is the square of an irreducible polynomial in $K[T]$. Show that if $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$ are any three non-zero vectors in V, then at least two of the three subspaces spaned by the sets $\{T^i\overrightarrow{u}\}_{i\geq 0}$, $\{T^i\overrightarrow{v}\}_{i\geq 0}$ and $\{T^i\overrightarrow{w}\}_{i\geq 0}$ coincide.

(*Stanford*)

Solution.

V can be viewed as a module over the polynomial ring $F[\lambda]$ simply by $f(\lambda)\cdot x = f(T)\cdot(x)$ for any $x \in V$, $f(\lambda) \in F[\lambda]$. Let $\{u_1, u_2, \cdots, u_n\}$ be a base of V over F, $A = (a_{ij})_{n\times n}$ be the matrix of T relative to the base. In general, a normal form for $\lambda I - A$ in $M_n(F[\lambda])$ has the form

$$\text{diag}\{1, \cdots, 1, d_1(\lambda), \cdots, d_s(\lambda)\}$$

where the $d_i(\lambda)$ are monic of positive degree and $d_i(\lambda)|d_j(\lambda)$ if $i \leq j$. By the structure theory of finite generated modules over P.I.D., there exist $z_i(i = 1, 2, \cdots, s) \in V$ such that $V = F[\lambda]\cdot z_1 \oplus F[\lambda]\cdot z_2 \oplus \cdots \oplus F[\lambda]\cdot z_s$ where $\text{Ann}(z_i) = (d_i(\lambda))$. Here, according to the assumptions, the minimal polynomial $m(\lambda)$ of T is $\det(\lambda I - A)$, so

$$m(\lambda) = d_s(\lambda) = \det(\lambda I - A).$$

Hence $s = 1$ and

$$V = F[\lambda]\cdot z_1 \cong F[\lambda]/(m(\lambda))$$

is cyclic. Since $m(\lambda)$ is the square of some irreducible polynomial, $V = F[\lambda]\cdot z_1$ has exactly two non-zero submodules. Obviously, the three subspaces generated by the sets $\{T^i\overrightarrow{u}\}_{i\geq 0}$, $\{T^i\overrightarrow{V}\}_{i\geq 0}$ and $\{T^i\overrightarrow{w}\}_{i\geq 0}$ are non-zero submodules of V over $F[\lambda]$. So at least two of them coincide.

1106

Let V be a finite dimensional vector space over $\mathbb{C}$ with basis $\{v_1, \cdots, v_n\}$. Let σ be a permutation on $\{v_1, \cdots, v_n\}$ and thus induce a linear transformation A on V. Show that A is diagonalizable.

(*Harvard*)