1105

Suppose V is a finite dimensional vector space over a field K, $T : V \to V$ a linear map such that the minimal polynomial of T coincides with the characteristic polynomial, which is the square of an irreducible polynomial in $K[T]$. Show that if $\vec{u}$, $\vec{v}$ and $\vec{w}$ are any three non-zero vectors in V, then at least two of the three subspaces spaned by the sets $\{T^i\vec{u}\}_{i\geq 0}$, $\{T^i\vec{v}\}_{i\geq 0}$ and $\{T^i\vec{w}\}_{i\geq 0}$ coincide.

(*Stanford*)

Solution.

V can be viewed as a module over the polynomial ring $F[\lambda]$ simply by $f(\lambda)\cdot x = f(T)\cdot(x)$ for any $x \in V$, $f(\lambda) \in F[\lambda]$. Let $\{u_1, u_2, \cdots, u_n\}$ be a base of V over F, $A = (a_{ij})_{n\times n}$ be the matrix of T relative to the base. In general, a normal form for $\lambda I - A$ in $M_n(F[\lambda])$ has the form

$$\mathrm{diag}\{1, \cdots, 1, d_1(\lambda), \cdots, d_s(\lambda)\}$$

where the $d_i(\lambda)$ are monic of positive degree and $d_i(\lambda)|d_j(\lambda)$ if $i \leq j$. By the structure theory of finite generated modules over P.I.D., there exist $z_i (i = 1, 2, \cdots, s) \in V$ such that $V = F[\lambda]\cdot z_1 \oplus F[\lambda]\cdot z_2 \oplus \cdots \oplus F[\lambda]\cdot z_s$ where $\mathrm{Ann}(z_i) = (d_i(\lambda))$. Here, according to the assumptions, the minimal polynomial $m(\lambda)$ of T is $\det(\lambda I - A)$, so

$$m(\lambda) = d_s(\lambda) = \det(\lambda I - A).$$

Hence $s = 1$ and

$$V = F[\lambda]\cdot z_1 \cong F[\lambda]/(m(\lambda))$$

is cyclic. Since $m(\lambda)$ is the square of some irreducible polynomial, $V = F[\lambda]\cdot z_1$ has exactly two non-zero submodules. Obviously, the three subspaces generated by the sets $\{T^i\vec{u}\}_{i\geq 0}$, $\{T^i\vec{V}\}_{i\geq 0}$ and $\{T^i\vec{w}\}_{i\geq 0}$ are non-zero submodules of V over $F[\lambda]$. So at least two of them coincide.

1106

Let V be a finite dimensional vector space over $\mathbb{C}$ with basis $\{v_1, \cdots, v_n\}$. Let σ be a permutation on $\{v_1, \cdots, v_n\}$ and thus induce a linear transformation A on V. Show that A is diagonalizable.

(*Harvard*)

Solution.

By re-ordering the elements $v_1, v_2, \cdots, v_n$, we assume that

$$\sigma = (v_1 \cdots v_{i_1})(v_{i_1+1} \cdots v_{i_2}) \cdots (v_{i_s+1} \cdots v_n), \quad (1 \le i_1 < i_2 < \cdots < i_s < n),$$

when σ is expressed as the product of disjoint cycles (This decomposition may have 1-cycles). Let W_j be the subspace of V generated by $\{V_{i_{j-1}} + 1, \cdots, v_{i_j}\}$ for $j = 1, 2, \cdots, s+1$ ($i_0 = 0, i_{s+1} = n$). Then the W_j are invariant subspaces of A and $V = W_1 \oplus W_2 \oplus \cdots \oplus W_{s+1}$. Let M_j be the matrix of $A|_{W_j} : W_j \rightarrow W_j$ relative to the base $\{v_{i_{j-1}+1}, \cdots, v_{i_j}\}$ of W_j over $\mathbb{C}$. Then $M = \text{diag}\{M_1, \cdots, M_{s+1}\}$ is the matrix of A relative to the base $\{v_1, v_2, \cdots, v_n\}$. So it suffices to prove that every M_j is diagonalizable.

Hence, without loss of generality, we may assume that σ is the n-cycle $(v_1, v_2, \cdots, v_n)$. The matrix of A relative to the base $\{v_1, v_2, \cdots, v_n\}$ is

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

It is easy to see that the minimal polynomial of M is $\lambda^n - 1$, and thus M is diagonalizable.

This completes the proof that A is diagonalizable.

1107

Let V be a finite dimensional vector space over the field of rational numbers. Suppose T is a non-singular linear transformation of V such that $T^{-1} = T^2 + T$. Prove that 3 divides the dimension of V, and prove that if $\dim V = 3$, then all such $T's$ are similar.

(*Harvard*)

Solution.

Since $T^{-1} = T^2 + T$, T is annihilated by the polynomial $\lambda^3 + \lambda^2 - 1$. Obviously, $\lambda^3 + \lambda^2 - 1$ is irreducible over the field Q of rational numbers. Thus $\lambda^3 + \lambda^2 - 1$ is the minimal polynomial $m(\lambda)$ of T.

Now let n be the dimension of V over Q, A be the matrix of T relative to some base of V, $\text{diag}\{\overbrace{1, \cdots, 1}^{n-s}, d_1(\lambda), \cdots, d_s(\lambda)\}$ be the normal form for $\lambda I - A$